Recall

$\mathfrak{g}$ = type ADE

$${}_{\mathcal{A}}U_q^- \cong \mathcal{H}^{tw} \quad \text{twisted version of Ringel–Hall algebra}$$

$$\text{s.t.} \quad \begin{array}{ccc} \cup & & \cup \\ f_i^{(n)} & \mapsto & q^{*}[S_i^{\oplus n}] \end{array}$$

$$L(\vec{c}, \vec{h}) \longmapsto q^{*}[\, S_{i_1}^{\oplus c_1} \oplus \Phi_{i_1}^{-}(S_{i_2})^{\oplus c_2} \oplus \cdots]$$

if $\vec{h}$ is adapted with Q

<u>Problem</u>. What is the image of $b(\vec{c}, \vec{h})$?
(canonical base elements)

ADE : Lusztig 1990

In fact, this was achieved already when he introduced the canonical base

This theory uses perverse sheaves on $E_Q(V) = \bigoplus_{h \in Q_1} \mathrm{Hom}(V_{o(h)}, V_{i(h)})$ ↶ $GL_Q(V) = \prod_{i \in Q_0} GL(V_i)$

<u>Application</u>

positivity
- $b(\vec{c}, \vec{h}), L(\vec{c}, \vec{h})$ ← transition matrix
- structure constants w.r.t $b(\vec{c}, \vec{h})$ — entries of nonnegative coefficients

Ringel-Hall algebra

$$\mathcal{H}^{tw} = \bigoplus_{[M]} \mathbb{Z}[q, q^{-1}][M]$$

$[M]$: isom. classes of representations of Q

↗ We replace this by perverse sheaves.

<u>1st approx.</u> Replace $\bigoplus \mathbb{Z}[q,q^{-1}][M]$ by

$$\bigoplus_{\substack{V:\\ Q_0\text{-graded}\\ \text{vector spaces/isom.}}} \left\{ \mathbb{Z}[q,q^{-1}]\text{-valued } GL_Q(V)\text{-invariant functions on } \mathbb{E}_Q(V) \right\}$$

This step is trivial : (convolution)

$[M] \longmapsto$ characteristic function of the $GL_Q(V)$-orbit corresponding to M

<u>2nd approx.</u> Replace functions by complexes of $GL_Q(V)$-equivariant sheaves on $\mathbb{E}_Q(V)$

($\mathbb{C}$-vector spaces or $\overline{\mathbb{Q}_\ell}$-vector spaces)

○ Philosophical background

X : top. space $\rightsquigarrow$ $H^{\cdot}(X, \mathbb{C})$ cohomology

$$\begin{array}{ccc} F & \to & X \\ & & \downarrow \pi \\ & & B \end{array}$$

fiber bundle $\rightsquigarrow$ some relations among $H^{\cdot}(X,\mathbb{C})$, $H^{\cdot}(F,\mathbb{C})$, $H^{\cdot}(B,\mathbb{C})$

Very roughly, one should study how $H^{\bullet}(\pi^{-1}(b), \mathbb{C})$ vary when b varies in B.

More generally $f: X \longrightarrow Y$.
fibers may change ...

☆ use the language of sheaves

to go further to analyze cohomology groups in relative situations.

Moreover, better to work on the level of derived categories, instead of sheaves of graded vector spaces

$$\cdots \xrightarrow{d} E^1 \xrightarrow{d} E^2 \xrightarrow{d} \cdots \qquad d \circ d = 0$$

complexes (modulo quasi-isom.)

derived category $\longrightarrow$ functions

$\left(\sum (-1)^i \dim H^i(\text{stalks}) \right)$

constant sheaves on orbits $\to$ char. func

This looses q. (Recall $q = \sqrt{\# k}$)

$\# \mathbb{P}^1(k) = \# k + 1$

$\mathbb{P}^1(k) = k \cup pt$

work on $E_Q(V)$ for $\overline{\mathbb{F}_q}$ with $\mathbb{F}_q$-rational structure

This is the most subtle part of the construction.
But we omit further detail.

Rem. In the current situation, q is also identified with cohomological degree shift.
This is a nontrivial assertion.

Finally canonical base elements are identified with special complexes in $D_{GL_Q(V)}(\mathbb{E}_Q(V))$:
intersection cohomology (IC) complexes of $GL_Q(V)$-orbits.

$\nwarrow$ simple objects in $\mathrm{Perv}_{GL_Q(V)}(\mathbb{E}_Q(V))$

perverse sheaves $\nearrow$

$$\mathrm{Perv}_{GL_Q(V)}(\mathbb{E}_Q(V)) \subset D_{GL_Q(V)}(\mathbb{E}_Q(V))$$

Then the construction of canonical base elements from PBW base elements follows from the defining property of IC complexes.

$$b(\vec{c},\vec{i}) = L(\vec{c},\vec{i}) + \sum_{\vec{c}' > \vec{c}} \underbrace{a_{\vec{c}\vec{c}'}}_{\in\, q\mathbb{Z}[q]} \cdot b(\vec{c}',\vec{i})$$

$\overline{}$ $\longleftrightarrow$ Verdier duality