Supplement

1221 **Prop** $\mathrm{Hom}_Q(\Phi^-_{i_1}\cdots\Phi^-_{i_{p-1}}(S_{i_p}), \Phi^-_{i_1}\cdots\Phi^-_{i_{q-1}}(S_{i_q}))=0$

if $p>q$

Prop. $\mathrm{Ext}^1_Q(\Phi^-_{i_1}\cdots\Phi^-_{i_{p-1}}(S_{i_p}), \Phi^-_{i_1}\cdots\Phi^-_{i_{q-1}}(S_{i_q}))=0$

if $p\leq q$

$\because$ Reflection functors

$$\mathrm{Ext}^1_Q(\ ,\ )\cong \mathrm{Ext}^1_{s_{i_{p-1}}\cdots s_{i_1}Q}(S_{\underbrace{i_p}_{\text{sink}}}, \Phi^-_{i_p}\cdots\Phi^-_{i_{q-1}}(S_{i_q}))$$

$S_{i_p}=P(i_p)$ projective

$\underset{\cdots}{\mathrm{Ext}^1}(S_{i_p}, \cdots)=0$ //

§ Ringel-Hall algebra

Assume k : finite field

Q : ADE quiver

$\mathcal{H}(Q,k) \equiv \mathcal{H}$ free abelian group with base = isom. classes of representations of Q

$M_1, M_2, L \in \mathrm{Rep}_k Q$

$$F^L_{M_1,M_2} := \#\left\{ \begin{array}{c} \text{subrepresentations } S \text{ of } L \\ \text{s.t. } S \cong M_2,\ L/S \cong M_1 \end{array} \right\}$$

$$= \frac{\#\{\text{short exact sequence } 0 \to M_2 \to L \to M_1 \to 0\}}{\#\mathrm{Aut}\, M_2 \times \#\mathrm{Aut}\, M_1}$$

Define a multiplication on $\mathcal{H}$ by

$$[M_1] * [M_2] = \sum_{[L]} F^L_{M_1,M_2} [L]$$

finite sum

<u>Prop</u> $(\mathcal{H}, *)$ is an associative algebra with unit $[0]$

$\because$ $([M_1]*[M_2])*[M_3] \overset{?}{=} [M_1]*([M_2]*[M_3])$

Both count $\left\{ L \supset X_1 \supset X_2 \;\middle|\; \begin{array}{l} L/X_1 \cong M_1 \\ X_1/X_2 \cong M_2 \\ X_2 \cong M_3 \end{array} \right\}$

//

Def. $(\mathcal{H}, *)$: Ringel-Hall algebra

Note $\mathcal{H}$ is graded by $\mathbb{Z}_{\geq 0}^{Q_0}$

Example

A_1-quiver $\bullet$ $\quad$ S : simple module

$$[S]*[S] = \#\{ M \subset S^{\oplus 2} \mid M : \text{1-dim subspace in } S^{\oplus 2} = \mathbb{k}^2 \} \times [S^{\oplus 2}]$$

$$\# \mathbb{P}^1(\mathbb{k}) = \# \underset{[1:*]}{\mathbb{k}} + \underset{[0:1]}{1}$$

$$= (\#\mathbb{k}+1)[S^{\oplus 2}] = q(q+q^{-1})[S^{\oplus 2}] = q[2]_q[S^{\oplus 2}]$$

$$q = \sqrt{\#\mathbb{k}}$$

more generally $[S^{\oplus n}]*[S^{\oplus m}]$

$$= \#Gr(m, m+n)\,[S^{\oplus(m+n)}]$$

Grassmannian mfd of m dim subspaces in $\mathbb{k}^{m+n}$

$$= \begin{bmatrix} m+n \\ m \end{bmatrix}_q \times q^{mn}\,[S^{\oplus(m+n)}]$$

$$\begin{bmatrix} m+n \\ m \end{bmatrix}_q = \frac{[m+n]_q!}{[m]_q!\,[n]_q!}$$

$mn = \dim Gr(m, m+n)$

twisted version of multiplication

$$[M_1]\cdot[M_2] = q^{\langle \dim M_1, \dim M_2\rangle}[M_1]*[M_2]$$

This is an associative algebra
defined over image $\left(\begin{array}{ccc} \mathbb{Z}[q,q^{-1}] & \longrightarrow & \mathbb{C} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ q & \longmapsto & \sqrt{\#k} \end{array}\right)$

<u>Fact</u> (Ringel)
structure constants $F^L_{M_1,M_2}$
are polynomials in $\#k$

twisted version : alg. over $\mathbb{Z}[q,q^{-1}] = \mathcal{A}$
$\mathcal{H}$: 〃 $\mathbb{Z}[q_k]$
$q_k = \#k$

