

Supplement

$$1221 \quad \underline{\text{Prop}} \quad \text{Hom}_Q(\overline{\Phi}_{i_1}^- \cdots \overline{\Phi}_{i_{p-1}}^-(S_{i_p}), \overline{\Phi}_{i_1}^- \cdots \overline{\Phi}_{i_f}^-(S_{i_f})) = 0 \\ \text{if } p > f$$

$$\underline{\text{Prop}} \quad \text{Ext}_Q^1(\overline{\Phi}_{i_1}^- \cdots \overline{\Phi}_{i_{p-1}}^-(S_{i_p}), \overline{\Phi}_{i_1}^- \cdots \overline{\Phi}_{i_f}^-(S_{i_f})) = 0 \\ \text{if } p \leq f$$

⊙ Reflection functors

$$\text{Ext}_Q^1(\quad, \quad) \cong \text{Ext}_{S_{i_{p-1}} \cdots S_{i_1} Q}^1(S_{i_p}, \overline{\Phi}_{i_p}^- \cdots \overline{\Phi}_{i_f}^-(S_{i_f}))$$

↑ sink

$S_{i_p} = P(i_p)$  projective

$$\text{Ext}_{\dots}^1(S_{i_p}, \dots) = 0 //$$

§ Ringel-Hall algebra

Assume  $k$ : finite field

$Q$ : ADE quiver

$\mathcal{H}(Q, k) \equiv \mathcal{H}$  free abelian group  
with base = isom. classes of representations of  $Q$

$M_1, M_2, L \in \text{Reps } Q$

$F_{M_1, M_2}^L := \# \left\{ \begin{array}{l} \text{subrepresentations } S \text{ of } L \\ \text{s.t. } S \cong M_2, L/S \cong M_1 \end{array} \right\}$

$= \frac{\# \{ \text{short exact sequence } 0 \rightarrow M_2 \rightarrow L \rightarrow M_1 \rightarrow 0 \}}{\# \text{Aut } M_2 \times \# \text{Aut } M_1}$

Define a multiplication on  $\mathcal{H}$  by

$$[M_1] * [M_2] = \sum_{[L]} F_{M_1, M_2}^L [L]$$

↳ finite sum

Prop  $(\mathcal{H}, *)$  is an associative algebra  
with unit  $[0]$

$$\textcircled{!} ([M_1] * [M_2]) * [M_3] \stackrel{?}{=} [M_1] * ([M_2] * [M_3])$$

Both count  $\{ L \supset X_1 \supset X_2 \mid \begin{array}{l} L/X_1 \cong M_1 \\ X_1/X_2 \cong M_2 \\ X_2 \cong M_3 \end{array} \}$  //

Def.  $(\mathcal{H}, *)$  : Ringel - Hall algebra

Note  $\mathcal{H}$  is graded by  $\mathbb{Z}_{\geq 0}^{\mathcal{O}_0}$

Example

$A_1$ -quiver •

$S$ : simple module

$$[S] * [S] = \# \{ M \subset S^{\oplus 2} \mid M: 1\text{-dim subspace in } S^{\oplus 2} = \mathbb{R}^2 \}$$

$$\uparrow$$

$$\# \mathbb{P}^1(\mathbb{R}) \times [S^{\oplus 2}]$$

$\parallel$

$$\# \mathbb{R} + 1$$

$$\left. \begin{array}{l} [1: *] \\ [0: 1] \end{array} \right\}$$

$$= (\# \mathbb{R} + 1) [S^{\oplus 2}] = q(q + q^{-1}) [S^{\oplus 2}] = q [2]_q [S^{\oplus 2}]$$

$$q = \sqrt{\# \mathbb{R}}$$

more generally  $[S^{\oplus n}] * [S^{\oplus m}]$

$$= \# \text{Gr}(m, m+n) [S^{\oplus (m+n)}]$$

(Grassmannian wfd of  $m$  dim subspaces in  $\mathbb{R}^{m+n}$ )

$$= \left[ \begin{array}{c} m+n \\ m \end{array} \right]_q \times q^{mn} [S^{\oplus (m+n)}]$$

