

Recall

$\mathfrak{g}$  = type ADE

$\mathbb{A}U_{\mathfrak{g}}^- \cong \mathcal{H}^{tw}$  twisted version of Ringel-Hall algebra

s.t.  $f_i^{(n)} \mapsto \mathfrak{g}^{\star} [S_i^{\oplus n}]$

$L(\vec{c}, \vec{r}) \mapsto \mathfrak{g}^{\star} [S_{i_1}^{\oplus c_1} \otimes \Phi_{i_1}^-(S_{i_2})^{\oplus c_2} \otimes \dots]$

if  $\vec{r}$  is adapted with  $Q$

Problem. What is the image of  $b(\vec{c}, \vec{r})$ ?  
(canonical base elements)

ADE : Lusztig 1990

In fact, this was achieved already when he introduced the canonical base

This theory uses **perverse sheaves** on  $\mathbb{F}_Q(V) = \bigoplus_{i \in Q_0} \text{Hom}(V_{i-1}, V_i)$   
 $GL_Q(V) = \prod_{i \in Q_0} GL(V_i)$

Application

positivity •  $b(\vec{c}, \vec{r}), L(\vec{c}, \vec{r})$

• structure constants w.r.t  $b(\vec{c}, \vec{r})$  ← transition matrix entries of

Ringel-Hall algebra

$\mathcal{H}^{tw} = \bigoplus_{[M]} \mathbb{Z}[\mathfrak{g}, \mathfrak{g}^{-1}][M]$

Hom. classes of representations of  $Q$

**nonnegative coefficients**

We replace this by **perverse sheaves**.

1st approx. Replace  $\oplus \mathbb{Z}[f, f^{-1}][M]$  by

$\oplus$   $\left\{ \mathbb{Z}[f, f^{-1}]\text{-valued } GL_Q(V)\text{-invariant} \right.$   
 $V:$   $\left. \text{functions on } \mathbb{F}_Q(V) \right\}$

$\mathbb{Q}_0$ -graded  
vector spaces/isom.

This step is trivial : (convolution)

$[M] \mapsto$  characteristic function

of the  $GL_Q(V)$ -orbit corresponding  
to  $M$

2nd approx. Replace functions by  
complexes of  $GL_Q(V)$ -equivariant  
sheaves on  $\mathbb{F}_Q(V)$

$\downarrow$   
 $\mathbb{C}$ -vector spaces or  $\overline{\mathbb{Q}}$ -vector spaces

○ Philosophical background

$X$ : top. space  $\rightsquigarrow H^i(X, \mathbb{C})$  cohomology

$F \rightarrow X$  fiber bundle

$\downarrow \pi$   
 $B$

$\rightsquigarrow$  some relations

among  $H^i(X, \mathbb{C})$

$H^i(F, \mathbb{C})$

$H^i(B, \mathbb{C})$

Very roughly, one should study

how

$H^0(\pi^{-1}(b), \mathbb{C})$

vary when

$b$  varies in  $B$ .

More generally  $f: X \rightarrow Y$ .  
 fibers may change ...

★ use the language of sheaves

to go further to analyze cohomology groups  
 in relative situations.

Moreover, better to work on the level of  
 derived categories, instead of sheaves of graded  
 vector spaces

$$\dots \xrightarrow{d} E^1 \xrightarrow{d} E^2 \xrightarrow{d} \dots \quad d \circ d = 0$$

Complexes (modulo quasi-isom.)

derived category  $\xrightarrow{\text{constant sheaves}}$  functions  $\xrightarrow{\text{char. func}}$   
 (  $\sum (-1)^i \dim H^i(\text{stalks})$  )  
 $n$  bits

This loses  $q$ . (Recall  $q = \sqrt{\# \mathbb{R}}$ )  $\# \mathbb{P}^1(\mathbb{R}) = \# \mathbb{R} + 1$   
 $\mathbb{P}^1(\mathbb{R}) = \mathbb{R} \cup \{\infty\}$

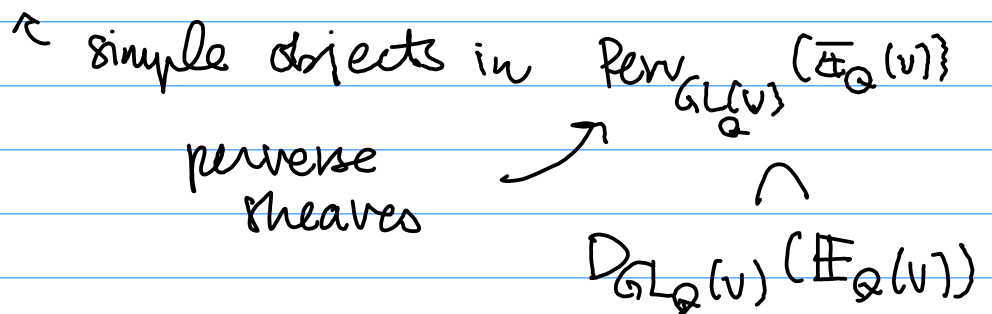
work on  $\mathbb{F}_q(V)$  for  $\overline{\mathbb{F}_q}$  with  $\mathbb{F}_q$ -rational structure

This is the most subtle part of the construction.  
 But we omit further detail.

Rem. In the current situation,  
 $\mathfrak{g}$  is also identified with  
 cohomological degree shift.  
 This is a nontrivial assertion.

Finally canonical base elements are  
 identified with special complexes  
 in  $D_{GL_Q(V)}(\mathbb{F}_Q(V))$ :

intersection cohomology (IC) complexes  
 of  $GL_Q(V)$ -orbits.



Then the construction  
 of canonical base elements  $+ \sum_{\vec{c} > \vec{c}'} a_{\vec{c}, \vec{c}'} b(\vec{c}', \vec{c})$   
 from PBW base elements  $b(\vec{c}', \vec{c})$   
 follows from the defining property  
 of IC complexes.  $\mathfrak{g} \mathbb{Z}[\mathfrak{f}]$

$\longleftrightarrow$  Verdier  
 duality