Def Take $i \in I$

$U_q^-[i] :=$ subalgebra generated by $ad^*(f_i^{(m)})(f_j)$ $j \neq i$, $0 \leqq m \leqq -a_{ij}$

${}^*U_q^-[i] :=$ " $ad(f_i^{(m)})(f_j)$

Th1. (1) $U_q^-[i] = \{x \in U_q^- \mid T_i^{-1}(x) \in U_q^-\} = \mathrm{Ker}\, {}_ir$

${}^*U_q^-[i] = \{x \in U_q^- \mid T_i(x) \in U_q^-\} = \mathrm{Ker}\, r_i$

(2) $U_q^- = \bigoplus_{n \geqq 0} f_i^n U_q^-[i] = \bigoplus_{n \geqq 0} {}^*U_q^-[i] f_i^n$

Note

- 1026 Cor 8 $\quad T_i^{-1} = * T_i *$
- 1109 Lem 7 $\quad {}_ir = * r_i *$

Thus it is enough to show the case without $*$

Recall $\quad T_i^{-1}(ad^*(f_i^{(m)})(f_j)) = \underbrace{ad(f_i^{(-a_{ij}-m)})(f_j)}_{\in U_q^-}$

(1026 Prop 6)

$\therefore \quad U_q^-[i] \subset \{x \in U_q^- \mid T_i^{-1}(x) \in U_q^-\}$

We prepare several lemmas

Lemma 2 $U_{\mathfrak{f}}^- = \sum_{n\geq 0} f_i^n U_{\mathfrak{f}}^-[i]$

$\because$ $ad^*(f_i^{(m)}) f_j f_i$

$= ad^*(f_i)(ad^*(f_i^{(m)}) f_j) + f_i t_i ad^*(f_i^{(m)})(f_j) t_i^{-1}$

$ad^*(f_i)(x) = x f_i - f_i t_i x t_i^{-1}$

1019 definition of ad

1^{st} term $= [m+1]_i \; ad^*(f_i^{(m+1)})(f_j)$

2^{nd} term $= q_i^{\langle h_i, m\alpha_i + \alpha_j\rangle} f_i \, ad^*(f_i^{(m)})(f_j)$

Thus we can move f_i to the left. //

Lemma 3

$\{x \in U_{\mathfrak{f}}^- \mid T_i^{-1}(x) \in U_{\mathfrak{f}}^-\} \subset \operatorname{Ker} {}_i r$

(proof) $x \in$ LHS

Write $r_i(x) = \sum_{n\geq 0} f_i^n T_i(u_n)$ $\exists$

${}_i r(x) = \sum_{n\geq 0} f_i^n T_i(v_n)$ $\exists$

(Lemma 2 + $U_{\mathfrak{f}}^-[i] \subset U_{\mathfrak{f}}^- \cap T_i U_{\mathfrak{f}}^-$)

$e_i x - x e_i = \dfrac{r_i(x) t_i - t_i^{-1}\, {}_i r(x)}{q_i - q_i^{-1}}$ 1109 Lemma 1 1

T_i^{-1}

$-t_i^{-1} f_i T_i^{-1}(x) + T_i^{-1}(x) t_i^{-1} f_i \in t_i^{-1} U_{\mathfrak{f}}^-$ by assumption

$$= \frac{1}{\xi_i - \xi_i^{-1}} \sum_{n \geq 0} \Big(\underset{\substack{\Uparrow \\ e_i^n t_i^{n-1} U_\xi^-}}{(-e_i t_i)^n u_n t_i^{-1}} - \underset{\substack{\Uparrow \\ e_i^n t_i^{n+1} U_\xi^-}}{t_i (-e_i t_i)^n v_n} \Big)$$

$\therefore$ triangular decomposition of U_ξ^-

$$\Rightarrow \quad u_n = 0 \quad \text{for } \forall n > 0$$

$$v_n = 0 \quad \text{for } \forall n \geq 0 \qquad //$$

(Proof of Th 1)

We have $U_\xi^-[i] \subset U_\xi^- \cap T_i U_\xi^- \subset \mathrm{Ker}\, {}_i r$ so far.

$$\therefore \quad U_\xi^- = \sum_{n \geq 0} f_i^n \, \mathrm{Ker}_i r$$

We show that this is a direct sum decomposition.

