Recall $\{L(\vec{c},\vec{i})\}$ is an $\mathcal{A}=\mathbb{Z}[q,q^{-1}]$-base of ${}_{\mathcal{A}}U_q^-$

($\mathbb{Z}[q,q^{-1}] \rightsquigarrow \mathbb{Q}[q,q^{-1}]$)
The following construction works also

- $\overline{} : U_q^- \circlearrowleft$ $\overline{f_i}=f_i$, $\overline{q}=q^{-1}$
upper unitriangular w.r.t. $\{L(\vec{c},\vec{i})\}$

$\vec{i}$: reduced expr. of w_0 (fix)

<u>Th 1</u>. (canonical base)
$\exists 1$ $\{b(\vec{c},\vec{i})\}$ character by the following properties

(1) $\overline{b(\vec{c},\vec{i})} = b(\vec{c},\vec{i})$

(2) $L(\vec{c},\vec{i}) = b(\vec{c},\vec{i}) + \sum_{\vec{d}>\vec{c}} a_{\vec{d}}\, b(\vec{d},\vec{i})$

s.t. $a_{\vec{d}} \in q\mathbb{Z}[q]$

<u>cf</u> Kazhdan-Lusztig base of Hecke algebra.
(analog of canon. base) std base (analog of PBW base)
same algorithm as above

(proof) descending induction on each weight space

If $\vec{c}$: maximal, $\overline{L(\vec{c},\vec{i})} = L(\vec{c},\vec{i})$

$b(\vec{c},\vec{h}) := L(\vec{c},\vec{h})$ (1), (2) are satisfied.

Suppose $b(\vec{d},\vec{h})$ is defined $\vec{d} > \vec{c}$.

We will define $b(\vec{c},\vec{h})$.

$$L(\vec{c},\vec{h}) - \overline{L(\vec{c},\vec{h})} = \sum_{\vec{d}>\vec{c}} \overset{\exists}{b_{\vec{d}}}\, L(\vec{d},\vec{h}) \underset{\substack{\uparrow \\ \text{induction} \\ \text{hypo}}}{=} \sum_{\vec{d}>\vec{c}} b'_{\vec{d}}\, b(\vec{d},\vec{h})$$

Consider $\overline{}$ of both sides

- $\overline{\text{LHS}} = -\text{LHS}$
- $\overline{\text{RHS}} = \sum_{\vec{d}>\vec{c}} \overline{b'_{\vec{d}}}\, b(\vec{d},\vec{h})$

$$\therefore \overline{b'_{\vec{d}}} = -b'_{\vec{d}}$$

$$\therefore b'_{\vec{d}} = \underbrace{a_1 q + a_2 q^2 + \cdots\cdots}_{=\,a_{\vec{d}}} - a_1 q^{-1} - a_2 q^{-2} - \cdots\cdots \qquad a_i \in \mathbb{Z}$$

$$\therefore b'_{\vec{d}} = a_{\vec{d}} - \overline{a_{\vec{d}}}$$

$$a_{\vec{d}} \in q\mathbb{Z}[q]$$

Let

$$b(\vec{c},\vec{h}) = L(\vec{c},\vec{h}) - \sum_{\vec{d}>\vec{c}} a_{\vec{d}}\, b(\vec{d},\vec{h})$$

Then $\overline{b(\vec{c},\vec{h})} = \overline{L(\vec{c},\vec{h})} - \sum_{\vec{d}>\vec{c}} \overline{a_{\vec{d}}}\, b(\vec{d},\vec{h})$

$\therefore\ b(\vec{c},\vec{i}) = \overline{b(\vec{c},\vec{i})} \iff b'_{\vec{d}} = a_{\vec{d}} - \overline{a_{\vec{d}}}$ //

<u>Def:</u> $\{b(\vec{c},\vec{i})\} \subset {}_{\mathcal{A}}U_q^- \subset U_q^-$
canonical base

☆ ADE $\{b(\vec{c},\vec{i})\}$ is independent of the choice of $\vec{i}$ $\quad (\vec{c} \in \mathbb{Z}^{\nu})$

