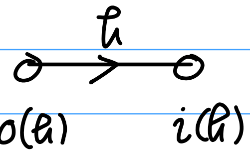
§ quiver and its representation

Def

A (finite) quiver is

$Q=(Q_0,Q_1)$ finite oriented graph

Q_0: set of vertices, Q_1: set of oriented edges



k: field

Def. A representation of Q (over k) is

- $V=\bigoplus_{i\in Q_0} V_i$ Q_0-graded (finite dimensional) vector space / k
- $B=\bigoplus_{h\in Q_1} B_h$ $B_h : V_{o(h)} \longrightarrow V_{i(h)}$ linear

$$\dim V \equiv \dim(V,B) = (\dim V_i)_{i\in Q_0} \in \mathbb{Z}^{Q_0}$$

(dimension vector)

$$\mathrm{Hom}_Q((V,B),(V',B')) = \{ \xi : V \to V' \text{ linear, } Q_0\text{-grading preserving} \}$$

$$\begin{tikzcd}
V_{o(h)} \arrow[r, "B_h"] \arrow[d, "\xi_{o(h)}"'] \arrow[dr, phantom, "\circlearrowright"] & V_{i(h)} \arrow[d, "\xi_{i(h)}"] \\
V'_{o(h)} \arrow[r, "B_h"] & V'_{i(h)}
\end{tikzcd}$$

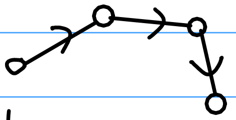
- isomorphic representations
- direct sum
- $\mathrm{Ker}\,\xi$, $\mathrm{Im}\,\xi$, $\mathrm{Cok}\,\xi$ are naturally defined as a quiver rep.

$\mathrm{Rep}_k Q$ = category of representations of Q

abelian category

kQ = path algebra associated with Q

= k-vector space with base = oriented path



+ multiplication given by concatenation of paths
(zero if two paths cannot be concatenated)

$$\overset{h}{\circ \to \circ}\overset{h'}{\to \circ} = hh'$$

$i \in Q_0$ is also regarded as a path (of length $= 0$)

$h \in Q_1$: length 1 path

unit $1 = \sum_{i \in Q_0} i$ kQ : associative algebra with unit 1

$\mathrm{Rep}\, kQ$ = category of rep. of kQ $\simeq$ $\mathrm{Rep}_k Q$

equiv. of abelian categories

$$\begin{matrix} V \\ \circlearrowleft \\ kQ \end{matrix} \longmapsto V_i = iV$$

$B_h = h$ as an operator on V

V : fix (Q_0-graded v. sp.

$$\mathbb{E}_Q(V) \overset{\text{def.}}{=} \underset{h \in Q_1}{\mathrm{Hom}}(V_{o(h)}, V_{i(h)})$$

$$GL_Q(V) = \prod_{i \in Q_0} GL(V_i) \circlearrowleft_{\text{conj}}$$

<u>Rem.</u> $k^\times \subset GL_Q(V)$ diagonal embedding

(scalar matrices

acts trivially on $\mathbb{E}_Q(V)$

$$GL_Q(V)/k^\times \curvearrowright \mathbb{E}_Q(V)$$

$$\mathbb{E}_Q(V)/GL_Q(V) \underset{1:1}{\longleftrightarrow} \text{isom. classes of rep } M \text{ of } Q \text{ with } \dim V = \dim M$$

orbit space

(

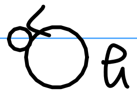
Use equivariant geometry

Examples

(1) $\circ$ one vertex, no edge

$\mathrm{Rep}_k Q =$ category of finite dim. vector spaces

$kQ = k$

(2)  one vertex, one edge loop

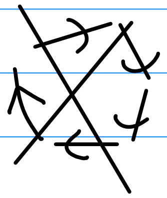
Jordan quiver

$\mathrm{Rep}_k Q =$ cat. of fin. dim. vector sp. V + end. B_h on V.

