- $T_i \curvearrowright$ integrable modules
- $\mathrm{Ad}\,T_i \curvearrowright U_{\mathfrak{g}}$

  denoted by T_i hereafter

braid relation

$$\begin{tikzcd} \widetilde{W} \arrow[r, two heads] & W \arrow[l, bend left] \end{tikzcd}$$

braid group

$w \mapsto \widetilde{w}$

$\cancel{s_i^2 = 1}$

$\ell(ww') = \ell(w) + \ell(w')$
$\Rightarrow \widetilde{ww'} = \widetilde{w}\,\widetilde{w'}$

T_w = operator corr. to $\widetilde{w}$.

<u>Prop</u> (1) $w \in W$ and $w(\alpha_i) \in \Delta^+ \Rightarrow T_w(f_i) \in U_{\mathfrak{g}}^-$
(2) $w(\alpha_i) = \alpha_k \Rightarrow T_w(f_i) = f_k$

(proof) <u>case 1°</u> $\mathfrak{g}$: rank 2 $\quad I = \{i, j\}$ $\quad$ o

$a_{ij} = -1 \quad a_{ji} = -1$

Exercise $\rightarrow$ $\left(\begin{array}{l} -2 \\ -3 \end{array} \right.$

<u>case 2°</u> general case

Consider a coset $w\langle s_i, s_j \rangle \in W/\langle s_i, s_j \rangle$

$\in$

w' : minimum length representative

$w = w' \underbrace{w''}_{\in \langle s_i, s_j \rangle}$

<u>Claim</u> (1) $\ell(w's_i) = \ell(w') + \ell(s_i)$
$\ell(w's_j) = \ell(w') + \ell(s_j)$

(2) $\ell(w) = \ell(w') + \ell(w'')$

proof : Exercise

$$T_w(f_i) = T_{w'} \underbrace{T_{w''}(f_i)}_{\in U_q^- \text{ by case } 1^\circ}$$

By induction on $\ell(w)$, may assume the assertion is true for $T_{w'}$

$\longrightarrow$ (1) is OK for T_w

(2) $w(\alpha_i) = \alpha_k$

enough to show that $w''(\alpha_i) = \alpha_i$ or α_j

$$w''(\alpha_i) = a\alpha_i + b\alpha_j \qquad a, b \in \mathbb{Z}_{\geq 0}$$

$$\Rightarrow \underset{\substack{\| \\ \alpha_k}}{w(\alpha_i)} = w'w''(\alpha_i) = a\underbrace{w'(\alpha_i)}_{\in \Delta^+} + b\underbrace{w'(\alpha_j)}_{\in \Delta^+}$$

Assumption

by Claim

$$\Longrightarrow \begin{matrix} a = 0 & \text{or} & b = 0 \\ b = 1 & & a = 1 \end{matrix}$$

//

Cor. $w = s_{i_1} \cdots s_{i_k}$ reduced expr. of $w \in W$

$$\Longrightarrow T_{i_1} T_{i_2} \cdots T_{i_{k-1}}(f_{i_k}) \in U_q^-$$

$\because$ $s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}) \in \Delta^+$ //

w_0 : longest element $\quad l(w_0) = \nu = \# \Delta^+$

$\parallel$ $s_{i_1} \cdots s_{i_\nu}$ reduced expr

$$\vec{h} = (i_1 \cdots i_\nu) \in I^\nu$$
$$\vec{c} = (c_1, \cdots, c_\nu) \in \mathbb{Z}_{\geq}^\nu$$

$L(\vec{c}, \vec{h}) \equiv L(\vec{c})$

if $\vec{h}$ is clear from the context.

$$:= f_{i_1}^{(c_1)} T_{i_1}(f_{i_2}^{(c_2)}) \cdots T_{i_1} \cdots T_{i_{\nu-1}}(f_{i_\nu}^{(c_\nu)})$$

$$f_i^{(c)} = \frac{f_i^c}{[c]!} \quad \text{divided power}$$

$\mathfrak{q}_i = \mathfrak{q}^{d_i}$

$wt = \sum s_{i_1} \cdots s_{i_{p-1}}(\alpha_{i_p}) \times c_p$

Th 1. Fix $\vec{h}$. $\{ L(\vec{c}) \mid \vec{c} \in \mathbb{Z}_{\geq 0}^\nu \}$ is a base of $U_\mathfrak{q}^-$ as a $\mathbb{Q}(\mathfrak{q})$-vector space

(PBW base)

(proof) Usual PBW thm for $U(n^-)$

and $\displaystyle \lim_{\mathfrak{q}\to 1} U_\mathfrak{q}^- = U(n^-)$ $\qquad \displaystyle \bigoplus_{\alpha \in \Delta^-} \mathfrak{g}_\alpha$

$$\Longrightarrow \dim_{\mathbb{Q}(\mathfrak{q})} (U_\mathfrak{q}^-)_\xi = \dim U(n^-)_\xi$$

$= \#$ of ways to express ξ as a sum of Δ^-

$\therefore$ enough to show that $L(\vec{c})$ is linearly independent

By the induction from above, we show $\{L(\vec{c}) \mid c_1 = \cdots = c_p = 0\}$ is linealy independent.