$\mathcal{B}(\infty)$ (as a set) $\quad \nu = \# \Delta^+$

In fact,

$$\begin{array}{ccc} \{b(\vec{c},\vec{i})\} & \longleftrightarrow & \{L(\vec{c},\vec{i})\} \\ & & \updownarrow \\ \{b(\vec{c}',\vec{i}')\} & \longleftrightarrow & \{L(\vec{c}',\vec{i}')\} \end{array}$$

trans. matrix $\in \mathbb{Z}[q]$
id at $q=0$
(piecewise linea bijection)

$$\therefore\ b(\vec{c},\vec{i}) = \sum \underbrace{p_{\vec{c},\vec{c}'}}\ b(\vec{c}',\vec{i}')$$

$\in \mathbb{Z}[q]$, "id" at $q=0$

Consider $\overline{}$

$\therefore\ p_{\vec{c},\vec{c}'} \in \mathbb{Z}[q] \cap \overline{\mathbb{Z}[q]} = \mathbb{Z}$

$\overline{\mathbb{Z}[q]} = \mathbb{Z}[q^{-1}]$

$\therefore\ (p_{\vec{c},\vec{c}'}) =$ "id"

For BCFG $\Leftarrow$ Th2 (11?) for B_2 & G_2
$\llcorner$ Shoji-Zhou

Alternatively we give a characterization of $\{\pm b(\vec{c}, \vec{h})\} = \widetilde{B(\infty)}$ signed canonical base in terms of $(,)$

<u>Th 2</u>. Suppose $x \in {}_{\mathbb{A}}U_q^-$ satisfies

- $(x, x) - 1 \in q\mathbb{A}_0$, where $\mathbb{A}_0 = \{f(q) \in \mathbb{Q}(q) \mid f \text{ is regular at } q=0\}$
- $\overline{x} = x$

$\Longrightarrow \exists\, x = \pm b(\vec{c}, \vec{h})$

(proof) Recall Prop 7 (11/30)

$$\mathcal{L}(\infty) = \bigoplus_{\vec{c}} \mathbb{A}_0 L(\vec{c}, \vec{h}) \underset{\text{Th 1}}{=} \bigoplus_{\vec{c}} \mathbb{A}_0 b(\vec{c}, \vec{h})$$

$$= \{x \in U_q^- \mid (x, x) \in \mathbb{A}_0\}$$

$\therefore$ Assumption $\Longrightarrow$ $x \equiv \pm L(\vec{c}, \vec{h}) \equiv \pm b(\vec{c}, \vec{h})$ mod $q\mathcal{L}(\infty)$

$\left(\begin{array}{l} \overline{x} = x \\ \overline{b(\vec{c}, \vec{h})} = b(\vec{c}, \vec{h}) \end{array} \right.$ $\therefore =$ modulo $\mathbb{Z}[q, q^{-1}] \cap q\mathbb{A}_0 \cap \overline{q\mathbb{A}_0} = \{0\}$ //

$$\mathcal{L}_{\mathbb{Z}}(\infty) := \bigoplus \mathbb{Z}[q] L(\vec{c}, \vec{h}) = \bigoplus \mathbb{Z}[q] b(\vec{c}, \vec{h})$$

<u>Prop 3</u>. $\mathcal{L}_{\mathbb{Z}}(\infty) \cap \overline{\mathcal{L}_{\mathbb{Z}}(\infty)} \longrightarrow \mathcal{L}_{\mathbb{Z}}(\infty) / q\mathcal{L}_{\mathbb{Z}}(\infty)$

$\quad \mathrel{\rotatebox{90}{$\in$}}\ \mathcal{L}_{\mathbb{Z}}(\infty) \nearrow$

is an isom. of $\mathbb{Z}$-modules

$$b(\vec{c}, \vec{i}) \longmapsto b(\vec{c}, \vec{i}) \bmod q\mathcal{L}_{\mathbb{Z}}(\infty)$$
$$\parallel$$
$$L(\vec{c}, \vec{i}) \bmod q\mathcal{L}_{\mathbb{Z}}(\infty)$$

q: variable $\longrightarrow$ $q = 0$

$(\mathbb{Z}[q] \cap \overline{\mathbb{Z}[q]})$ crystal

$\overline{}$-invariant

canonical base $\longleftarrow$ melting

We ~may~ regard $\mathcal{B}(\infty)$ or $\tilde{\mathcal{B}}(\infty)$ as a ~~subset~~ base of $\mathcal{L}_{\mathbb{Z}}(\infty)/q\mathcal{L}_{\mathbb{Z}}(\infty)$