$(\mathcal{H}^{tw}, \cdot)$ $\mathcal{A}$-algebra

<u>Ex</u>. A_1-quiver $\bullet$ $[S]^n = \underbrace{[S]\cdots[S]}_{n}$

$$= q^{1+2+\cdots+(n-1)}\underbrace{[S]*\cdots*[S]}_{n}$$

$$= q^{n(n-1)}[n]_q!\,[S^{\oplus n}]$$

$$\frac{[S]^n}{[n]_q!} = q^{n(n-1)}[S^{\oplus n}] \quad \therefore (\mathcal{H}^{tw},\cdot) \cong {}_{\mathcal{A}}U_q^-(\mathfrak{sl}_2)$$

more general case :

1° $\quad \overset{i}{\bullet} \qquad \overset{j}{\bullet}$ $\qquad [S_i]*[S_j] = [S_i \oplus S_j] = [S_j]*[S_i]$

2° $\quad \bullet \longrightarrow \bullet$ $\qquad [k \xrightarrow{1} k] =: [M]$

$$[k^{\oplus 2} \xrightarrow[{[1\,0]}]{} k] = [M \oplus S_i] =: [X]$$

$$[k^{\oplus 2} \xrightarrow[0]{} k] = [S_i^{\oplus 2} \oplus S_j] =: [Y]$$

$$[S_i]*[S_j] = \underbrace{F^{M}_{S_i,S_j}}_{=1}[M] + \underbrace{F^{S_i\oplus S_j}_{S_i,S_j}}_{=1}[S_i \oplus S_j]$$

$$\left(\begin{array}{ccccc} S_i & = & [k & & 0] \\ \twoheaduparrow & & \uparrow & & \\ M & = & [k & \longrightarrow & k] \\ \hookuparrow & & & & \uparrow \\ S_j & = & [0 & & k] \end{array}\right.$$

$$[S_i]\cdot[S_j] = q^{-1}\big([M] + [S_i \oplus S_j]\big)$$

$$[S_j]*[S_i] \;=\; [S_i \oplus S_j]$$

$$\begin{array}{ccc} \| & & [0 & & k] \\ & & & & \uparrow \\ [S_j]\cdot[S_i] & & [k & \longrightarrow & k] \\ & & \hookuparrow & & \\ & & [k & & 0] \end{array}$$

$$\therefore\ [M] = q[S_i]\cdot[S_j] - [S_j]\cdot[S_i]$$
$$= q\big([S_i]\cdot[S_j] - q^{-1}[S_j]\cdot[S_i]\big)$$

Cf. $T_i^{-1}(f_j) = f_i f_j - q f_j f_i$

$T_i(f_j) = f_j f_i - q f_i f_j$

$$[S_i] * [M] = \underbrace{F^{X}_{S_i,M}}_{\#k}[X] + \underbrace{F^{Y}_{S_i,M}}_{=0}[Y]$$

$$\begin{tikzcd}
S_i & {[k \quad 0]} \\
X \arrow[u, two heads] & {[k^{\oplus 2} \to k]} \arrow[u, two heads] \\
M \arrow[u, hook] & {[k \xrightarrow{1} k]} \arrow[u, hook, "{[1\,0]}"] \arrow[u, phantom, shift left=6, "\parallel"]
\end{tikzcd}$$

$$\begin{bmatrix} 1 \\ \underset{\in k}{*} \end{bmatrix}$$

$$[S_i] \cdot [M] = [X]$$

$$[S_i] * [S_i \oplus S_j] = [X] + (\#k + 1)[Y]$$

$$[S_i \oplus S_j] * [S_i] = (\#k + 1)[Y]$$

In summary :

$$[S_i]^{*2} * [S_j] = (\#k+1)[X] + (\#k+1)[Y]$$

$$[S_i] * [S_j] * [S_i] = [X] + (\#k+1)[Y]$$

$$[S_j] * [S_i]^{*2} = (\#k+1)[Y]$$

$$\therefore \ [S_i]^{*2} * [S_j] - (\#k+1)[S_i] * [S_j] * [S_i] + \#k [S_j] * [S_i]^{*2} = 0$$

twisted version : q-Serre relation

$$[S_i]^2 \cdot [S_j] - (q+q^{-1})[S_i]\cdot[S_j]\cdot[S_i] + [S_j]\cdot[S_i]^2 = 0$$

