$Q = (Q_0, Q_1)$: quiver

$\Phi_i^{\pm}$: reflection functor

$+$: i : sink $\qquad$ $-$: i source

$\longrightarrow \circ \longleftarrow \qquad \longleftarrow \circ \longrightarrow$

$$(V, B) \cong \Phi_i^- \Phi_i^+ (V, B) \oplus \underset{\substack{\uparrow \\ \text{supported at } i \\ \mathrm{Coker}(\oplus V_{o(a)} \longrightarrow V_i)}}{(V, B)(i)}$$

(V, B) : indecomposable $\Rightarrow$ $(V, B) \cong \Phi_i^- \Phi_i^+ (V, B)$ or S_i

<u>Def.</u> $\vec{h} = (i_1, \cdots, i_N)$: reduced expression of w_0 (longest elem.)

$\overset{\text{def.}}{\Longleftrightarrow}$ adapted with Q

i_1 : sink of Q

i_2 : $\quad$ " $\quad$ $s_{i_1} Q$

i_3 : $\quad$ " $\quad$ $s_{i_2} s_{i_1} Q$

$\vdots$

<u>Ex.</u> $\underset{1}{\circ} \longrightarrow \underset{2}{\circ}$ $\quad \vec{h} = (212)$ $\quad$ adapted

$(V, B) \in \mathrm{Rep}_k Q$

Consider $(V^p, B^p) = \Phi_{i_p}^+ \Phi_{i_{p-1}}^+ \cdots \Phi_{i_1}^+ (V, B)$

<u>Prop</u>. $\vec{h}$: adapted with Q $\qquad \nu = |\Delta_+|$

$$(V^\nu, B^\nu) = 0$$

∵ may assume indecomposable

$$(V^{p-1}, B^{p-1}) = S_{i_p} \Rightarrow \Phi^+_{i_p}(V^{p-1}, B^{p-1}) = 0$$

$$\Rightarrow (V^\nu, B^\nu) = 0$$

otherwise $\dim \Phi^+_{i_p}(V^{p-1}, B^{p-1}) = \dim \Phi^+_{i_p} \cdots \Phi^+_{i_1}(V, B)$

$$= s_{i_p} \cdots s_{i_1}(\dim(V, B))$$

If not 0

$$\dim(V^\nu, B^\nu) = s_{i_\nu} \cdots s_{i_1} \dim(V, B) = w_0 \dim(V, B)$$

$\mathbb{Z}^{Q_0}_{\le 0} \ni \dim(V^\nu, B^\nu)$, $\quad w_0 \dim(V,B)$ with $\dim(V,B) \in \mathbb{Z}^{Q_0}_{\ge 0}$ $\qquad$ contradiction //

From the proof,

$\exists p$ s.t. $(V^{p-1}, B^{p-1}) = S_{i_p}$ $\quad (\Phi^+_{i_p}(V^{p-1}, B^{p-1}) = 0)$

$$\therefore (V, B) = \Phi^-_{i_1} \Phi^-_{i_2} \cdots \Phi^-_{i_{p-1}}(S_{i_p})$$

$\dim = s_{i_1} \cdots s_{i_{p-1}}(\alpha_{i_p}) \in \Delta^+$

Conversely $\quad$ indecomposable

This proves Gabriel's thm $\quad$ indecomp $\overset{\text{bij.}}{\longleftrightarrow} \Delta^+$

under the assumption $\exists \vec{h}$ adapted with Q

For a later purpose, we prepare the following:

<u>Prop</u> $\mathrm{Hom}_Q(\bar{\Phi}_{i_1}^- \cdots \bar{\Phi}_{i_{p-1}}^-(S_{i_p}), \bar{\Phi}_{i_1}^- \cdots \bar{\Phi}_{i_{q-1}}^-(S_{i_q})) = 0$

if $p > q$

$\because$ Reflection functor, enough to show

$$\mathrm{Hom}_{s_{i_{q-1}} \cdots s_{i_1} Q}(\underbrace{\bar{\Phi}_{i_q}^- \cdots \bar{\Phi}_{i_{p-1}}^-(S_{i_p})}_{(V,B)}, S_{i_q}) = 0$$

(V,B)

$$\begin{tikzcd} \bigoplus_{i(h)=i_q} V_{o(h)} \arrow[r] \arrow[d] & V_{i_q} \arrow[d] \\ 0 \arrow[r, "0"'] & k \end{tikzcd}$$