$\left(\mathcal{L}(\infty)/q\mathcal{L}(\infty) \right)$

<u>Recall</u> the intermediate base appeared in A_2

$\parallel$ canonical base

<u>Cor. 4</u> $* : U_q^- \circlearrowleft$

(1) $\tilde{\mathcal{B}}(\infty)$ is invariant under $*$

(2) $\mathcal{B}(\infty)$ 〃

ADE   (<u>Fact</u> BCFG true also)

proof (1) $(x^*, y^*) = (x, y)$ (1109 Prop 8)

$+$ Th2 $\quad x = \overline{x} \Rightarrow \overline{x^*} = x^*$ OK

(2) Lemma 5 $\quad \vec{h} = (i_1, \cdots, i_\nu)$, $\vec{c} = (c_1, \cdots, c_\nu)$

Let $\vec{h}^* = (\iota(i_\nu), \iota(i_{\nu-1}) \cdots, \iota(i_1))$

$\vec{c}^* = (c_\nu, c_{\nu-1}, \cdots, c_1)$

ι : involution of I = index set of simple roots

$w_0(\alpha_i) = -\alpha_{\iota(i)}$

$\Longrightarrow \quad L(\vec{c}, \vec{h})^* = L(\vec{c}^*, \vec{h}^*)$

$\because \quad w_0 = s_{i_1} \cdots s_{i_\nu} = s_{i_\nu} \cdots s_{i_1}$

$\quad \| $

$\quad w_0^{-1}$

$$s_{i_{\nu-1}} \cdots s_{i_1}(\alpha_{\iota(i_\nu)}) = s_{i_\nu} w_0(\alpha_{\iota(i_\nu)})$$
$$= \alpha_{i_\nu}$$
$$s_{i_{\nu-1}} \cdots s_{i_1} s_{\iota(i_\nu)} = s_{i_\nu} s_{i_{\nu-1}} \cdots s_{i_1} = w_0$$

By induction
$$\begin{cases} s_{i_{p-1}} \cdots s_{i_1} s_{\iota(i_\nu)} \cdots s_{\iota(i_p)} = w_0 \\ s_{i_{p-1}} \cdots s_{i_1} s_{\iota(i_\nu)} \cdots s_{\iota(i_{p+1})}(\alpha_{\iota(i_p)}) = \alpha_{i_p} \end{cases}$$

1102 Prop
$$T_{i_{p-1}} \cdots T_{i_1} T_{\iota(i_\nu)} \cdots T_{\iota(i_{p+1})}(f_{\iota(i_p)})$$
$$= f_{i_p}$$

$$T_{\iota(i_\nu)} \cdots T_{\iota(i_{p+1})}(f_{\iota(i_p)}) = T_{i_1}^{-1} T_{i_2}^{-1} \cdots T_{i_{p-1}}^{-1}(f_{i_p})$$
$$= (T_{i_1} \cdots T_{i_{p-1}}(f_{i_p}))^*$$

1026 Cor 8 //

11:22~

§ canonical base on representations

<u>Def 1</u>.

$b \in \widetilde{\mathcal{B}}(\infty) \overset{\text{Define}}{\rightsquigarrow} \varepsilon_i(b), \varepsilon_i^*(b) \in \mathbb{Z}_{\geq 0}$

$b \in f_i^{\varepsilon_i(b)} U_q^- \setminus f_i^{\varepsilon_i(b)+1} U_q^-$

$\varepsilon_i^*(b) = \varepsilon_i(b^*)$ i.e. $b \in U_q^- f_i^{\varepsilon_i^*(b)} \setminus U_q^- f_i^{\varepsilon_i^*(b)+1}$

Take a reduced expr. of w_0 starting from i

(i.e. $i_1 = i$)

$b = \pm b(\vec{c}, \vec{i})$

<u>Lemma 2</u> $\varepsilon_i(b) = c_1$

∵ $b(\vec{c}, \vec{i}) = \underset{\substack{\uparrow \\ f_i^{c_1} U_q^-[i] \\ \setminus \pm f_i U_q^-}}{L(\vec{c}, \vec{i})} + \sum_{\vec{d} > \vec{c}} a_{\vec{d}} L(\vec{d}, \vec{i})$

$\vec{d} > \vec{c} \Rightarrow d_1 \geq c_1$

//

<u>Remark 3</u> To compute $\varepsilon_j(b(\vec{c}, \vec{i}))$ $(j \neq i)$

we first use piecewise linear bijection

$\rightsquigarrow b(\vec{c}, \vec{i}) = \pm b(\vec{c}', \vec{i}')$

$\vec{i}'$ starting from j.