$kQ = k[h]$: polynomial ring in variable h with coeff. in k

$$\begin{tikzcd} \text{isom. classes of rep. of } Q \arrow[r, leftrightarrow] & \mathrm{End}(V)/GL(V) \arrow[d, leftrightarrow, "k=\overline{k}"'] \\ & \text{Jordan normal forms} \end{tikzcd}$$

Prop. If Q has no oriented loops cycles



kQ : finite dimensional as a k-vector space

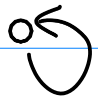
Def. ○ A $\overset{\wedge}{\ }$quiver representation is **simple** if

there is no nontrivial subrepresentation.

○ A quiver repr. M is <u>indecomposable</u> if 直既約

$$M \not\cong M_1 \oplus M_2 \quad \text{unless} \quad M_1 = 0 \text{ or } M_2 = 0$$

Examples

(1)  indecomposable

$k = \bar{k}$

$$V = k^n, \quad B = \underbrace{\begin{bmatrix} x & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & x \end{bmatrix}}_{n \times n}$$

simple, $n = 1$ $\quad V = k \quad B = x$

(2) $\underset{V_1}{\circ} \overset{B}{\xrightarrow{k}} \underset{V_2}{\circ}$ $\quad (V, B), (V', B')$ are isom.

$\iff V \cong V'$ graded vector sp.

s.t. $B \longleftrightarrow B'$

Suppose

$V = V'$ isom. $\iff \exists\, P \in GL(V_2)$ s.t. $B' = PBQ^{-1}$

$Q \in GL(V_1)$

$\therefore \dim V_1, \dim V_2, \operatorname{rank} B$ are the only invariants

indecomposable $\underset{\substack{\uparrow \\ \text{simple}}}{k \to 0}$, $\underset{\nearrow}{0 \to k}$, $\underset{\substack{\uparrow \\ \text{not simple}}}{k \xrightarrow{1} k}$

Def. S_i : 1-dim vector sp. at vertex i
0 at other vertices
$B_h = 0$

$i \in Q_0$
s.t. no edge
loop at i

simple repr. of Q

Krull-Schmidt thm

$\forall$ representation of a quiver is a direct sum of indecomposable ones

Moreover the decomposition is unique up to permutation of summands

i.e. If $M_1 \oplus \cdots \oplus M_p \cong N_1 \oplus \cdots \oplus N_q$

M_i, N_i : indecomposable

$\Rightarrow p = q \quad \exists$ bij $\sigma : \{1, \cdots, p\} \to \{1, \cdots, q\}$

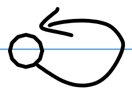
$M_i \cong N_{\sigma(i)}$

(proof is omitted) Derksen-Weyman text book

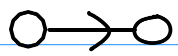
Def. A quiver Q is of *finite type*

$\Leftrightarrow$ there is only finitely many isom. classes of indecomposable representations

Ex.



not finite type



: finite type

Kac ~~for~~ holds for more general ~~graph~~ quiver

Th [Gabriel]

A connected quiver Q is of finite type

$\Longleftrightarrow$ the underlying unoriented graph

(replace $\circ \to \circ \rightsquigarrow \circ - \circ$)

is of type ADE Dynkin diagram

11:23~

Moreover ADE case

$$\text{indecomposable rep.} \overset{}{\underset{\text{bij.}}{\longleftrightarrow}} \text{positive roots of } \mathfrak{g}_{ADE}$$

$$M \longmapsto \dim M \in \mathbb{Z}^{Q_0} \subset \mathfrak{h}^*$$

Ex. $\circ \to \circ$

$k \quad 0$	α_1
$0 \quad k$	α_2
$k \xrightarrow{1} k$	$\alpha_1 + \alpha_2$

$\Longrightarrow$ (easier direction)

Def. Tits form $B_Q(\vec{v}) \overset{\text{def.}}{=} \sum_{i \in Q_0} v_i^2 - \sum_{h \in Q_1} v_{i(h)} v_{o(h)}$

