Last week:

We introduced Hecke algebra $H(W)$

and Kazhdan–Lusztig base $C_w$

( 〃 polynomial $P_{y,w}(q)$)

§ Toy model convolution algebra

based on finite sets

$X$: finite set

$\mathcal{F}[X] = \{ \mathbb{C}\text{-valued functions on } X \}$

$\mathcal{F}[X \times X]$ convolution product

$f, g \in \mathcal{F}[X\times X]$

$$(f * g)(x,z) = \sum_{y \in X} f(x,y) g(y,z)$$

$$\begin{array}{ccccc} & & X\times X\times X & & \\ & \swarrow p_{12} & \downarrow p_{23} & \searrow p_{13} & \\ X\times X & & X\times X & & X\times X \end{array}$$

$= p_{13*}(p_{12}^* f \cdot p_{23}^* g)$

sum over fibers of $p_{13}$

associative algebra with unit $= \chi_\Delta$ characteristic func. of $\Delta_X \subset X\times X$

If $|X| = n \Rightarrow \mathcal{F}[X\times X] \cong \mathrm{Mat}_{n\times n}(\mathbb{C})$

matrix algebra

$\mathcal{F}[X]$ is a module of $\mathcal{F}[X\times X]$

$\mathcal{F}[X] \cong \mathbb{C}^n$

$$f * g(x) = \sum_{y\in X} f(x,y) g(y)$$

<u>variant</u> $G \curvearrowright X$ finite group $\quad \mathcal{F}[X\times X]^G = \{\mathbb{C}\text{-valued } G\text{-invariant functions on } X\times X\}$

$\cap$ subalg.

$\mathcal{F}[X\times X]$

<u>Ex.</u> $G \curvearrowright G = X$ left multiplication

$$\mathcal{F}[G\times G]^G \subset \mathcal{F}[G\times G]$$

$$\varphi(x,y) \mapsto \Phi(x) := \varphi(x,e) \qquad \mathcal{F}[G\times G]^G \cong \mathcal{F}[G] \text{ group ring}$$

Conversely $\Phi(x) \rightsquigarrow \varphi(x,y) := \Phi(y^{-1}x)$

$$\Phi, \Psi \qquad (\Phi * \Psi)(x) = \sum_{g\in G} \Phi(g^{-1}x)\Psi(g)$$

<u>Rem</u> $\mathbb{C}[G] \ni \chi_g$ (charact. func) $\quad \chi_g * \chi_h = \chi_{gh}$

usual convention for the group ring

This gives the opposite multiplication

group rings play important roles in representation theory of finite groups

$\mathcal{F}[G]$ is a representation of $G\times G$ by left and right multiplication

**Th.** $\mathcal{F}[G] \cong \bigoplus_{\rho:\text{irreducible rep.}} \underbrace{\rho^* \otimes \rho}_{\cong\, \mathrm{End}_{\mathbb{C}}(\rho)}$

$G\times G$-equivariant algebra isomorphism

$$\mathcal{F}[G] \ni \Phi \longmapsto \bigoplus_{\rho} \Phi(g^{-1})\rho(g)$$

- $\mathcal{F}[G]$ is a semisimple algebra
  ($\Leftrightarrow$ $\forall$rep is a directsum of irreducible rep's)
- $Z(\mathcal{F}[G]) = \bigoplus \mathbb{C}\,\mathrm{id}_\rho$
  center
  $\bigoplus \mathbb{C}\,\mathrm{id}_\rho = \bigoplus \mathrm{End}_G(\rho)$ Schur

$\Delta : G \to G\times G$ $\quad \mathcal{F}[G]^{\Delta G} \cong \bigoplus \mathrm{End}_G(\rho)$

$\mathcal{F}[G]^{\Delta G} = \{$class functions $\Phi(x)$ s.t. $\Phi(gxg^{-1}) = \Phi(x)$ $\forall g\}$

e.g. character of representations

**ex.** $H \subset G$ subgroup

$$H(G,H) = \{\Phi \in \mathcal{F}[G] \mid \Phi(hgh'^{-1}) = \Phi(g)\ \ \forall h, h' \in H\}$$

$\Phi = \varphi(\cdot, e) \in \mathcal{F}[G\times G]^{\Delta G}$

called also Hecke algebra

$$\mathcal{H}(G,H) = \mathbb{C}[G\times G]^{\Delta G,\, H\times H}$$
invariant

$$= \mathbb{C}[{}_{\Delta G}\backslash G\times G / H\times H]$$

$$= \mathbb{C}[\underbrace{G/H}_{\substack{\shortparallel \\ X}} \times \underbrace{G/H}_{\substack{\shortparallel \\ X}}]^{\Delta G}$$

$\therefore$ $\mathcal{H}(G,H)$ is an example of convolution algebras

induced module

$\rho : H \to GL(V)$ representation of $H$

$$\mathrm{Ind}_H^G \rho = \{ \varphi : G \to V \mid \varphi(xh^{-1}) = \rho(h)\varphi(x) \ \forall h \in H \}$$

repr. of $G$.