$\varepsilon_j(b(\vec{c}, \vec{i})) = c_1'$

Exercise compute $\varepsilon_1, \varepsilon_2$ for canonical base
$\mathfrak{g} = A_2$

Th4 (1) $\{ b \in \widetilde{B}(\infty) \mid \varepsilon_i(b) \geq n \}$ is a signed base of $f_i^n U_\mathfrak{g}^-$

(2) $\{ b \in \widetilde{B}(\infty) \mid \varepsilon_i^*(b) \geq n \}$ " of $U_\mathfrak{g}^- f_i^n$

(3) λ: dominant integral weight
$V(\lambda)$ = corresponding irreducible rep.

$$= U_\mathfrak{g}^- / \underbrace{\sum_i U_\mathfrak{g}^- f_i^{\langle \lambda, h_i \rangle + 1}}_{=: I_\lambda}$$

$\widetilde{B}(\lambda) = \{ b \bmod I_\lambda \mid \varepsilon_i^*(b) \leq \langle \lambda, h_i \rangle \ \forall i \}$

is a signed base of $V(\lambda)$

canonical base of $V(\lambda)$

(proof) (1) Choose $\vec{h}$ as above.

$\{ L(\vec{c}, \vec{h}) \mid c_1 \geq n \}$: base of $f_i^n U_\mathfrak{g}^-$

$\xrightarrow{\text{Th1}}$ $\{ b(\vec{c}, \vec{h}) \mid \underbrace{c_1 \geq n}_{\Updownarrow} \}$! "

b " $\quad \varepsilon_i(b) \geq n$

(2) $*$

(3) $\widetilde{\mathcal{B}}(\infty) \setminus \widetilde{\mathcal{B}}(\lambda) = \{ \cdots \mid \varepsilon_i^*(b) \geq \langle \lambda, h_i \rangle + 1 \ \exists i \}$

base of I_λ

Exercise Compute $\widetilde{\mathcal{B}}(\lambda)$ for "small" λ.

$\mathfrak{g} = A_2$

Th 5. $b \in \widetilde{\mathcal{B}}(\infty)$ s.t. $\varepsilon_i(b) = 0$

$\exists 1 \ b' \in \widetilde{\mathcal{B}}(\infty)$ s.t. $f_i^{(n)} b \equiv b' \bmod f_i^{n+1} U_{\mathfrak{g}}^-$

Moreover $\cdot \ \varepsilon_i(b') = n$

$\cdot \ \{ b \mid \varepsilon_i(b) = 0 \} \longrightarrow \{ b' \mid \varepsilon_i(b') = n \}$ is a bijection

(!) $\vec{h}$: as above , $b = \pm b(\vec{c}, \vec{h})$

$c_1 = 0$ by assumption.

$$f_i^{(n)} b(\vec{c}, \vec{h}) = f_i^{(n)} \Big(L(\vec{c}, \vec{h}) + \sum a_{\vec{d}} L(\underset{\substack{\shortparallel \\ (d_1, d_2, \cdots)}}{\vec{d}}, \vec{h}) \Big)$$

$$= L(\underset{\substack{\shortparallel \\ (n, c_2, \cdots)}}{\vec{c}'}, \vec{h}) + \sum a_{\vec{d}} \underset{\mathfrak{g} \mapsto \mathfrak{g}_i}{\begin{bmatrix} d_1 + n \\ n \end{bmatrix}_i} L(\underset{\substack{\shortparallel \\ (d_1 + n, d_2, \cdots)}}{\vec{d}'}, \vec{h})$$

$$= L(\vec{c}', \vec{h}) + \sum_{\vec{d} > \vec{c}} a_{\vec{d}} L(\vec{d}', \vec{h})$$

$$\underbrace{+ \sum_{\substack{d_1 > 0 \\ \vec{d} > \vec{c}}} a_{\vec{d}} \left(\begin{bmatrix} d_1 + n \\ n \end{bmatrix}_i - 1 \right) L(\vec{d}', \vec{h})}_{= y}$$