$$\parallel \frac{[m+n]_q!}{[m]_q! [n]_q!}$$

$$mn = \dim \text{Gr}(m, m+n)$$

twisted version of multiplication

$$[M_1] \cdot [M_2] = f^{\langle \dim M_1, \dim M_2 \rangle} [M_1] * [M_2]$$

This is an associative algebra

defined over  $\mathbb{Z}[f, f^{-1}] \rightarrow \mathbb{C}$   
 image  $\left( \begin{array}{ccc} \mathbb{Z}[f, f^{-1}] & \xrightarrow{\psi} & \mathbb{C} \\ & \downarrow f & \downarrow \\ & f & \mathbb{N} \# \mathbb{R} \end{array} \right)$

Fact (Ringel)

structure constants  $F_{M_1, M_2}^L$   
 are polynomials in  $\# \mathbb{R}$

twisted version : alg. over  $\mathbb{Z}[f, f^{-1}] = \mathbb{A}$   
 $\mathcal{H}$  : "  $\mathbb{Z}[\# \mathbb{R}]$   
 $\# \mathbb{R} = \# \mathbb{R}$

$(\mathcal{H}^{tw}, \cdot)$   $\mathbb{A}$ -algebra

Ex.  $\mathbb{A}_1$ -quiver  $[S]^n = \underbrace{[S] \cdots [S]}_n$

$$= f^{1+2+\dots+(n-1)} \underbrace{[S] * \dots * [S]}_n$$

$$= f^{n(n-1)} [n]_f! [S^{\oplus n}]$$

$$\frac{[S]^n}{[n]_f!} = f^{n(n-1)} [S^{\oplus n}] \quad \therefore (\mathcal{H}^{tw}, \cdot) \cong \mathbb{A} \overline{U_f(\mathcal{R}_2)}$$

more general case :

$$1^\circ \quad \begin{matrix} i & j \\ \bullet & \bullet \end{matrix} \quad [S_i] * [S_j] = [S_i \oplus S_j] = [S_j] * [S_i]$$

$$2^\circ \quad \bullet \longrightarrow \bullet \quad [R \xrightarrow{1} R] =: [M]$$

$$[R^{\oplus 2} \xrightarrow{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} R] = [M \oplus S_i] =: [X]$$

$$[R^{\oplus 2} \xrightarrow{0} R] = [S_i^{\oplus 2} \oplus S_j] =: [Y]$$

$$[S_i] * [S_j] = F_{S_i, S_j}^M [M] + F_{S_i, S_j}^{S_i \oplus S_j} [S_i \oplus S_j]$$

$$\left( \begin{array}{l} S_i = [R \quad 0] \\ \uparrow \\ M = [R \rightarrow R] \\ \uparrow \\ S_j = [0 \quad R] \end{array} \right) \begin{array}{l} \parallel \\ \uparrow \\ \uparrow \\ \uparrow \end{array} \begin{array}{l} 1 \\ \\ \\ 1 \end{array}$$

$$[S_i] \cdot [S_j] = f^{-1}([M] + [S_i \oplus S_j])$$

$$[S_j] * [S_i] = [S_i \oplus S_j]$$

$$\begin{array}{l} \parallel \\ [S_j] \cdot [S_i] \end{array} \begin{array}{l} [0 \quad R] \\ \uparrow \\ [R \rightarrow R] \\ \uparrow \\ [R \quad 0] \end{array}$$

$$\begin{aligned} \therefore [M] &= f[S_i] \cdot [S_j] - [S_j] \cdot [S_i] \\ &= f([S_i] \cdot [S_j]) - f^{-1}[S_j] \cdot [S_i] \end{aligned}$$

$$\left( \begin{array}{l} \text{cf. } T_i^{-1}(f_j) = f_i f_j - \beta f_j f_i \\ T_i(f_j) = f_j f_i - \beta f_i f_j \end{array} \right.$$

$$[S_i] * [M] = F_{S_i, M}^X [X] + F_{S_i, M}^Y [Y]$$

$$\begin{array}{c} S_i \\ \uparrow \\ X \\ \uparrow \\ M \end{array} \quad \begin{array}{c} [\mathbb{R} \ 0] \\ \uparrow \\ [\mathbb{R}^{\oplus 2} \rightarrow \mathbb{R}] \\ \uparrow \text{ (10) } \parallel \\ [\mathbb{R} \rightarrow \mathbb{R}] \end{array}$$

