11.16
11.23 | no lectures

Recall

$\mathfrak{g} = A_2 = \mathfrak{sl}_3$ $\qquad w_0 = s_1 s_2 s_1 = s_2 s_1 s_2$

$\vec{i} = (121)$, $\vec{i}' = (212)$

$\{L(\vec{c}, \vec{i})\}$ $\qquad \mathcal{B}(\infty)$ $\qquad \{L(\vec{c}', \vec{i}')\}$

PBW base $\qquad$ canonical base

$\subset U_q^-$

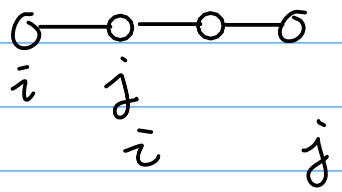
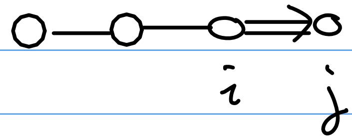
<u>Th 2</u>. (2) $\mathbb{Z}[q]$-submodule spanned by $\{L(\vec{c}, \vec{i})\}$
$=$ $\qquad$ " $\qquad$ $\mathcal{B}(\infty)$
$=$ $\qquad$ " $\qquad$ $\{L(\vec{c}', \vec{i}')\}$
$=: \mathcal{L}_{\mathbb{Z}}(\infty)$

(3) $\mathbb{Z}$-base of $\mathcal{L}_{\mathbb{Z}}(\infty) / q\mathcal{L}_{\mathbb{Z}}(\infty)$ induced from $\{L(\vec{c}, \vec{i})\}$
$=$ $\qquad$ " $\qquad$ $B(\infty)$
$=$ $\qquad$ " $\qquad$ $\{L(\vec{c}', \vec{i}')\}$

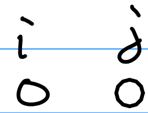
Th 2$'$ = statements (2), (3) for $\{L(\vec{c}, \vec{i})\}$, $\{L(\vec{c}', \vec{i}')\}$

$\mathfrak{g}$: finite type (cpx simple Lie alg)

<u>Th 3</u> $\vec{i}, \vec{i}'$: reduced expr. of w_0
PBW bases $\{L(\vec{c}, \vec{i})\}$, $\{L(\vec{c}', \vec{i}')\}$
Assume Th 2$'$ holds for $\{i, j\} \subset I$ = index set of simple roots
$i \neq j$

(e.g.  

$\Longrightarrow$ Th 2' holds for $\{L(\vec{c}, \vec{h})\}$, $\{L(\vec{c}', \vec{h}')\}$ of $\mathfrak{g}$

In particular, Th 2' holds for A_2, $A_1 + A_1$ 
└ obvious

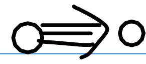
$\Longrightarrow$ true for $\mathfrak{g}$: ADE

(proof) $\mathcal{B}_W$ = braid group assoc. with W
$= \langle \sigma_i \mid \sigma_i \sigma_j \cdots = \sigma_j \sigma_i \cdots \rangle$

← rank 2 relation involving $\{i, j\} \subset I$, $i \neq j$

reduced expr's $\vec{h}, \vec{h}'$ are transformed by successive applications rank 2 relations.

Assumption $\Longrightarrow$ true for $\mathfrak{g}$ //

general $\mathfrak{g}$ true $\Longleftarrow$ true B_2 
G_2 

<u>Facts</u> Th 2' true for B_2, G_2

complicated calculation → B_2

Xi. J. of Alg. <u>218</u> (1999)

G_2 ← Kashiwara ground loop argument

Shoji - Zhou 1810.04378

elementary 1910.05532

alternative proof

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§ bilinear form on U_q^-

<u>Aim</u> PBW base : orthogonal base w.r.t. (,)

(not orthonormal) (bilinear form inner product)

$(L(\vec{c},\vec{h}), L(\vec{c}',\vec{h}'))$

= explicit

categorification quiver Hecke algebra

category of fin. dim. modules

Grothendieck group $[M]$ <u>rel</u> $0 \to M_1 \to M_2 \to M_3 \to 0$

$\Leftrightarrow [M_2] = [M_1] + [M_3]$

$\cong$

$_{A}U_q^-$ integral form of U_q^- Khovanov-Lauda Rouquier

{simple module} $\leftrightarrow$ dual canonical base Varagnolo-Vasserot

{Certain module} $\leftrightarrow$ PBW base S. Kato

$\mathrm{Ext}^{\bullet}(,)$ vanishing $\longleftrightarrow$ orthogonality

${}'U_q^- =$ free $\mathbb{Q}(q)$-ass. alg.
generated by $\{f_i\}_{i\in I}$
(not imposing q-Serve rel.)

$$\mathrm{wt}(f_{i_1}\cdots f_{i_\ell}) \overset{\text{def.}}{=} -\alpha_{i_1} - \cdots - \alpha_{i_\ell}$$

Define a <u>twisted</u> multiplication on ${}'U_q^- \otimes {}'U_q^-$

$$(x_1\otimes x_2)\cdot(x_1'\otimes x_2') := q^{-(\mathrm{wt}\, x_2, \mathrm{wt}\, x_1')} x_1 x_1' \otimes x_2 x_2'$$

associative

Define a twisted comultiplication

$$r : {}'U_q^- \longrightarrow {}'U_q^- \otimes {}'U_q^-$$ alg. hom. w.r.t. <u>twist mult.</u> on RHS

$$r(f_i) = f_i \otimes 1 + 1 \otimes f_i$$

<u>Rem</u>. $\Delta : U_q \longrightarrow U_q \otimes U_q$ coproduct

usual mult. componentwise

not cocommutative

r ; twist mult.