Write $y = \sum_{\substack{\vec{e} > \vec{c} \\ e_1 > n}} c_{\vec{e}}\, b(\vec{e}, \vec{h})$

$$c_{\vec{e}} = c^{+}_{\vec{e}} + c^{0}_{\vec{e}} + c^{-}_{\vec{e}}, \qquad c^{+}_{\vec{e}} \in q\mathbb{Z}[q],\quad c^{0}_{\vec{e}} \in \mathbb{Z},\quad c^{-}_{\vec{e}} \in q^{-1}\mathbb{Z}[q^{-1}]$$

$$f_i^{(n)} b(\vec{c}, \vec{h}) - \sum \left(c^{0}_{e} + c^{-}_{e} + \overline{c^{-}_{e}}\right) b(\vec{e}, \vec{h})$$

$$= L(\vec{c}\,', \vec{h}) + \sum a_{\vec{a}} L(\vec{a}\,', \vec{h}) + \sum \left(c^{+}_{\vec{e}} - \overline{c^{-}_{\vec{e}}}\right) b(\vec{e}, \vec{h})$$

- $\overline{\mathrm{LHS}} = \mathrm{LHS}$
- $\mathrm{RHS} \in \mathcal{L}_{\mathbb{Z}}(\infty)$ & $\mathrm{RHS}\big|_{q=0} = L(\vec{c}\,', \vec{h})$

$\therefore\ \mathrm{LHS} = b(\vec{c}\,', \vec{h})$

$$f_i^{(n)} b(\vec{c}, \vec{h}) = b(\vec{c}\,', \vec{h}) + \sum_{e_1 > n} \left(c^{0}_{e} + c^{-}_{e} + \overline{c^{-}_{e}}\right) b(\vec{e}, \vec{h})$$

//

Kashiwara operator

$$\begin{tikzcd}
\{b \in \widetilde{B}(\infty) \mid \varepsilon_i(b) = n\} \arrow[r, leftrightarrow] \arrow[dr, "\tilde{f}_i"'] & \{b \in \widetilde{B}(\infty) \mid \varepsilon_i(b) = 0\} \arrow[d, leftrightarrow] \\
 & \{b \in \widetilde{B}(\infty) \mid \varepsilon_i(b) = n+1\}
\end{tikzcd}$$

$$\tilde{e}_i \overset{\text{def}}{=} \tilde{f}_i^{-1} : \{\varepsilon_i(b) = n\} \longrightarrow \{\varepsilon_i(b) = n-1\} \quad \text{for } n > 0$$

$\tilde{e}_i b \overset{\text{def.}}{=} 0 \quad (n=0)$

$\tilde{e}_i : \tilde{B}(\infty) \longrightarrow \tilde{B}(\infty) \sqcup \{0\}$

$b = b(\vec{c}, \vec{h})$ start from i

$\tilde{f}_i b = b(\vec{c}', \vec{h})$, $\vec{c}' = (c_1+1, c_2, \cdots)$

$\tilde{e}_i b = b(\vec{c}'', \vec{h})$, $\vec{c}'' = (c_1-1, c_2, \cdots)$

<u>Th 6</u>. $\tilde{B}(\infty) = \{ \pm \tilde{f}_{i_1} \cdots \tilde{f}_{i_p} 1 \mid p \geq 0,\ i_1, \cdots, i_p \in I \}$

$\tilde{B}(\infty) \ni b$

$\because$ may assume $b \neq \pm 1$

$\exists i$ s.t. ${}_i r(b) \neq 0$ (1109 Lemma 2)

$b = \pm b(\vec{c}, \vec{h})$ start from i

Recall $U_q^- = \bigoplus_n f_i^n \underbrace{U_q^-[i]}_{= \operatorname{Ker}{}_i r}$

$\therefore c_1 > 0$

$\therefore \tilde{e}_i^{c_1} b = \pm b(0, c_2, \cdots)$

$b = \pm \tilde{f}_i^{c_1} b(0, c_2, \cdots)$ apply the same argument //

$\overline{\mathcal{B}}(\infty)$ ($\mathcal{B}(\infty)$) $\longleftarrow$ <u>crystal</u>

$+ \tilde{e}_i, \tilde{f}_i$ combinatrial skelton of representation.

<u>Execise</u> compute $\tilde{e}_i, \tilde{f}_i$ on $\overline{\mathcal{B}}(\lambda)$ for $\mathfrak{g} = A_2$

"small"