S_{i_q}

commutativity $\Longrightarrow 0$ //

<u>Def</u> a Coxeter element $c \in W$ (Weyl group)

$c = s_1 \cdots s_n$ $\quad \{1, \ldots, n\} = Q_0$

numbering of Q_0

any Coxeter elements are conjugate to each other

<u>Def.</u> $h =$ order of c $\quad$ Coxeter number

Suppose $Q_0 = Q_0^+ \sqcup Q_0^-$

s.t. $\forall$ edge h (unoriented) $\quad Q_0^+ \ni \circ \overset{h}{\text{———}} \circ \in Q_0^-$

Q : tree $\Rightarrow$ $\exists$

$$c_\pm = \prod_{i \in Q_0^\pm} s_i \quad \text{(independent of the ordering on } Q_0^+, Q_0^- \text{)}$$

$c = c_+ c_-$: Coxeter element

<u>Fact</u>. (cf. Humphreys §3.17) $\qquad c^h = 1$

$$w_0 = \cdots\cdots c_+ \underbrace{c_- c_+} \quad (h \text{ factors})$$

$$= \cdots\cdots c_- c_+ c_- \quad (\quad \text{"} \quad)$$

$$w_0^2 = c^h = 1$$

<u>Cor.</u> $\underbrace{n}_{\# Q_0} h = 2\ell(w_0) = |\Delta|$

<u>Prop.</u> $\forall Q$ quiver of ADE type

$\exists \vec{h}$: adapted sequence

(proof) $Q_0 = Q_0^+ \sqcup Q_0^-$ as above

Choose Q_1 (orientation of edges)

s.t. Q_0^+ : sink, Q_0^- : source

e.g. $\underset{+}{\circ} \longleftarrow \underset{-}{\circ} \longrightarrow \underset{+}{\circ} \longleftarrow \underset{-}{\circ}$

$\Longrightarrow$ $\cdots C_+ \underbrace{C_- C_+}_{\text{return back to the original orientation}}$: adapted to Q with this orientation

- $\forall$ orientations are connected each other by successive applications of $s_{i_1}, \dots, s_{i_q}$

(↑ easy to show)

$\vec{i} = (i_1, \dots, i_\nu)$ adapted with Q

$(i_2, \dots, i_\nu, j)$ 〃 $s_{i_1} Q$

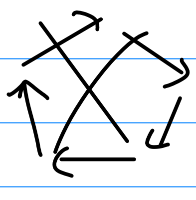
$-w_0(\alpha_{i_1}) = \alpha_j$

//

11:08 ~

§ representation theory of quiver

Assume Q has no oriented cycles



Lemma 1. $\{S_i\}_{i \in Q_0}$ is a complete list of isom. classes of simple representations of $\mathbb{C}Q$

(∵) $\longrightarrow \overset{i}{\circ} \leftarrow$, $\exists$ sink among supp M

M: simple repr.

$\Rightarrow \quad S_i \hookrightarrow M$ $\quad$ M: simple $\Rightarrow M = S_i$ //

$\mathcal{C}$ = abelian category

Grothendieck group $K(\mathcal{C})$ is an abelian group generated by $[M]$ $\quad M \in \mathrm{Obj}(\mathcal{C})$

relation $\quad 0 \to A \to B \to C \to 0$ (exact seq.)

$\Rightarrow \quad [B] = [A] + [C]$

in particular $\quad A \cong B \Rightarrow [A] = [B]$ in $K(\mathcal{C})$

$[A \oplus C] = [A] + [C]$

<u>Lemma 2</u>. $\quad K(\mathrm{Rep}_{\Bbbk} Q) \cong \mathbb{Z}^{Q_0}$

$$[M] \mapsto \dim M$$

(∵) $\longrightarrow$ well-defined

$0 \to A \to B \to C \to 0$

$\Rightarrow \quad \dim B = \dim A + \dim C$

$$[S_i] \mapsto \alpha_i \quad \begin{cases} 1 & \text{at } i \\ 0 & \text{otherwise} \end{cases}$$