$\cdot\ \sigma\circ r = r$

σ : $1 \leftrightarrow 2$

But Δ and r are <u>essentially</u> same

Prop 1. $\exists 1$ $\mathbb{Q}(q)$-valued bilinear form $(\,,\,)$ on ${}'U_q^-$

s.t. (1) $(f_i, f_j) = \delta_{ij} / 1 - q_i^2$

(2) $(x, yy') = (r(x), y \otimes y')$

(3) $(xx', y) = (x \otimes x', r(y))$

where $(\,,\,)$ on ${}'U_q^- \otimes {}'U_q^-$ is given by

$$(x_1 \otimes x_2, y_1 \otimes y_2) = (x_1, y_1)(x_2, y_2)$$

Moreover $(\,,\,)$ is symmetric.

(proof) $({}'U_q^-)^* =$ graded dual of ${}'U_q^-$

$$= \bigoplus_{\xi} ({}'U_q^-)^*_{\xi}$$

$\llcorner$ wt space

r is compatible with weights

$r^* =$ transpose of r : $({}'U_q^-)^* \otimes ({}'U_q^-)^* \to ({}'U_q^-)^*$

multiplication.

associative $\longleftarrow$ r : coassociative

$(({}'U_q^-)^*, r^*)$: $\mathbb{Q}(q)$-alg.

Define $\phi : {}'U_q^- \longrightarrow ({}'U_q^-)^*$ alg. hom preserving wts

$$f_i \longmapsto \xi_i$$

where $\xi_i \in ({}'U_q^-)^*_{-\alpha_i}$ with $\langle \xi_i, f_i \rangle = \dfrac{1}{1-q_i^2}$

Then we set $(x, y) := \langle \phi(y), x \rangle$

properties (1) : OK

(2) $(x, yy') = \langle \phi(yy'), x \rangle$

$= \langle \phi(y)\phi(y'), x \rangle$ mult. on $({}'U_f^-)^*$

$= \langle \phi(y) \otimes \phi(y'), r(x) \rangle = (r(x), y \otimes y')$

def. of mult. = transpose of r

(3) $(xx', y) = (x \otimes x', r(y))$

induction on wt y

<u>step 1°</u> $y = f_i \implies x = f_i,\ x' = 1$

or $x = 1,\ x' = f_i$

OK

<u>step 2°</u> (3) is true for $y, y' \implies$ (3) is true for yy'

(Exercise)

$(x, y) = 0$ unless wt x = wt y

- uniqueness ((1).(2) are enough)
- symmetric $\Longleftarrow$ uniqueness

$(x, y)' = (y, x)$ satisfies (1),(2),(3)

$\implies (x, y)' = (x, y)$ //

<u>Lemma 2</u> (1) $r(f_i^{(p)}) = \sum_{t+s=p} q_i^{-ts} f_i^{(t)} \otimes f_i^{(s)}$

(2) $(f_i^{(p)}, f_i^{(p)}) = \prod_{s=1}^{p} \frac{1}{1 - q_i^{2s}}$

(proof) (1) induction

(2) $\Leftarrow$ (1) //

<u>Lemma 3</u>. $r(\mathrm{ad}^*(f_i^{(m)})(f_j))$

$$= 1 \otimes \mathrm{ad}^*(f_i^{(m)})(f_j) + \sum_{k=0}^{m} \prod_{h=0}^{k-1} (1 - q_i^{2h+2(1-m-a_{ij})}) q_i^{-k(m-k)} \mathrm{ad}^*(f_i^{(m-k)})(f_j) \otimes f_i^{(k)}$$

(proof) (Exercise)

Use q-binom thm //

<u>Prop 4</u>. $r(\mathrm{ad}^*(f_i^{(1-a_{ij})})(f_j)) = 1 \otimes \mathrm{ad}^*(f_i^{(1-a_{ij})})(f_j)$

$+ \ \mathrm{ad}^*(f_i^{(1-a_{ij})})(f_j) \otimes 1$

(proof) Take $m = 1 - a_{ij}$ in Lemma 3.

$$q_i^{2h+2(1-m-a_{ij})} = q_i^{2h} \underset{h=0}{=} 1$$

$\therefore$ 2nd term vanish unless $k=0$ //

11:33 ~

<u>Cor 5</u>. r induces $U_q^- \longrightarrow U_q^- \otimes U_q^-$ alg. hom.