$\# \mathbb{R}$

$\parallel 0$

$$\begin{bmatrix} 1 \\ \oplus \end{bmatrix} \in \mathbb{R}$$

$$[S_i] \cdot [M] = [X]$$

$$[S_i] * [S_i \oplus S_j] = [X] + (\# \mathbb{R} + 1) [Y]$$

$$[S_i \oplus S_j] * [S_i] = (\# \mathbb{R} + 1) [Y]$$

In summary :

$$[S_i]^{*2} * [S_j] = (\# \mathbb{R} + 1) [X] + (\# \mathbb{R} + 1) [Y]$$

$$[S_i] * [S_j] * [S_i] = [X] + (\# \mathbb{R} + 1) [Y]$$

$$[S_j] * [S_i]^{*2} = (\# \mathbb{R} + 1) [Y]$$

$$\therefore [S_i]^{*2} * [S_j] - (\# \mathbb{R} + 1) [S_i] * [S_j] * [S_i] + \# \mathbb{R} [S_j] * [S_i]^{*2} = 0$$

twisted version :  $q$ -Serre relation

$$[S_i]^2 \cdot [S_j] - (q + q^{-1}) [S_i] \cdot [S_j] \cdot [S_i] + [S_j] \cdot [S_i]^2 = 0$$

Similarly we have  $q$ -Serre relation for  $i \leftarrow j$

$$\begin{aligned} \therefore \mathbb{A} U_q^- &\xrightarrow[\text{alg. hom}]{\exists} (\mathcal{H}^{\text{tw}}, \bullet) && \text{grading preserving} \\ f_i^{(n)} = \frac{f_i^n}{[n]_{q_i}!} &\mapsto q^{n(n-1)} [S_i^{\oplus n}] \end{aligned}$$

Th. This is an isomorphism

(proof) Choose  $\vec{p}$  : reduced expression of  $w_0$  adapted with  $Q$

$$\{ \Phi_{i_1}^- \cdots \Phi_{i_{p-1}}^- (S_{i_p}) \}_{1 \leq p \leq \nu} \quad \text{complete list of ism. classes of indecomposable reps of } Q$$

$$\vec{c} = (c_1, \dots, c_\nu) \in \mathbb{Z}_{\geq 0}^\nu$$

$$M(\vec{c}, \vec{p}) := S_{i_1}^{\oplus c_1} \oplus \Phi_{i_1}^- (S_{i_2})^{\oplus c_2} \oplus \dots$$

$\mathcal{H}^{\text{tw}} \ni [M(\vec{c}, \vec{p})]$  gives an  $\mathbb{A}$ -base

PBW base of  $\mathbb{A} U_q^-$  :  $\mathbb{A}$ -base  $\uparrow$  same size

$\therefore$  enough to show the surjectivity

Recall Prop. of  $\text{Ext}^1$ -vanishing

$\mathbb{P}$  short exact seq. splits

$$[S_{i_1}^{\oplus C_1}] * [\Phi_{i_1}^-(S_{i_2})^{\oplus C_2}] * \dots$$

$$= [S_{i_1}^{\oplus C_1} \otimes \Phi_{i_1}^-(S_{i_2})^{\oplus C_2} \otimes \dots]$$

$$= [M(\vec{c}, \vec{c}')] ]$$

$\therefore$  enough to show  $[\Phi_{i_1}^- \dots \Phi_{i_{p-1}}^-(S_{i_p})^{\oplus C_p}] \in \text{image}$   $\star$

$$n = C_p \quad \frac{q^{n(n-1)}}{[n]_q!} [\Phi_{i_1}^- \dots \Phi_{i_{p-1}}^-(S_{i_p})^{\oplus C_p}]^n$$