$\therefore \longrightarrow$ is surjective

$\forall M$ has a filtration whose succ. quot. are simple

$$\therefore \quad \begin{array}{ccc} K(\mathrm{Rep}_k Q) & \cong & \mathbb{Z}^{Q_0} \\ \cup & & \cup \\ [S_i] & \longmapsto & \alpha_i \end{array} \quad \text{bijection} \quad //$$

<u>Def.</u> P: projective module

$\overset{}{\underset{\text{def.}}{\iff}}$ $\mathrm{Hom}_{kQ}(P, \bullet)$ is exact

$0 \to A \to B \to C \to 0$

$\Longrightarrow \quad 0 \to \mathrm{Hom}_{kQ}(P, A) \to \mathrm{Hom}_{kQ}(P, B) \to \mathrm{Hom}_{kQ}(P, C) \to 0$

$i \in Q_0 \qquad P(i) := kQ \cdot i$

length 0 path

$= \langle \text{path starting from } i \rangle$

$\bigoplus_i P(i) = kQ$

($P(i)$: direct summand of kQ.)

<u>Lemma 3</u>. $\begin{array}{ccc} \mathrm{Hom}_{kQ}(P(i), M) & \xrightarrow{\cong} & M_i \\ \cup & & \downarrow \\ \varphi & \longmapsto & \varphi(i) \end{array}$ is an isom.

length 0 path

In particular $P(i)$: projective

(proof) $m \in M_i$ Given $\quad \varphi(p) := pm$, $p \in P(i)$

this determines $\varphi \in \mathrm{Hom}_{kQ}(P(i), M)$

This gives the inverse hom. //

<u>Prop 4.</u> $\{P(i)\}_{i \in Q_0}$ forms a complete list of isom classes of indecomposable projective modules of kQ.

(proof) $\mathrm{Hom}_{kQ}(P(i), P(i)) \underset{\text{Lemma 3}}{\cong} P(i)_i \cong k \quad (k \cdot i)$ ← length 0

no oriented cycles

$\Rightarrow P(i)$: indecomposable

Let

P : projective module

$$P' := \bigoplus_i P(i) \otimes_k \mathrm{Hom}_{kQ}(P, S_i)$$

$$\bigoplus_i P(i)^{\oplus \dim \mathrm{Hom}(P, S_i)}$$

$$\mathrm{Hom}(P', S_j) \cong \bigoplus_i \underbrace{\mathrm{Hom}_{kQ}(P(i), S_j)}_{(S_j)_i = \delta_{ij} k} \otimes_k \mathrm{Hom}_{kQ}(P, S_i)$$

$$\overset{\star}{\cong} \mathrm{Hom}(P, S_j)$$

M : $\forall$ rep ($\exists$ filtr. by S_i)

$$\underset{\text{projectivity}}{☆} \Longrightarrow \quad Hom(P', M) \cong Hom(P, M)$$

$$0 \to A \to B \to C \to 0$$

$$\begin{tikzcd}
0 \arrow[r] & Hom(P,A) \arrow[r] & Hom(P,B) \arrow[r] & Hom(P,C) \arrow[r] & 0 \\
0 \arrow[r] & Hom(P',A) \arrow[r] \arrow[u, "\cong"] & Hom(P',B) \arrow[r] \arrow[u, dashed, "\cong"] & Hom(P',C) \arrow[r] \arrow[u, "\cong"] & 0
\end{tikzcd}$$

$$\Longrightarrow \quad P \cong P' \quad //$$

$\forall M = (V, B) \in \mathrm{Rep}\, kQ$

consider the following exact sequence : (projective resolution)

(☆)

$$0 \to \bigoplus_{h \in Q_1} P(i(h)) \otimes kh \otimes V_{o(h)} \xrightarrow{d_1} \bigoplus_{i} P(i) \otimes_k V_i \xrightarrow{d_0} V \to 0$$

s.t. $\quad d_0(p \otimes v) = pv$

$\quad d_1(p \otimes h \otimes v) = ph \otimes v - p \otimes B_h v$

$$(d_0 d_1 = 0)$$

<u>Prop 5</u> ☆ is an exact sequence.

