Plan

semisimple cases

T^*flag variety

- Springer representation of Weyl group
- integrable highest weight representations of Kac-Moody

quiver variety

non-semisimple case

$/\mathbb{C}$

Z : space (algebraic variety)

$\circlearrowleft T$ (algebraic) torus $\mathbb{C}^\times \times \cdots \times \mathbb{C}^\times$

$H_T^*(Z)$ equivariant cohomology

realization

$H_T^*(pt) = \mathbb{C}[\mathfrak{t}]$ $\mathfrak{t} = \mathrm{Lie}\, T$

polynomial ring over $\mathfrak{t}$

$= \mathrm{Sym}(\mathfrak{t}^*)$

A : noncommutative algebra defined over $\mathbb{C}[\mathfrak{t}]$

Often center$(A) = Z(A) = \mathbb{C}[\mathfrak{t}]$

L : simple A-module $\Longrightarrow$ $Z(A)$ acts on L by scalar multiplication

Schur's lemma

$$\Longrightarrow \quad \begin{array}{ccc} Z(A) & \xrightarrow[\shortparallel]{\chi} & \mathbb{C} \\ \shortparallel\!/ & & \\ & \chi_L & \\ \mathbb{C}[\mathfrak{t}] & & \end{array} \quad \text{algebra hom}$$

χ_L is given by evaluation at $\xi \in \mathfrak{t}$

L is a representation of the specialized algebra

$$A \otimes_{Z(A)} \mathbb{C}^{\chi}$$

$\mathbb{C}$-algebra

$A \leftsquigarrow H_T^*(Z)$ realize

$\xRightarrow{\text{specialize}}$ $A \otimes_{Z(A)} \mathbb{C} \leftsquigarrow H_T^*(Z) \otimes_{H_T^*(pt)} \mathbb{C}$ real.

$$H^*(Z^\xi) \quad // \quad \text{localization thm in equiv. cohomology}$$

$$Z^\xi = \{ x \in Z \mid \exp(t\xi) \cdot x = x \ \ \forall t \in \mathbb{C} \}$$

fixed pt w.r.t. ξ

☆ $H^*(Z^\xi)$ (or its variant)

can be analyzed by

powerful tools for

(co)homology

$\longrightarrow$ understand representation L.

○ Examples of applications of this method

- Kazhdan-Lusztig conjecture ← ∞-dim'l representations of $\mathfrak{g}$
  (Brylinski-Kashiwara
  Beilinson-Bernstein

  use D-modules over flag var.

  new proof

  Braverman-Finkelberg-N 2007.09799

  use convolution algebra defined via Zastava space

  ||

  based Maps ($\mathbb{P}^1$, flag variety)

  (completion of)

- affine Hecke algebra (Kazhdan-Lusztig, Ginzburg

  T^*flag cf. Chriss-Ginzburg

- quantum affine algebra N 1998 ~

  quiver variety

- Coulomb branches ....
  quantized

⋮

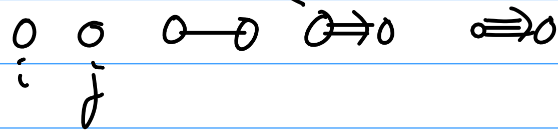
§ Very brief review of [KL, G]

W = Weyl group
$= \langle s_i \rangle$ $\qquad$ i : vertex of Dynkin diagram

$s_i^2 = 1$

braid relation $\quad s_i s_j \cdots = s_j s_i \cdots$

$\# = 2, 3, 4, 6,$



Iwahori-Hecke algebra $H(W)$ $= \mathbb{Z}[q^{1/2}, q^{-1/2}]$-algebra

generator : T_i

relations : $(T_i + 1)(T_i - q) = 0$

· braid relation

Fact $\quad w \in W \rightsquigarrow T_w = T_{i_1} \cdots T_{i_l}$ well-defined

$w = s_{i_1} \cdots s_{i_l}$ reduced expr

$$H(W) \otimes_{\mathbb{Z}[q^{1/2}, q^{-1/2}]} \mathbb{C} = \mathbb{C}[W]$$

$q = 1$ $\qquad$ group ring

Fact. $H(W)$ has a $\mathbb{Z}[q^{1/2}, q^{-1/2}]$-base $\{T_w\}$

affine Hecke algebra

P = weight lattice ($\supset Q$: root lattice)

$$P = \{ \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{R} \ni x \mid (x, \alpha_i) \in \mathbb{Z} \ \forall i \}, \qquad Q = \bigoplus \mathbb{Z}\alpha_i$$