We show  $\star$  by induction on dim. vectors

We choose  $i$ : sink among support of

$$\Phi_{i_1}^- \dots \Phi_{i_{p-1}}^-(S_{i_p})^{\oplus C_p} =: (V, B)$$

Then

$$0 \rightarrow S_i \otimes_k V_i \rightarrow (V, B) \rightarrow \text{Cok} \rightarrow 0$$

$$[\text{Cok}] * [S_i \otimes_k V_i] = [(V, B)] + \sum M [M]$$

$$\dim M = \dim V$$

$$M \not\cong (V, B)$$

$\nearrow$   $\nearrow$   
image

$$\frac{f_i^n}{[n]_q!} q^{n(n-1)}$$

$$n = \dim V_i$$

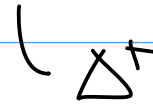
by induction hypothesis



$$\dim M = \dim V = c_p \cdot \alpha$$

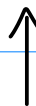
$$\alpha = s_{i_1} \cdot \dots \cdot s_{p-1}(\alpha_{i_p}) \in \Delta^+$$

M is a direct sum of indecomposables



contains at least two non-isomorphic direct summands

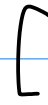
$$\overline{\Phi}_{i_1} \dots \overline{\Phi}_{i_{q-1}} (S_{i_f})^{\oplus n_f} \oplus \overline{\Phi}_{i_1} \dots \overline{\Phi}_{i_{q-1}} (S_{i_{f'}})^{\oplus n_{f'}} \oplus \dots$$



$(f \neq f')$



$\oplus \dots$



image



image

by induction hypothesis



image

$$\therefore [(V, B)] \in \text{image} //$$

Suppose  $i = \cancel{i_1} \setminus (\text{sink})$

$$\mathcal{H}_i^+ \equiv \mathcal{H}_i^+(Q, k) \stackrel{\text{def}}{=} \{ [(V, B)] \in \mathcal{H}^{\text{tw}} \mid$$

$$\bigoplus_{i(a)=i} V_0(a) \rightarrow V_i \}$$

this condition

is preserved under extensions

$\therefore \mathcal{H}_i^+$  is a subalgebra of  $\mathcal{H}^{\text{tw}}$

Assume  $\bar{i} = i_1$

$$\{ \underbrace{[M(\vec{c}, \vec{r}')] ]}_{\parallel} \mid c_1 = 0 \} : \mathbb{A}\text{-base of } \mathcal{H}_i^+$$

$$\Phi_{i_1}^-(S_{i_2}) \otimes \dots$$

Apply reflection functor  $\overline{\Phi}_i^+$

$$\mathcal{H}_i^- = \mathcal{H}_i^-(s_i Q, k)$$

injective

$$= \{ [(V, B)] \in \mathcal{H}^{\text{tw}} \mid V_i \xrightarrow{\downarrow} \bigoplus_{a(i)=i} \overline{V}_i(a) \}$$

$$= \overline{\Phi}_i^+(\mathcal{H}_i^+)$$

$$= \{ [M(\vec{c}', \vec{r}')] ] \mid c'_1 = 0 \}$$

Assume  $\bar{i} = i_1$

$(i_2, i_3, \dots, i_w, i^*)$

$$\begin{array}{ccc}
 \text{Th.} & A U_f^- & \xrightarrow{\cong} \mathcal{H}^w \\
 \text{i: sink} & \cup & \cup \\
 & \underline{A U_f^-(i)} & \xrightarrow{\cong} \mathcal{H}_i^+ \\
 \tau_i^{-1} & \cong \downarrow & \hookrightarrow \cong \downarrow \Phi_i^+ * f^{-(\alpha_i, \dim \bullet)} \\
 & \underline{A^* U_f^-(i)} & \xrightarrow{\cong} \mathcal{H}_i^-
 \end{array}$$

$$(*) \quad \underline{A U_f^-(i)} = \text{subalg. generated by } ad^{(*)}(f_i^{(m)})(f_j)$$

$$(i \neq j) \quad 0 \leq m \leq -a_{ij}$$

(proof) Claim.  $\mathcal{H}_i^+ = \text{generated by } [S_j^{\oplus n}]_{j \neq i}, [(\mathbb{k} \leftarrow \mathbb{k})]_{i, j}^{\oplus n} = \mathcal{H}_i^-$