Suppose $\quad 0 = \sum_{n=0}^{n_0} f_i^n x_n \qquad x_n \in \mathrm{Ker}_i r$

We show $\forall x_n = 0$ by the induction on n_0

$n_0 = 0 \Rightarrow$ obvious

Suppose the assertion is true for $n_0 - 1$

$$0 = {}_i r \Big(\sum_{n=0}^{n_0} f_i^n x_n \Big) = \sum_{n=1}^{n_0} [n]_i \, \xi_i^{n-1} f_i^{n-1} x_n$$

$\therefore \; x_1 = \cdots = x_n = 0 \quad$ by induction hypothesis

$$\therefore\ 0=\sum_{n=0}^{n_0} f_i^n x_n = x_0 \quad //$$

Cor 4 $U_q^- = U_q^-[i] \overset{\perp}{\oplus} f_i U_q^- =^{*} U_q^-[i] \overset{\perp}{\oplus} U_q^- f_i$

(∵) · $\oplus$ is already shown.

· 1109 Lemma 10 : $(f_i y, x) = (f_i, f_i)(y, {}_i r(x))$

$\Longrightarrow \quad \perp$ //

Th 5. (1) Let $x, y \in U_q^-[i]$.

$$(x, y) = (T_i^{-1}(x), T_i^{-1}(y))$$

(2) (orthogonality of PBW base)

Fix a reduced expression $w_0 = s_{i_1} \cdots s_{i_\nu}$

$\vec{i} = (i_1, \cdots, i_\nu)$

$$(L(\vec{c}_1, \vec{i}), L(\vec{c}_2, \vec{i})) = \delta_{\vec{c}_1 \vec{c}_2} \prod_{p=1}^{\nu} (f_{i_p}^{(c_{1,p})}, f_{i_p}^{(c_{1,p})})$$

$$f_{i_1}^{(c_1)} T_{i_1}(f_{i_2}^{(c_2)}) \cdots \qquad = \quad \text{''} \quad \prod_{p=1}^{\nu} \prod_{d=1}^{c_{1,p}} \frac{1}{1-q_{i_p}^{2d}}$$

The proof of Th5(1) is elementary, but requires some computation. $\Longrightarrow$ We skip it.

(2) follows from (1)

References Lusztig: Prop 38.2.1

Jantzen: Prop. 8.28

Recall We considered $\mathbb{Z}[q] \subset \mathbb{Q}(q)$ on 1102.
Introduce its "rational" version :

$$A_0 \overset{\text{def.}}{=} \{ f \in \mathbb{Q}(q) \mid f \text{ is regular at } q=0 \}$$

not 1

Recall also $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$

1102 $\quad {}_{\mathcal{A}}U_q^- = \mathcal{A}$-subalg generated by $f_i^{(n)}$

$$\parallel \bigoplus_{\vec{c}} \mathcal{A}\, L(\vec{c}, \vec{h}) \quad \text{for type } A_2$$

1109 Th 2 about $\mathbb{Z}[q]$-submodule & base on $q=0$

Th 3.
Th2 holds for rank 2 $\Rightarrow$ holds in general

The same argument applies to

$${}_{\mathcal{A}}U_q^- = \bigoplus_{\vec{c}} \mathcal{A} L(\vec{c}, \vec{h})$$

true for A_2 ✓

B_2 : not difficult

G_2 : Xi On PBW bases of the quantum group $U_v(G_2)$, Alg. Colloq. $\underline{2}$ (1995)

(This is easier than th 2 for G_2)

We accept this result and

Th 6

$${}_{\mathcal{A}}U_q^- = \bigoplus_{\vec{c}} \mathcal{A} \cdot L(\vec{c}, \vec{h})$$

Prop 7 Let $\mathcal{L}(\infty) := \{x \in U_q^- \mid (x,x) \in A_0\}$

(1) $\mathcal{L}(\infty)$ is an A_0-submodule of U_q^- with base $\{L(\vec{c}, \vec{i})\}$

(2) $x \in {}_A U_q^-$ satisfies $(x,x) - 1 \in q A_0$

$\Longrightarrow x \equiv \pm L(\vec{c}, \vec{i}) \mod q\mathcal{L}(\infty)$

(proof) (1) Write $\underset{\substack{\rotatebox{90}{$\in$} \\ U_q^-}}{x} = \sum_{\vec{c}} a_{\vec{c}} L(\vec{c}, \vec{i})$

Decompose $a_{\vec{c}} = \underset{\substack{\rotatebox{90}{$\in$} \\ \mathbb{Q}}}{\underbrace{p_{\vec{c}}}} q^{-t} + (\text{higher order in } q)$

And take minimum t among all $\vec{c}$

Then $(x,x) \equiv q^{-2t} \sum_{\vec{c}} p_{\vec{c}}^2 \mod q^{-2t+1} A_0$

$\therefore (x,x) \in A_0 \iff t \leqq 0 \iff a_{\vec{c}} \in A_0 \ \forall \vec{c}$

(2) Assume further $\vec{a_c} \in A = \mathbb{Z}[q, q^{-1}]$
$(x,x) \equiv 1 \mod q A_0$