$W \ltimes P = W_{aff}$ affine Weyl group

$H(W_{aff})$: affine Hecke alg. ↙ deformation

base $\{ e^{\lambda} \cdot T_w \mid w \in W, \lambda \in P \}$

- $\langle T_w \rangle \cong H(W)$ (T_1 = unit)
- $\langle e^{\lambda} \rangle \cong \mathbb{Z}[P][q, q^{-1}]$
- $T_i e^{s_i(\lambda)} - e^{\lambda} T_i = (1-q) \dfrac{e^{\lambda} - e^{s_i(\lambda)}}{1 - e^{-\alpha_i}}$

Prop (Bernstein) $Z(H(W_{aff})) = \mathbb{Z}[P][q, q^{-1}]^W$

(KL, G)

geometric realization of $H(W_{aff})$

$X = G/B$ flag variety

T^*X : cotangent bundle

$$\begin{array}{ccc} T^*X = G \times^B (\mathfrak{g}/\mathfrak{b})^* & \longrightarrow & \mathfrak{g}^* \\ \ni [g, x] & \longmapsto & gx \end{array}$$ moment map

$$Z = T^*X \times_{\mathfrak{g}^*} T^*X$$ Steinberg variety

$$\circlearrowright$$
$$G \times \mathbb{C}^*$$

scalar mult. on fibers

$$\begin{tikzcd}
 & T^*X \times_{\mathfrak{g}^*} T^*X \times_{\mathfrak{g}^*} T^*X \arrow[dl, "p_{12}"'] \arrow[d, "p_{23}"] \arrow[dr, "p_{13}"] & \\
Z & Z & Z
\end{tikzcd}$$

$$K^{G\times\mathbb{C}^*}(Z) \ni c, c'$$

$$\underbrace{p_{13*}}_{\text{"integration along fibers"}}\Big(\underbrace{p_{12}^*}_{\text{"pull-back"}} c \underbrace{\overset{L}{\otimes}}_{\text{"product"}} p_{23}^* c'\Big) = cc'$$ convolution product

[KL, G]
$$\begin{array}{ccc}
H(W_{aff}) & \cong & K^{G\times\mathbb{C}^*}(Z) \\
T_i & \longmapsto & \cdots\cdots \text{ (explicit)} \\
e^{\lambda} & \longmapsto & [\mathcal{O}(-\lambda)|_{\Delta}] \\
\cup & & \cup \\
Z(H(W_{aff})) & \cong & K^{G\times\mathbb{C}^*}(pt)\underbrace{[\mathcal{O}_{\Delta}]}_{\text{unit}} \\
 & & \wr\wr \\
 & & R(T)^W[q, q^{-1}]
\end{array}$$

$\Delta \subset T^*X \times T^*X$ diagonal

$$R(T)^W[q, q^{-1}] \xrightarrow{\chi} \mathbb{C}$$ central character

$$\text{m-Spec } R(T)^W[q, q^{-1}] \cong T/W \times \mathbb{C}^\times$$

$$\left(\mathbb{C}[T/W \times \mathbb{C}^\times] = R(T)^W[q, q^{-1}] \right)$$

χ = evaluation at $a = (s, q_0)$, $s \in T(/W)$, $q_0 \in \mathbb{C}^\times$

$s \in T \curvearrowright T^*X$, Z

$q_0 \in \mathbb{C}^* \curvearrowright$

specialized alg is understood by $K(Z^a)$

$\parallel$ fixed pts wrt a

$$Z^a = (T^*X)^a \times_{(\mathfrak{g}^*)^a} (T^*X)^a$$

Example $\mathfrak{g} = \mathfrak{sl}(n)$

$s = \mathrm{diag}(s_1, \cdots, s_n)$

$\xi \in \mathfrak{g}^*$ $\qquad$ ξ is fixed by a

$\iff s\xi = q_0 \xi s$

s_i-eigenspace $\to$ $q_0 s_i$-eigenspace

Assume all eigenvalues are $\in q_0^{\mathbb{Z}} s_1$

$V_i :=$ eigenspace for $q_0^{i-1} s_1$

$$\cdots \longrightarrow V_0 \xrightarrow{\xi} V_1 \xrightarrow{\xi} V_2 \longrightarrow \cdots \qquad q_0 \notin \sqrt{1}$$

$$\longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow$$

$$V_0 \xrightarrow{\xi} V_1 \xrightarrow{\xi} V_2 \xrightarrow{\xi} \cdots \longrightarrow V_0$$

$f_0 \in V_7$

$\therefore$ ξ is a representation of a quiver

$\longrightarrow$ Lusztig's canonical base

$\rightsquigarrow$ representations of $H(W_{aff})$ ($\mathfrak{g} = \mathfrak{sl}_n$)

can be understood via canonical base of U_q (type A affine type A)

11:40 ~

Kazhdan–Lusztig conjecture

$\mathfrak{g}$ = cpx simple Lie algebra
$\cup$
$\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$