① similar to the proof of the previous thm

$$0 \rightarrow S_j \otimes_{\mathbb{k}} V_i \rightarrow (V, B) \rightarrow \text{Cok} \rightarrow 0$$

$\Phi_{i_1}^- \dots \Phi_{i_{p-1}}^- (S_{i_p})^{\oplus n}$   
 // may assume

$j$ : sink among support of  $(V, B)$

~~we cannot use this in this case module~~

~~Instead consider  $\Phi_i^+(V, B)$~~

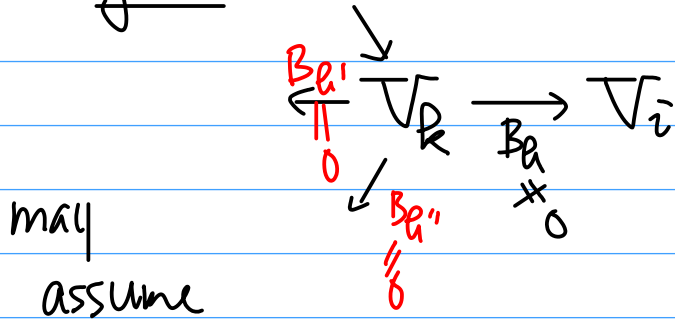
~~Rep  $\mathbb{k} = \mathbb{Q}$~~

~~$i$ : source~~

~~$\Phi_i^+$~~

If  $j \neq i$ ,  $[S_j] \in \mathcal{H}_i^+$   $\Rightarrow$  OK by induction

$$\underline{j=i}$$



Define  $Y := [V_k \rightarrow \widehat{\text{Im } B_a}]$   
 $\widehat{V_i}$   
 0 for other vertices

$$0 \rightarrow Y \rightarrow (V, B) \rightarrow \text{Cok} \rightarrow 0$$

Then  $[Cok] * [Y] = [(V, B)] + \sum C_M [M]$   
 $\dim M = \dim V$   
 $M \neq (V, B)$

$[Y] \in \mathcal{H}_i^+$   
 $[Cok] \in \mathcal{H}_i^+$   $\therefore [Y], [Cok] \in \mathcal{H}_i^+$   
 $\oplus^? \uparrow \oplus^? \uparrow$   
 $S_{\mathbb{R}} \oplus [\mathbb{R} \rightarrow \mathbb{R}]$  by induction hyp.

$[M] \in \mathcal{H}_i^+ \rightarrow [M] \in \mathcal{H}_i^+$   
 by induction

$$\begin{aligned}
 [M] = [e_i \xrightarrow{1} e_j] & \stackrel{\text{calculation}}{=} f([s_i][s_j] - f^{-1}([s_j][s_i])) \\
 \parallel & \qquad \qquad \qquad \parallel \\
 \Phi_i^+([s_j]) & \qquad \qquad \qquad \overline{f \operatorname{ad}(f_i)(f_j)} \\
 & \qquad \qquad \qquad = \overline{f T_i^-(f_j)}
 \end{aligned}$$

$$\begin{aligned}
 [M'] = [e_i \xleftarrow{1} e_j] & = f([s_j][s_i] - f^{-1}([s_i][s_j])) \\
 & = \overline{f T_i^-(f_j)} //
 \end{aligned}$$

Cor.  $\vec{h} = (i_1, \dots, i_r)$ : adapted with  $Q$

$$\Rightarrow [L(\vec{c}, \vec{h})] = \overline{f} [S_{i_1}^{\oplus c_1} \oplus \overline{\Phi_{i_1}}(S_{i_2})^{\oplus c_2} \oplus \dots]$$

$\text{PBW base} \longleftrightarrow \text{base of Ringel-Hall alg.}$   
 given by isom. classes of modules