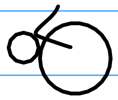
$\vec{v} \in \mathbb{Z}^{Q_0}$

$\| \; (v_i)$

$= \dim GL_Q(V) - \dim E_Q(V)$

if $\dim V = \vec{v}$

Prop (1) if Q is of finite type

$\Rightarrow$ Q has no edge loop 

(2) If Q is of finite type

$\Rightarrow B_Q(\vec{v}) \geq 1$ for $\forall \vec{v} = \dim V \in \mathbb{Z}_{\geq 0}^{Q_0}$

(proof) (1) Ok

(2) $\mathbb{E}_Q(V) \curvearrowleft GL_Q(V)$ finitely many orbits

$\exists\, \mathcal{O}$: open orbit

$$\dim \mathcal{O} = \dim \mathbb{E}_Q(V) \qquad M \in \mathcal{O}$$

$$\| $$

$$\dim GL_Q(V) - \underbrace{\dim \mathrm{Stab}_{GL_Q(V)} M}_{\substack{\vee\!\shortparallel \\ 1}}$$

Lemma

$\forall \vec{v} \in \mathbb{Z}^{Q_0}$ or $\mathbb{Q}^{Q_0}$ $\qquad \underline{B_Q(\vec{v}) \geq 0 \quad \text{"="} \Leftrightarrow \vec{v} = \vec{0}}$

(∵) $\vec{v} = \underset{\mathbb{Z}_{\geq 0}^{Q_0}}{\underset{\cap}{\vec{v}_+}} + \underset{\mathbb{Z}_{\leq 0}^{Q_0}}{\underset{\cap}{\vec{v}_-}}$

$$B_Q(\vec{v}) = B_Q(\vec{v}_+) + B_Q(\vec{v}_-) + \underbrace{\text{crossterm}}_{\substack{\vee\!\shortparallel \\ 0}}$$

//

$$B_Q + B_{\overline{Q}} = \begin{bmatrix} 2 & & \cdot \\ & \ddots & \\ & & 2 \end{bmatrix}$$ — # of edges in the underlying oriented graph

symmetric quiver with opposite orientations

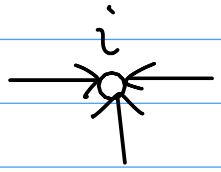
$\overset{\longrightarrow}{\underset{Q_1}{\uparrow}}$ $\qquad$ $\overset{\longleftarrow}{\underset{\overline{Q}_1}{\uparrow}}$

= Cartan matrix of the unoriented graph

pos. def. ⟺ type ADE

§ reflection functor

Def $i \in Q_0$ is a sink



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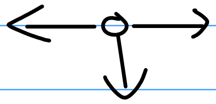
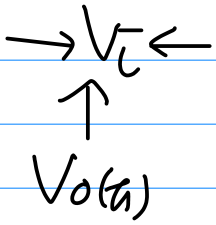


i: sink or source $\quad s_iQ$ = quiver obtained from Q by reversing orientations of edges from/to i

i: sink

$$\Phi_i^+ : \mathrm{Rep}\,Q \longrightarrow \mathrm{Rep}\,s_iQ$$
$$(V,B) \longmapsto (V',B')$$

$$\begin{cases} V'_k = V_k & \text{if } k \neq i \\ V'_i = \mathrm{Ker}\left(\bigoplus\limits_{\substack{i(h)=i \\ h\in Q_1}} V_{o(h)} \xrightarrow[\sum B_h]{} V_i \right) \end{cases}$$



$$\begin{cases} B'_{\bar h} = B_{\bar h} & i(h) \neq i \\ B'_{\bar h} = \text{composite}\left(V_i' \hookrightarrow \bigoplus V_{o(h)} \xrightarrow{\text{proj}} V_{o(h)}\right) \\ i(h) = i \end{cases}$$

$$\begin{tikzcd} & V_i' \arrow[l] \arrow[r] \arrow[d, "\bar h"] & {} \\ & V_{o(h)} & \end{tikzcd}$$

i : source

Φ_i^- : Rep $Q \longrightarrow$ Rep $s_i Q$

$$\begin{tikzcd} {} & V_i \arrow[l] \arrow[r] \arrow[d] & {} \\ & V_{i(h)} & o(h)=i \end{tikzcd}$$

similarly defined

- $V_i' = \mathrm{Cok}\left(V_i \longrightarrow \bigoplus_{o(h)=i} V_{i(h)}\right)$