Similarly we have q-Serre relation for $\overset{i}{\circ} \leftarrow \overset{j}{\bullet}$

$$\therefore \quad {}_{\mathcal{A}}U_q^- \xrightarrow[\text{alg. hom}]{\exists} (\mathcal{H}^{tw}, \circ) \qquad \text{grading preserving}$$

$$f_i^{(n)} = \frac{f_i^n}{[n]_q!} \longmapsto q^{n(n-1)}[S_i^{\oplus n}]$$

<u>Th</u>. This is an isomorphism

(proof) Choose $\vec{h}$: reduced expression of w_0 adapted with Q

$$\{\Phi_{i_1}^- \cdots \Phi_{i_{p-1}}^-(S_{i_p})\}_{1 \le p \le \nu}$$ complete list of isom. classes of indecomposable rep's of Q

$$\vec{c} = (c_1, \ldots, c_\nu) \in \mathbb{Z}_{\ge 0}^{\nu}$$

$$M(\vec{c}, \vec{h}) := S_{i_1}^{\oplus c_1} \oplus \Phi_{i_1}^-(S_{i_2})^{\oplus c_2} \oplus \cdots$$

$\mathcal{H}^{tw} \ni [M(\vec{c}, \vec{h})]$ gives an $\mathcal{A}$-base

PBW base of ${}_{\mathcal{A}}U_q^-$: $\mathcal{A}$-base

same size

$\therefore$ enough to show the surjectivity

Recall Prop. of Ext^1-vanishing

$\Downarrow$ short exact seq. splits

$$[S_{i_1}^{\oplus c_1}] * [\Phi_{i_1}^{-}(S_{i_2})^{\oplus c_2}] * \cdots$$
$$= [S_{i_1}^{\oplus c_1} \oplus \Phi_{i_1}^{-}(S_{i_2})^{\oplus c_2} \oplus \cdots]$$
$$= [M(\vec{c}, \vec{h})]$$

$\therefore$ enough to show $[\Phi_{i_1}^{-} \cdots \Phi_{i_{p-1}}^{-}(S_{i_p})^{\oplus c_p}] \in \text{image}$ ☆

$$\| $$

$$n = c_p \qquad \frac{q^{n(n-1)}}{[n]_q!} [\Phi_{i_1}^{-} \cdots \Phi_{i_{p-1}}^{-}(S_{i_p})]^n$$

We show ☆ by induction on dim. vectors

We choose i : sink among support of

$$\Phi_{i_1}^{-} \cdots \Phi_{i_{p-1}}^{-}(S_{i_p})^{\oplus c_p} =: (V, B)$$

Then

$$0 \to S_i \otimes_k V_i \to (V, B) \to \mathrm{Cok} \to 0$$

$$[\mathrm{Cok}] * [S_i \otimes_k V_i] = [(V,B)] + \sum c_M [M]$$

$\dim M = \dim V$

$M \ncong (V, B)$

$\cap$ $\cap$

image

by induction hypothesis

$\frac{f_i^n}{[n]!} q^{n(n-1)}$ $\quad n = \dim V_i$

$\dim M = \dim V = c_p \cdot \alpha \qquad \alpha = s_{i_1} \cdots s_{i_{p-1}}(\alpha_{i_p}) \in \Delta^+$

M is a direct sum of indecomposables $\in \Delta^+$

$\uparrow$

contains at least two non-isomorphic direct summands

$$\Phi_{i_1}^- \cdots \Phi_{i_{q-1}}^-(S_{i_q})^{\oplus n_q} \oplus \Phi_{i_1}^- \cdots \Phi_{i_{q'-1}}^-(S_{i_{q'}})^{\oplus n_{q'}} \oplus \cdots$$

$(q \neq q')$

$$\underbrace{[\quad] \quad [\quad]}_{\text{image}}$$

$[\quad]$ image; $[\quad]$ image by induction hypothesis

$\therefore [(V, B)] \in$ image //

Suppose $i = \cancel{i_1}$ (sink)

$$\mathcal{H}_i^+ \equiv \mathcal{H}_i^+(Q, \mathbf{e}) \overset{\text{def}}{=} \{ [(V,B)] \in \mathcal{H}^{tw} \mid \bigoplus_{i(a)=i} V_{o(a)} \twoheadrightarrow V_i \}$$

this condition is preserved under extensions

$\therefore \mathcal{H}_i^+$ is a subalgebra of $\mathcal{H}^{tw}$

Assume $i = i_1$

$$\{ [\underline{M(\vec{c}, \vec{i})}] \mid c_1 = 0 \} \; : \; A\text{-base of } \mathcal{H}_i^+$$

$$\underline{M(\vec{c},\vec{i})} = \Phi_{i_1}^-(S_{i_2})^{\oplus c_2} \oplus \cdots$$

Apply reflection functor Φ_i^+

$$\mathcal{H}_i^- = \mathcal{H}_i^-(s_i Q, \mathbf{e})$$

$$= \{ [(V,B)] \in \mathcal{H}^{tw} \mid V_i \xrightarrow{\text{injective}} \bigoplus_{o(a)=i} V_{i(a)} \}$$

$$= \Phi_i^+(\mathcal{H}_i^+)$$

$$= \{ [M(\vec{c}', \underbrace{\vec{i}'}_{})] \mid c_\nu' = 0 \}$$

$$\vec{i}' = (i_2, i_3, \cdots, i_\nu, i^*)$$

Assume $i = i_1$

Th.