Then $t = 0$, $p_{\vec{c}} \in \mathbb{Z} \ \forall \vec{c}$

$\therefore 1 = \sum p_{\vec{c}}^2 \Rightarrow p_{\vec{c}} = 0$ except one $\vec{c}$

& $p_{\vec{c}} = \pm 1$ //

Note that we only used $\{L(\vec{c},\vec{b})\}$ is an orthogonal base

Thus $\displaystyle\bigoplus_{\vec{c}} A_0 L(\vec{c},\vec{b}) = \mathcal{L}(\infty) = \bigoplus_{\vec{c}'} A_0 L(\vec{c}',\vec{b}')$

$\uparrow$ independent of the choice of the reduced expression!

follows from an easier argument

Also

$\left(\{\pm L(\vec{c},\vec{b}) \bmod q\mathcal{L}(\infty)\} \subset \mathcal{L}(\infty)/q\mathcal{L}(\infty) \right.$ is independent of $\vec{b}$

is proved by an easier argument.

§ canonical base

We fix $\vec{i}$

Recall the bar involution $\overline{}$ on U_q^-

$\bar{q} = q^{-1}$, $\bar{f_i} = f_i$ (Lusztig cf. Beck–N affine)

<u>Th 1</u> If we express $\overline{}$ as a matrix by the PBW base for $\vec{i}$,
it is upper <u>uni</u>triangular w.r.t. the lexicographic order
(diagonal $= 1$)

$$\left(\vec{c} < \vec{d} \overset{\text{def}}{\iff} \exists p = 0,1,\dots,\nu \text{ s.t. } c_1 = d_1, \dots, c_{p-1} = d_{p-1},\ c_p < d_p \qquad \vec{c} = (c_1,\dots,c_\nu),\ \vec{d} = (d_1,\dots,d_\nu) \right)$$

Define $\overline{T_i} := \overline{} \circ T_i \circ \overline{} : U_q \circlearrowleft$

<u>Lemma 2</u> V: integrable representation of U_q
s.t. V has $\overline{} : V \circlearrowleft$ s.t. $\bar{\bar{v}} = v$, $\overline{(xv)} = \bar{x} \cdot \bar{v}$ ($x \in U_q$, $v \in V$)

e.g. $V(\lambda) = U_q^- / (f_i^{\langle h_i, \lambda \rangle + 1})$

Define $\Pi : V \longrightarrow V$ by

$$c(m) = q^{-m(m+3)/2} (q - q^{-1}) \cdots (q^m - q^{-m})$$

$$\Pi(v) = \sum_{m \geq 0} \overline{c(m)}_{q \to q_i} f_i^{(m)} e_i^{(m)} t_i^m v$$

Then (1) $T_i = \Pi \circ \overline{T_i}$

(2) $x \in U_q$, $v \in V$ $\quad T_i(x) \Pi(v) = \Pi(\overline{T_i}(x) v)$

(3) If $e_i v = 0$, $T_i(x) v \equiv \overline{T_i}(x) v \mod f_i V$

(proof) (1) enough to show the case $\mathfrak{g} = \mathfrak{sl}_2$
$\Rightarrow$ direct calculation. $\leftarrow$ Exercise

(2) compatibility of T_i for algebra and representation
integrable

(3) $T_i(x)v = T_i(x)\,\Pi(v)$ as $e_i v = 0$
$\overset{(2)}{=} \Pi\, T_i(xv)$
$\equiv \overline{T_i}(xv)$ modulo $f_i V$ by the definition of Π. //

Let ${}^i\pi : U_q^- \longrightarrow U_q^-[i]$, $\pi^i : U_q^- \longrightarrow {}^*U_q^-[i]$
orthogonal projection given by Cor. 4

<u>Lemma 3</u>. $x \in {}^*U_q^-[i] \Rightarrow T_i \circ \pi^i(\overline{x}) = {}^i\pi(\overline{T_i(x)})$

∵ Let $y = T_i(x) \in U_q^-[i]$

$$\overline{x} = \underset{\substack{\cap \\ {}^*U_q^-[i]}}{\pi^i(\overline{x})} + \underset{\substack{\cap \\ U_q^- f_i}}{x' f_i}$$

$V_\lambda = V(\lambda) \ni v_\lambda$

$${}^i\pi(\overline{y})v_\lambda \equiv \overline{y} v_\lambda \quad \text{mod } f_i V$$
$$= \overline{T_i(\overline{x})} v_\lambda$$
$$\equiv T_i(\overline{x}) v_\lambda \text{ by Lemma 2 (3)}$$

$$= (T_i \pi^i(\overline{x}) + T_i(x')(-t_i^{-1}e_i))v_\lambda$$
$$= T_i\ \pi^i(\overline{x})v_\lambda$$

$\therefore\ {}^i\pi(\overline{T_i(x)}) - T_i\,{}^i\pi(\overline{x}) \in f_i U_q^- + \sum U_q^- f_j^{\lambda_j+1}$

Take sufficiently large λ,

$$\underbrace{{}^i\pi(\overline{T_i(x)})}_{\in\, U_q^-[i]} - \underbrace{T_i\,{}^i\pi(\overline{x})}_{\in\, U_q^-[i]} \in f_i U_q^-$$

but

$\therefore = 0$ //

(proof of Th 1)

$$\overline{L(\vec{c},\vec{i})} = L(\vec{c},\vec{i}) + \sum_{\vec{d} > \vec{c}} a_{\vec{d}} L(\vec{d},\vec{i})$$

We show this by induction on p s.t.