☹ · surjectivity of d_0

$$P(i) \ni \underset{\text{length } 0}{i} \qquad d_0(i \otimes \underset{V_i}{v}) = v$$

· $\operatorname{Ker} d_0 = \operatorname{Im} d_1$

Suppose $d_0(\sum p_n \otimes v_n) = \sum p_n v_n = 0$

$$\sum p_n \otimes v_n = \sum_{\text{length } p_n = 0} p_n \otimes v_n + \sum \underbrace{p_n' e_n}_{\substack{\| \\ \text{length } p_n' > 0}} \otimes v_n$$

$$= \sum_{\text{length} = 0} p_n \otimes v_n + \sum \underset{\substack{\text{length } p_n' \\ = \text{length } p_n - 1}}{p_n'} \otimes B_{e_n} v_n + \underbrace{\sum (p_n' e_n \otimes v_n - p_n' \otimes B_{e_n} v_n)}_{\in \operatorname{Im} d_1}$$

$\rightsquigarrow$ modulo $\operatorname{Im} d_1$, may assume $\sum \underset{\text{length } 0}{p_n} \otimes v_n$.

But $\sum \underset{\text{length } 0}{p_n} v_n = 0 \implies \sum p_n \otimes v_n = 0$

injectivity of d_1

$$0 = d_1\left(\sum p_n \otimes \ell_n \otimes v_n\right) = \sum p_n \ell_n \otimes v_n - p_n \otimes B_{\ell_n} v_n$$

Suppose

$\ell = $ max length of p_n

Take length $= \ell + 1$ terms

$$0 = \sum \underbrace{p_\ell}_{\text{length} = \ell} \ell_\ell \otimes v_n \qquad \Rightarrow \quad 0 = \sum p_\ell \otimes \ell_\ell \otimes v_n$$

//

$\rightsquigarrow$ $\mathrm{Ext}^i(M,N)$ is defined

$$0 \to P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

$$0 \to \mathrm{Hom}(P_0, N) \xrightarrow{d_1} \mathrm{Hom}(P_1, N) \longrightarrow 0$$

$$\mathrm{Ext}^0(M,N) \overset{\text{def.}}{=} \mathrm{Ker}\, d_1 = \mathrm{Hom}(M,N)$$

$$\mathrm{Ext}^1(M,N) \overset{\text{def.}}{=} \mathrm{Cok}\, d_1$$

$$\mathrm{Ext}^i(M,N) = 0 \quad \text{for } i > 1$$

(kQ : hereditary)

$0 \to A \to B \to C \to 0 \qquad N \in \mathrm{Rep}\, kQ$

$$\Rightarrow \quad 0 \to \mathrm{Hom}(C,N) \to \mathrm{Hom}(B,N) \to \mathrm{Hom}(A,N)$$

long exact seq

$$\to \mathrm{Ext}^1(C,N) \to \mathrm{Ext}^1(B,N) \to \mathrm{Ext}^1(A,N) \to 0$$

☆ Take $M=(V,B)=S_i$

$$0 \to \bigoplus_{e_i : i \to j} P(j) \longrightarrow P(i) \longrightarrow S_i \longrightarrow 0$$

$$\mathrm{Ext}^1(S_i, S_j) = \bigoplus_{e_i : i \to j} k$$

<u>Tits form</u> $\langle M, N\rangle \overset{\text{def.}}{=} \sum (-1)^i \dim \mathrm{Ext}^i(M,N)$

$$= \dim \mathrm{Hom}(M,N) - \dim \mathrm{Ext}^1(M,N)$$

<u>Th 6.</u> $\langle M, N\rangle$ depends only on $\dim M, \dim N \in K(\mathrm{Rep}_k Q)$

and given by $\sum_i \dim M_i \dim N_i - \sum_{e_i \in Q_1} \dim M_{o(e_i)} \dim N_{i(e_i)}$

Ringel form $M=N$

(proof) $0 \to A \to B \to C \to 0$

$\Rightarrow \langle B, N\rangle = \langle A, N\rangle + \langle C, N\rangle$

long exact seq

$$0 \to P_1 \to P_0 \to M \to 0$$

$\therefore$ $\langle M, N\rangle = \langle P_0, N\rangle - \langle P_1, N\rangle$

$= \cdots =$ above formula //

$$\langle S_i, S_j \rangle = \delta_{ij} - \#\underset{\text{oriented}}{\text{edges}}\ i \to j$$

Cartan matrix $\quad 2\delta_{ij} - \#\underset{\text{unoriented}}{\text{edges}}\ i - j$

$$= \langle S_i, S_j \rangle_Q + \langle S_i, S_j \rangle_{\overline{Q}}$$

orientation reversed quiver.