1° $p = \nu \Rightarrow L(\vec{c}) = 1$

2° Assume the assertion is true for p.

$c_1 = \cdots = c_{p-1} = 0$

$$L(\vec{c}) = T_{i_1} \cdots T_{i_{p-1}}(f_{i_p}^{(c_p)}) L(\vec{c}')$$

$$T_{i_p}^{-1} T_{i_{p-1}}^{-1} \cdots T_{i_1}^{-1} L(\vec{c})$$

$$= \frac{1}{[c_p]_{\xi_{i_p}}!} \underbrace{(-e_{i_p} t_{i_p})^{c_p}}_{U_\xi^+ U_\xi^0} \times \underbrace{T_{i_p}^{-1} \cdots T_{i_1}^{-1} L(\vec{c}')}_{U_\xi^-}$$

triangular decomp. $\quad U_\xi \cong U_\xi^+ \otimes U_\xi^0 \otimes U_\xi^-$

$\Longrightarrow$ linearly independence is true also for $p-1$.

//

<u>Rem</u>. PBW base depends on the choice of reduced expr. $w_0 = s_{i_1} \cdots s_{i_\nu}$.

$$-\mathrm{wt}\, T_{i_1} \cdots T_{i_{p-1}}(f_{i_p}) = \underbrace{s_{i_1} \cdots s_{i_{p-1}}(\alpha_{i_p})}_{\beta_p} \in \Delta^+$$

Another reduced expr.
$w_0 = s_{i'_1} \cdots s_{i'_\nu}$

$T_{i_1} \cdots T_{i_{p-1}}(f_{i_p})$

$\beta_p = s_{i'_1} \cdots s_{i'_{q-1}}(\alpha_{i'_q})$

$T_{i'_1} \cdots T_{i'_{q-1}}(f_{i'_q})$

they are not scalar multiple in general

$f \in \mathbb{Q}(q)$

$\text{LHS} \neq f \cdot \text{RHS}$

No notion of "root subspaces" for U_q^-

Prop (Levendorskii–Soibelman formula) (convexity)

$\beta_p :=$ as above

$F_{\beta_p} := T_{i_1} \cdots T_{i_{p-1}}(f_{i_p})$

Suppose $p > p'$

$$F_{\beta_p} F_{\beta_{p'}} = q^{-(\beta_p, \beta_{p'})} F_{\beta_{p'}} F_{\beta_p} + \sum_{\vec{c}} a_{\vec{c}}\, L(\vec{c})$$

If $a_{\vec{c}} \neq 0 \Rightarrow c_q \neq 0$ only if $p' < q < p$

$\vec{i} \rightsquigarrow$ total order on Δ^+

$$\cdots \beta_{p'} \underbrace{\cdots\cdots}_{\beta_q} \beta_p \cdots$$

(proof) Apply $T_{i_{p-1}}^{-1} \cdots T_{i_1}^{-1}$ to LHS.

$$f_{i_p} \left(T_{i_{p-1}}^{-1} \cdots T_{i_{p'+1}}^{-1} \right) \underbrace{(-t_{i_{p'}}^{-1} e_{i_{p'}})}_{U_q^+ U_q^0}$$

$$= q^{-(\alpha_{i_p}, s_{i_{p-1}} \cdots s_{i_{p'+1}}(\alpha_{i_{p'}}))}$$

$$\times \left(T_{i_{p-1}}^{-1} \cdots T_{i_{p'+1}}^{-1} \right)(-t_{i_{p'}}^{-1} e_{i_{p'}}) f_{i_p}$$

$$+ U_q^{\geq 0}$$

$$\left([f_i, e_j] = \delta_{ij} \frac{t_i - t_i^{-1}}{q_i - q_i^{-1}} \right)$$

$$= \underline{q^{-(\beta_p, \beta_{p'})}}\, T_{i_{p-1}}^{-1} \cdots T_{i_1}^{-1} \underline{(F_{\beta_{p'}} F_{\beta_p})} + U_q^{\geq 0}$$