$U(\mathfrak{g})$: universal enveloping algebra of $\mathfrak{g}$
$\lambda \in \mathfrak{h}^* \rightsquigarrow M(\lambda)$: Verma module
$\cup$
m_λ highest weight vector

- $h m_\lambda = \lambda(h) m_\lambda$
- $M(\lambda) = U(\mathfrak{g}) m_\lambda$ highest weight module
- $e_i m_\lambda = 0$
- universal among h.w. module

$$M(\lambda) = U(\mathfrak{g}) / U(\mathfrak{g})(h - \lambda(h)\mathrm{id},\ e_i)$$

$M(\lambda) \twoheadrightarrow L(\lambda)$ simple module (unique)

$$\mathrm{ch}\, M(\lambda) = e^\lambda \prod_{\alpha \in \Delta^+} \frac{1}{1 - e^{-\alpha}}$$

(PBW thm)

<u>Problem</u> $\mathrm{ch}\, L(\lambda) = ?$

<u>example</u> Weyl character formula
λ: integral dominant weight

$$\Rightarrow \quad \mathrm{ch}\, L(\lambda) = \sum_{w \in W} \frac{(-1)^{\ell(w)} e^{w(\lambda+\rho)-\rho}}{\prod (1 - e^{-\alpha})}$$

$\rho = \frac{1}{2}$ sum of positive roots (Weyl vector)

$$= \sum_{w \in W} (-1)^{\ell(w)} \operatorname{ch} M(w \cdot \lambda)$$

$$w \cdot \lambda = w(\lambda + \rho) - \rho$$

dot action

<u>Th</u> [KL conj. BK, BB]

Suppose $\lambda + \rho$: regular, dominant, integral

$w, y \in W$ $$\operatorname{ch} M(w\cdot\lambda) = \sum_{\substack{y \geq w \\ \uparrow \\ \text{Bruhat order}}} \underbrace{P_{w,y}(1)}_{q=1 \uparrow \ P_{w,y}(q)} \operatorname{ch} L(y \cdot \lambda)$$

Kazhdan-Lusztig polynomial

combinatorial object $\longrightarrow$

via $H(W) = \langle T_i, \text{relation} \rangle$

$\overline{}$ = bar involution on $H(W)$

$$q^{1/2} \longmapsto q^{-1/2}$$

$$T_i \longmapsto T_i^{-1} = q^{-1} T_i + (q^{-1} - 1).$$

$$\left(\overline{T_w} = (T_{w^{-1}})^{-1} \right)$$

<u>Lemma</u> $\overline{}$ is uni~~upper~~triangular w.r.t. the base $H_w := q^{-\ell(w)/2} T_w$

order = Bruhat

i.e. $$\overline{H_w} \in H_w + \sum_{y < w} \mathbb{Z}[q^{1/2}, q^{-1/2}] H_y$$

(proof) induction · true for T_i

w choose s s.t. $ws < w$

$\Rightarrow H_{ws} H_s = H_w$

Induction hypothesis $\overline{H_{ws}} \in H_{ws} + \sum_{\underset{\sim}{y<ws}} \cdots$

check $\times H_s$

$H_w + \sum_{\underset{\sim}{y<w}} \cdots$

detail is omitted. //

Prop $\exists 1$ $\overline{}$-invariant $C_w \in H(W)$ s.t.

$$C_w \in H_w + \sum_{y<w} \underset{\sim\sim\sim\sim}{q^{-1/2} \mathbb{Z}[q^{-1/2}] H_y}$$

ex. $C_i = \underbrace{q^{-1/2} T_i + q^{-1/2}}_{= H_i}$

(proof) Define C_w by induction on $<$

$$\overline{H_w} - H_w = \sum_{y<w} \underbrace{P_y(q)}_{\in \mathbb{Z}[q^{1/2}, q^{-1/2}]} C_y$$

defined already

$$\overline{P_y(q)} = P_y(q^{-1}) = -P_y(q)$$

$$\therefore P_y(q) = \underbrace{\cdots - a_2 q^{-1} - a_1 q^{-1/2}}_{\substack{\| \\ h_y(q)}} + a_1 q^{1/2} + a_2 q + \cdots$$

$$\therefore P_y(q) = h_y(q) - \overline{h_y(q)}$$

$$C_w \overset{\text{def.}}{=} H_w - \sum h_y(q)\, C_y$$

$$\overline{C_w} = \overline{H_w} - \sum \overline{h_y(q)}\, C_y$$

$$\therefore C_w = \overline{C_w} \quad //$$

<u>Def.</u> KL polynomial

$$C_w = H_w + \sum q^{\frac{\ell(w)}{2}} \underbrace{P_{y,w}(q^{-1})}_{\text{KL polynomial}} H_y$$

(∃ algorithm)

can be computed recursively