- $B'_{\bar h} =$ composite of $\left(V_{i(h)} \longrightarrow \bigoplus_{o(h)=i} V_{i(h)} \longrightarrow V_i'\right)$
proj.

These are functors (hom $\longrightarrow$ hom).

Th. $i \in Q_0$: sink
$\exists$ canonical exact sequence

$$0 \to \Phi_i^- \Phi_i^+ (V,B) \longrightarrow (V,B) \longrightarrow (V,B)(i) \longrightarrow 0$$

at vertex i
$\mathrm{Coker}\left(\bigoplus V_{o(h)} \longrightarrow V_i\right)$

$S_i \otimes_k \underset{\text{v.sp.}}{\underline{\qquad}}$

0 at other vertices

Moreover this exact sequence splits.

i.e. $(V,B) \cong \Phi_i^- \Phi_i^+ (V,B) \oplus (V,B)(i)$

(proof) $(V,B) \underset{\Phi_i^+}{\rightsquigarrow} (V',B') \underset{\Phi_i^-}{\rightsquigarrow} (V'',B'')$

$$V_i' = \mathrm{Ker}\left(\bigoplus V_{o(h)} \longrightarrow V_i\right)$$

$$V_i'' = \mathrm{Cok}\left(V_i' \longrightarrow \underbrace{\bigoplus V'_{o(h)}}_{\substack{\parallel \\ \bigoplus V_{o(h)}}}\right)$$

$$\begin{tikzcd}
0 \arrow[r] & V_i' \arrow[r] & \bigoplus V_{o(h)} \arrow[r] \arrow[d, "\sum B_h"] & V_i'' \arrow[r] \arrow[dl, dotted] & 0 \\
 & & V_i & &
\end{tikzcd}$$

induced injective hom.

$\mathrm{coker} = \mathrm{Coke} \sum B_h$

$= (V,B)(i)$

This gives the s.e.s. as graded v. sp's.

—— also s.e.s. of quiver rep.

<u>splitting</u>

$$\begin{tikzcd}
\bigoplus V_{o(h)} \arrow[r] \arrow[d, shift right, "0"'] & V_i \arrow[d, two heads, shift right] & (V,B) \\
0 \arrow[r, "0"'] \arrow[u, shift right, red, "0"'] & \mathrm{Cok}(\ ) \arrow[u, shift right, red] & (V,B)(i)
\end{tikzcd}$$

splitting as vector spaces

automatically.

commutative diagram

Splitting as quiver rep. //

Cor. (V,B) : indecomposable rep.

i : sink $\Longrightarrow$ $\Phi_i^+(V,B)$: indecomposable

($(V,B)(i)=0$)

i.e. $\bigoplus V_{o(h)} \longrightarrow V_i$ surj.

or $(V,B) \cong S_i$

Moreover $\dim \Phi_i^+(V,B) = \underbrace{s_i}_{}(\dim(V,B))$

simple reflection of the Weyl group

$\mathbb{Z}^{Q_0} \subset \mathfrak{h}^*$

(proof)

$$\dim \Phi_i^+(V,B) = \sum_{j \neq i} \dim V_j \cdot \alpha_j + \left(\sum_{i(h)=i} \dim V_{o(h)} - \dim V_i \right) \alpha_i$$

$$\|$$

$$s_i \dim(V,B) \quad //$$

$$s_i \alpha_j = \alpha_j - \underbrace{\langle h_i, \alpha_j \rangle}_{} \alpha_i$$

Cartan matrix