1st term of RHS

$$\Rightarrow c_q = 0 \quad \text{if } q \geq p$$

Similarly we apply $T_{i_{p'}}^{-1} \cdots T_{i_1}^{-1}$ to LHS

$$\underbrace{T_{i_{p'+1}} \cdots T_{i_{p-1}}(f_{i_p})}_{\in U_q^-} \underbrace{(-t_{i_{p'}}^{-1} e_{i_{p'}})}_{\in U_q^{\geq 0}}$$

$$= \cdots + U_q^{\leq 0}$$

same argument shows $c_q = 0$ if $q \leq p'$

//

$\vec{k}$: fix $\quad L(\vec{c})^- = L(\vec{c})$ above

$L(\vec{c})^+ = L(\vec{c}) \quad e_i \leftrightarrow f_i$

$\hat{U}_q^+$

<u>Prop</u>. T_w : braid group op. on a tensor product of integrable modules

$$T_w = \sum_{\vec{c}} \prod_{k=1}^{p} (-1)^{c_k} q^{-c_k(c_k-1)/2} \prod_{a=1}^{c_k} (q_{i_k}^{a} - q_{i_k}^{-a})$$

$$\times L(\vec{c})^+ \otimes L(\vec{c})^- \circ (T_w \otimes T_w)$$

braid grp operators on tensor factors

(proof) 10.19 Prop 3. — sl_2-case

+ induction on the length on w.

Start from 11:21

§ change of base

Consider the case $sl_3 = A_2$ first

$\{i, j\} = \{1, 2\}$ $\qquad w_0 = s_1 s_2 s_1 = s_2 s_1 s_2$

$f_{12} = T_1(f_2) = f_2 f_1 - q f_1 f_2 \quad \vec{k} \quad \vec{k}'$

$f'_{12} = T_2(f_1) = f_1 f_2 - q f_2 f_1$

$$L(\vec{c},\vec{t}) = f_1^{(c_1)} f_{12}^{(c_2)} f_2^{(c_3)}$$

$$L(\vec{c}',\vec{t}') = f_2^{(c_1')} f_{12}^{(c_2')} f_1^{(c_3')}$$

<u>Q.</u> How they are related?

<u>Idea</u>: introduce an "intermediate" base
monomial
$\rightsquigarrow$ canonical base (for sl_3)

<u>Lemma</u> ($f_i^{(p)} = 0$ if $p<0$)

$$f_2^{(l)} f_1^{(m)} f_2^{(n)} = \sum_{k=0}^{l} q^{(l-k)(m-k)} \begin{bmatrix} l-k+n \\ l-k \end{bmatrix} f_1^{(m-k)} f_{12}^{(k)} f_2^{(l-k+n)}$$

$$f_1^{(l)} f_2^{(m)} f_1^{(n)} = \sum_{k=0}^{n} q^{(m-k)(n-k)} \begin{bmatrix} n-k+l \\ n-k \end{bmatrix} f_1^{(n-k+l)} f_{12}^{(k)} f_2^{(m-k)}$$

(proof) <u>Exercise</u>

Show $(\star)$ $f_2^{(l)} f_1^{(m)} = \sum_k q^{(l-k)(m-k)} f_1^{(m-k)} f_{12}^{(k)} f_2^{(l-k)}$

by double induction on l and m.

$(\star) \times f_2^{(n)} \longrightarrow$ upper equality

$f_1^{(l)} \times (\star)\Big|_{\substack{l\to m \\ m\to n}} \longrightarrow$ lower equality //

Observation 1 $\quad f_2^{(\ell)} f_1^{(\ell+n)} f_2^{(n)} = f_1^{(n)} f_2^{(\ell+n)} f_1^{(\ell)}$

$(m = \ell + n)$

Observation 2

$$q^{(\ell-k)(m-k)} \underbrace{\begin{bmatrix} \ell-k+n \\ \ell-k \end{bmatrix}}_{\substack{\uparrow \\ q^{-(\ell-k)n}(1+q\mathbb{Z}[q])}} \in q^{(\ell-k)(m-k-n)} \times (1+q\mathbb{Z}[q])$$

from the definition

q-binomial coeff. $\longleftrightarrow$ cohomology of Grassmannian mfd

$(\ell-k)n$ — cpx dim. of Grassmann

$q \leftrightarrow q^{-1} \quad \longleftrightarrow \quad$ Poincaré duality

Assume

$\begin{cases} \ell \geq k \\ m \geq \ell+n \ (\geq k+n) \end{cases} \Longrightarrow$ above $\in \delta_{k\ell} + q\mathbb{Z}[q]$

$$f_2^{(\ell)} f_1^{(m)} f_2^{(n)} = f_1^{(m-\ell)} f_{12}^{(\ell)} f_2^{(n)} \leftarrow k = \ell \text{ term}$$

$$+ \sum_{k=0}^{\ell-1} \underbrace{a_k}_{\substack{\uparrow \\ q\mathbb{Z}[q]}} f_1^{(m-k)} f_{12}^{(k)} f_2^{(\ell-k+n)}$$