$c_{p+1} = \cdots = c_\nu = 0$ (for any reduced expr)

☆ $p=0 \to$ obvious

☆ Suppose $c_1 = 0$. Then $L(\vec{c},\vec{i}) \in U_q^-[i_1]$ $\quad {}^{/\!/}\, f_{i_1}^{(c_1)} T_{i_1}(f_{i_2}^{(c_2)})$

$T_i^{-1} L(\vec{c},\vec{i}) = f_{i_2}^{(c_2)} T_{i_2}(f_{i_3}^{(c_3)}) \cdots \in {}^*U_q^-[i_1]$

$\therefore$ Lemma 3 $\Rightarrow \overline{L(\vec{c},\vec{i})} = \overline{T_{i_1}(\qquad)}$

$\equiv T_{i_1} \circ \pi^{i_1}(\overline{f_{i_2}^{(c_2)} T_{i_2}(f_{i_3}^{(c_3)}) \cdots})$ ☆

mod $f_{i_1} U_q^-$

Let $\vec{c}\,' = (c_2, c_3, \cdots, c_\nu, 0)$, $\vec{i}\,' = (i_2, i_3, \cdots, i_\nu, \tau(i_1))$

Defined by $w_0 = s_{i_2} \cdots s_{i_\nu} s_{\tau(i_1)}$

$\Longleftrightarrow \alpha_{\tau(i_1)} = -w_0 \alpha_{i_1}$

By induction hypothesis

$$\overline{L(\vec{c}',\vec{e}')} = L(\vec{c}',\vec{e}') + \sum_{\vec{d}'>\vec{c}'} a_{\vec{d}'} L(\vec{d}',\vec{e}')$$

Write

$$\vec{d}' = (d_2, d_3, \dots, d_\nu, d_1)$$

$$L(\vec{d}',\vec{e}') = f_{i_2}^{(d_2)} T_{i_2}(f_{i_3}^{(d_3)}) \cdots (T_{i_2}\cdots T_{i_\nu})(f^{(d_1)}_{\tau(i_1)})$$

$$s_{i_2}\cdots s_{i_\nu}(\alpha_{\tau(i_1)}) = w_0^2 \alpha_{i_1} = \alpha_{i_1}$$

$$\therefore f_{i_1}^{(d_1)}$$

If $d_1 \neq 0 \Rightarrow \pi^{i_1}(L(\vec{d}',\vec{e}')) = 0$

If $d_1 = 0 \Rightarrow \pi^{i_1}(L(\vec{d}',\vec{e}')) = L(\vec{d}',\vec{e}')$

$$\therefore \overline{L(\vec{c},\vec{e})} \overset{☆}{=} \underbrace{T_{i_1} L(\vec{c}',\vec{e}')}_{= L(\vec{c},\vec{e})} + \sum_{\substack{\vec{d}'>\vec{c}' \\ d_1=0}} a_{\vec{d}'} \underbrace{T_{i_1} L(\vec{d}',\vec{e}')}_{= L(\vec{d},\vec{e})}$$

$$\vec{d} = (0, d_2, \dots, d_\nu)$$

$$\vec{d}' > \vec{c}' \iff \vec{d} > \vec{c}$$

Also $\quad f_{i_1} U_q^- = \bigoplus_{d_1>0} \mathbb{Q}(q) L(\vec{d},\vec{e})$ ← always larger than $\vec{c}$

$\therefore$ OK in this case

For general $\vec{c}$: $L(\vec{c},\vec{a}) = f_{i_1}^{(c_1)} L(\underset{\substack{\parallel \\ (0,c_2,\cdots,c_n)}}{\vec{c}'},\vec{a})$

$$\overline{L(\vec{c},\vec{a})} = f_{i_1}^{(c_1)} \overline{L(\vec{c}',\vec{a})}$$

$$= f_{i_1}^{(c_1)} L(\vec{c}',\vec{a}) + f_{i_1}^{(c_1)} \sum_{\vec{d}' > \vec{c}'} a_{\vec{d}'} L(\vec{d}',\vec{c}')$$

this also satisfies the condition //