$$\begin{tikzcd}
{}_{\mathcal{A}}U_f^- \arrow[r, "\cong"] & \mathcal{H}^{tw} \\
{}_{\mathcal{A}}U_f^-[i] \arrow[u, phantom, "\cup"] \arrow[r, "\cong"] \arrow[d, "T_i^{-1}"', "\cong"] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{H}_i^+ \arrow[u, phantom, "\cup"] \arrow[d, "\cong"', "\Phi_i^+ \times \hat{q}^{-(\alpha_i, \dim \bullet)}"] \\
{}^{*}_{\mathcal{A}}U_f^-[i] \arrow[r, "\cong"] & \mathcal{H}_i^-
\end{tikzcd}$$

i: sink

(*) ${}_{\mathcal{A}}U_f^-[i]$ = subalg. generated by $\mathrm{ad}^{(*)}(f_i^{(m)})(f_j)$

$(i \neq j) \quad 0 \leq m \leq -a_{ij}$

(proof) Claim. $\mathcal{H}_i^+$ = generated by $[S_j^{\oplus n}]_{j \neq i}$, $[(\underset{i}{k} \leftarrow \underset{j}{k})^{\oplus n}]$

$\| \; \mathcal{H}_i'$

① similar to the proof of the previous thm

$0 \to S_j \otimes_k V_j \to (V,B) \to \mathrm{Cok} \to 0$ $\quad \Phi_{i_1}^- \cdots \Phi_{i_{p-1}}^-(S_{i_p}^{\oplus n})$ $\|$ may assume

j: sink among support of (V,B)

~~we cannot use this in this case~~

~~module~~

~~Instead consider $\Phi_i^+(V,B)$~~

Rep_k ~~S_iQ~~

~~i: source~~

~~Φ_i^+~~

If $j \neq \bar{i}$, $[S_j] \in \mathcal{H}_{\bar{i}}^{+}$ $\Longrightarrow$ OK by induction

$\underline{j = \bar{i}}$

may assume

$$\begin{tikzcd}
\arrow[r, "B_{a'}=0"'] & \arrow[l] V_k \arrow[r, "B_a \neq 0"'] \arrow[dl, "B_{a''}=0"] & V_{\bar{i}} \\
&
\end{tikzcd}$$

Define $Y := [V_k \twoheadrightarrow \operatorname{Im} B_a]$, with $\operatorname{Im} B_a \subset V_{\bar{i}}$, 0 for other vertices

$$0 \to Y \longrightarrow (V, B) \longrightarrow \mathrm{Cok} \longrightarrow 0$$

Then $[\mathrm{Cok}] * [Y] = [(V,B)] + \sum_{\substack{\dim M = \dim V \\ M \not\cong (V,B)}} c_M [M]$

$[Y] \in \mathcal{H}_{\bar{i}}^{+}$

$[\mathrm{Cok}] \in$

$\therefore$ $[Y], [\mathrm{Cok}] \in \mathcal{H}_{\bar{i}}'$ by induction hyp.

$S_k^{\oplus ?} \oplus [k \xrightarrow{1} k]^{\oplus ?}$

$[M] \in \mathcal{H}_{\bar{i}}^{+} \longrightarrow [M] \in \mathcal{H}_{\bar{i}}'$ by induction

$$[M] = [\underset{i}{k} \xrightarrow{1} \underset{j}{k}] \underset{\substack{\uparrow \\ \text{calculation} \\ \text{later}}}{=} q\left([S_i][S_j] - q^{-1}[S_j][S_i]\right)$$

$$\| \qquad\qquad\qquad\qquad \| $$

$$\Phi_i^{+}([S_j]) \qquad\qquad q\,\overline{\mathrm{ad}(f_i)(f_j)}$$

$$= q\,\overline{T_i^{-1}(f_j)}$$

$$[M'] = [\underset{i}{k} \xleftarrow{1} \underset{j}{k}] = q\left([S_j][S_i] - q^{-1}[S_i][S_j]\right)$$

$$= q\,\overline{T_i(f_j)} \qquad //$$

<u>Cor</u>. $\vec{h} = (i_1, \dots, i_N)$: adapted with Q

$$\Longrightarrow \quad [L(\vec{c}, \vec{h})] = q^{☆}\left[S_{i_1}^{\oplus c_1} \oplus \Phi_{i_1}^{-}(S_{i_2})^{\oplus c_2} \oplus \cdots\right]$$

PBW base $\longleftrightarrow$ twist base of Ringel-Hall alg. given by isom. classes of modules