$\because$ $U_q^- = {}'U_q^- / q\text{-Serre relation} \iff \mathrm{ad}^*(f_i^{(1-a_{ij})})(f_j) = 0$

10.19

Th 6. $(\, , \,)$ descends to U_q^-.
$(\, , \,)$ satisfies (1), (2), (3)
unique, symmetric

(proof) $\wedge \operatorname{ad}^*(f_i^{(1-a_{ij})})(f_j) \perp U_q^-$,
enough to show w.r.t. $(\, , \,)$

$$0 = (\operatorname{ad}^*(f_i^{(1-a_{ij})})(f_j), f_i^{(p)} f_j f_i^{(p')}) \quad p + p' = 1 - a_{ij}$$
$$= (r(r(\quad)), f_i^{(p)} \otimes f_j \otimes f_i^{(p')})$$
$$= 0$$
/ OK

//

Prop 4

Recall $* : U_q^- \circlearrowleft$ anti-auto $f_i^* = f_i$

Lemma 7 $r(x^*) = (t \circ r(x))^{* \otimes *}$

$t : 1 \longleftrightarrow 2$ swap factors

(proof)
Step 1° $x = 1, f_i$ OK

Step 2° true for x, x' $\Longrightarrow$ true for xx'
(Exercise)

Prop. 8 $(x, y) = (x^*, y^*)$

(proof) enough to show that RHS satisfies (1), (2), (3)

(1) : OK

(2) : $(x^*, (yy')^*) = (r(x)^{*\otimes *}, y^* \otimes y'^*)$

↑ follows from Lem 7.

(3) : symmetric + (2) //

Def. 9 $_ir : U_q^- \to U_q^-$, $r_i : U_q^- \to U_q^-$ by

Define

$$r(x) = f_i \otimes {}_ir(x) + \cdots$$
$$= r_i(x) \otimes f_i + \underbrace{\cdots\cdots}_{\substack{\text{wt of } 1^{st} \text{ (or } 2^{nd}) \\ \text{cmp.} \neq -\alpha_i}}$$

Lemma 10 $(f_i y, x) = (f_i, f_i)(y, {}_ir(x))$

$(y f_i, x) = (f_i, f_i)(y, r_i(x))$

(proof)

$$(f_i y, x) = (f_i \otimes y, \underbrace{r(x)}_{\substack{\parallel \\ f_i \otimes {}_ir(x) + \cdots}}) =$$

$\perp$

//

Note invariant inner product

$$([f_i, y], x) = (y, [f_i, x]) \quad =?$$

$x, y \in \mathfrak{g}$

${}_i r$, r_i are characterised by the following properties:

$$\begin{cases} {}_i r(1) = 0, & {}_i r(f_j) = \delta_{ij} \\ {}_i r(xx') = q^{(\mathrm{wt}\,x, \alpha_i)} x \cdot {}_i r(x') + {}_i r(x) x' \end{cases}$$

r_i : similar

$$r_i = * \circ {}_i r \circ *$$

<u>Lemma 11</u>. $$e_i x - x e_i = \frac{r_i(x) t_i - t_i^{-1}\, {}_i r(x)}{q_i - q_i^{-1}}$$

(proof) <u>Step 1°</u> true for $x = 1, f_j$

<u>Step 2°</u> true for $x, x' \Rightarrow$ true for xx' //

<u>Lemma 12</u> $$\bigcap_i \operatorname{Ker} {}_i r = \mathbb{Q}(q) 1 = \bigcap_i \operatorname{Ker} r_i$$

(proof) Using $*$, enough to show the assertion for ${}_i r$

Show

$${}_i r(x) = 0 \quad \forall i \implies x = 0$$

$\mathrm{wt}\, x \neq 0$ by induction on wt }

<u>Claim</u>. $${}_i r \circ r_j = r_j \circ {}_i r$$

(:) step 1 true for $1, f_k$

<u>step 2</u>. true for $x, x' \Rightarrow$ true for xx' //

$$ {}_i r \circ r_j(x) = r_j \circ \underbrace{{}_i r(x)}_{=0} \quad \text{by assumption} $$

Induction hypo $\Rightarrow r_j(x) = 0 \quad \forall j$

Lemma 11 $\Rightarrow [e_i, x] = 0 \quad \forall i$

$V(\lambda)$: irr. f.d. rep. of U_q

$\ni v_\lambda$

$$ e_i \underbrace{x v_\lambda}_{\text{not h.w. vector}} = x \underbrace{e_i v_\lambda}_{=0} \quad \forall i $$

$\therefore x v_\lambda = 0 \quad \lambda \to \infty \quad x = 0$ //

<u>Th</u>.13. $(\, , \,)$ is non-degenerate.

$\because$ $x \in$ radical of $(\, , \,)$ induction on wt x

- wt $x = 0 \Rightarrow$ ok
- $\underbrace{(f_i y, x)}_{=0 \text{ assump.}} = \underbrace{(f_i, f_i)}_{\neq 0} \underbrace{(y, {}_i r(x))}_{=0} \quad \forall y$

Indud. hypo $\Rightarrow {}_i r(x) = 0 \ \forall i$

Lem 12 $\Rightarrow x = 0$ //