$m \geq \ell + n$

$(\Leftrightarrow m - \ell \geq n)$

Similarly

$$f_1^{(l)} f_2^{(m)} f_1^{(n)} = f_1^{(l)} f_{12}^{(n)} f_2^{(m-n)} + \sum_{k=0}^{n-1} a_k' f_1^{(n-k+l)} f_{12}^{(k)} f_2^{(m-k)}$$

$m \geq l+n$

$(\Leftrightarrow m-n \geq l)$

$$\mathcal{B}(\infty) := \{ f_2^{(l)} f_1^{(m)} f_2^{(n)} \mid m \geq l+n \}$$
$$\cup \{ f_1^{(l)} f_2^{(m)} f_1^{(n)} \mid m \geq l+n \}$$

$\left(\cap \quad \begin{matrix} m = l+n \\ l \leftrightarrow n \end{matrix} \right)$

<u>Th 2</u> (1) $\mathcal{B}(\infty)$ is a base of U_q^-

(2) $\mathbb{Z}[q]$-submodule of U_q^- generated by $\mathcal{B}(\infty)$ $\leftarrow \mathcal{L}_{\mathbb{Z}}(\infty)$

= 〃 $\{L(\vec{c}, \vec{k})\}$

= 〃 $\{L(\vec{c}', \vec{k}')\}$

(3) $\mathbb{Z}$-base of $\mathcal{L}_{\mathbb{Z}}(\infty) / q\mathcal{L}_{\mathbb{Z}}(\infty)$ induced from $\mathcal{B}(\infty)$

= 〃 $\{L(\vec{c}, \vec{k})\}$

= 〃 $\{L(\vec{c}', \vec{k}')\}$

Recall we used $A_1 = \{ f \in \mathbb{Q}(q) \mid f \text{ is regular at } q=1 \}$

$\rightsquigarrow q = 1$

$\mathbb{Z}[q] \subset \mathbb{Q}(q)$

can be used for the specialization at $q=0$

$\mathcal{L}_{\mathbb{Z}}(\infty)/q\mathcal{L}_{\mathbb{Z}}(\infty)$ —— U_q^- specialization at $q=0$

(2), (3) $\Leftarrow$ PBW base is independent of the choice of the reduced expression at $q=0$!

Kashiwara: crystal base $\leftarrow$ (3)

q — temperature for solvable lattice models (original motivation of Jimbo to introduce U_q)

$q \to 0$ absolute temperature 0

(proof) base transition matrix

$\{L(\vec{c}, \vec{i})\} \longleftrightarrow B(\infty)$

$$\begin{bmatrix} 1 & \ddots & q\mathbb{Z}[q] \\ 0 & & 1 \end{bmatrix}$$

under suitable ordering on base elements

//

$\mathbb{A} = \mathbb{Z}[q, q^{-1}]$

${}_{\mathbb{A}}U_q^- = \mathbb{A}$-subalgebra generated by $f_i^{(n)}$ $\quad i = 1, 2,\ n \in \mathbb{Z}_{>0}$

<u>Cor</u> $${}_{\mathbb{A}}U_q^- = \bigoplus_{\vec{c}} \mathbb{A} L(\vec{c}, \vec{h}) = \bigoplus_{\vec{c}'} \mathbb{A} L(\vec{c}, \vec{h}') = \bigoplus_{b \in B(\infty)} \mathbb{A} b$$

$\because$ 2nd, 3rd equ. are conseq. of Th2.

$$L(\vec{c}, \vec{h}) = f_1^{(c_1)} \cdots$$
$$L(\vec{c}', \vec{h}') = f_2^{(c'_1)} \cdots$$

$\bigoplus \mathbb{A} L(\vec{c}, \vec{h})$: preserved under the left mult. of $f_1^{(n)}$

$\|$

$\bigoplus \mathbb{A} L(\vec{c}', \vec{h}')$: $\quad$ // $\quad f_2^{(n)}$ //

$$\{L(\vec{c}, \vec{h})\} \longleftrightarrow B(\infty) \longleftrightarrow \{L(\vec{c}', \vec{h}')\}$$

transition matrix give a bijection among $\vec{c}$, $\vec{c}'$

piecewise linear bijection
$$\begin{cases} c'_1 = c_2 + c_3 - \min(c_1, c_3) \\ c'_2 = \min(c_1, c_3) \\ c'_3 = c_1 + c_2 - \min(c_1, c_3) \end{cases}$$