$\mathrm{Res}_H^G$

$G$ $\downarrow$ $G/H$ H-principal bdle $\rightsquigarrow$ (ass. bdle) $G \times_{H,\rho} V$ $\downarrow$ $G/H$

space of sections

$G$-acts on $\mathrm{Ind}_H^G \rho$ by left multiplication

$$\mathcal{H}(G,H) \cong \underbrace{\mathbb{C}[G/H]}^{H}$$

$$\cong \left(\mathrm{Ind}_H^G \underbrace{1_H}_{\text{trivial } H\text{-rep.}}\right)^H \cong \mathrm{Hom}_H(1_H, \mathrm{Ind}_H^G(1_H))$$

$$\cong \mathrm{Hom}_H(1_H, \mathrm{Res}_H^G \mathrm{Ind}_H^G(1_H))$$

$$\underset{\text{Frobenius reciprocity}}{\cong} \quad \mathrm{Hom}_G(\mathrm{Ind}_H^G 1_H, \mathrm{Ind}_H^G 1_H)$$

multiplicity of $\rho$ in $\mathrm{Ind}_H^G 1_H$

$$\mathrm{Ind}_H^G 1_H = \bigoplus_{\rho} V_\rho \otimes \rho$$

irreducible rep

$$\cong \bigoplus_{\rho} \mathrm{End}_{\mathbb{C}}(V_\rho)$$

This is an algebra isomophism

$H(G,H)$ knows $\mathrm{Ind}_H^G 1_H$.

Observation $H(G,H)$ is commutative

$\iff \dim V_\rho = 1$ (or $0$) $\forall \rho$ multiplicity free

$(G,H)$ is called a Gelfand pair.

- Hecke alg. $H(W)$ $\longrightarrow$ noncommutative
- variant of affine Hecke algebra $\longrightarrow$ commutative

Satake isomorphism

$G$ = loop group

$H$ = Iwahori subgroup $\subset$ $H$ = sph. subgroup (max. parabolic subgroup)

affine Hecke alg

noncomm.

$\longrightarrow$ commutative

Coulomb branch $\mathcal{M}_C$ $\leftarrow$ affine algebraic variety

$$\mathcal{M}_C = \operatorname{Spec} \underbrace{\mathbb{C}[\mathcal{M}_C]}$$

$\nearrow$ coordinate ring

constructed as a convolution algebra. (geometric)

commutative!

## § Iwahori – Hecke algebra $H(W)$ and flag variety

(Ex. continued)

$k$ = finite field

$$G = GL_n(k) \supset B = B(k) = \left\{ \begin{bmatrix} \ddots & * \\ 0 & \ddots \end{bmatrix} \right\}$$

$(H)$

upper triangular matrices

$$G/B \cong \{ 0 \subset S_1 \subset S_2 \subset \cdots \subset S_{n-1} \subset k^n \}$$

flag variety

($k$-linear) subspace s.t. $\dim S_i = i$

$\mathcal{F}[G/B \times G/B]^{\Delta G}$ Hecke algebra

**Prop** (Iwahori)

$$H(W) \otimes \mathbb{C} \cong \mathcal{F}[G/B \times G/B]^{\Delta G}$$

$W = \mathfrak{S}_n$, $\mathbb{Z}[q, q^{-1}]$, $q \mapsto |k|$

$$= H(G, B)$$

special case $\underline{n=2}$ $G/B \cong \mathbb{P}^1(k)$

$G \curvearrowright G/B \times G/B$ orbits

unit

· $\Delta$ : diagonal

· complement of $\Delta$

Set $\chi :=$ constant $1$ on $G/B \times G/B$

$$(\chi * \chi)(x,z) = \sum_{\substack{y \in G/B \\ \shortparallel \\ \mathbb{P}^1}} \chi(x,y)\chi(y,z)$$

$$= |\mathbb{P}^1(k)| = 1 + |k| = (1+|k|)\chi$$

$H(\mathfrak{S}_2)$
$\ni T_i$
$(i = s_1)$

$$0 = (T_i + 1)(T_i - q)$$

$\Downarrow$

$$(T_i+1)^2 = (1+q)(T_i+1)$$

$$\therefore \quad \chi \longmapsto T_i + 1 \quad //$$

$\underline{11:28 \sim}$

<u>Schubert cell</u>

$e_1, \cdots, e_n$ : standard basis of $k^n$

$$\shortparallel$$

$$\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$w \in \mathfrak{S}_n \quad \underbrace{\langle e_{w(1)} \rangle}_{S_1} \subset \underbrace{\langle e_{w(1)}, e_{w(2)} \rangle}_{S_2} \subset \cdots \quad \in G/B$

Define $X_w$ ( Schubert cell )

$$X_w = BwB \subset G/B$$

- $X_w \cong k^{l(w)}$ $\quad l(w) = $ length of $w$

- $G/B = \coprod_{w \in \mathfrak{S}_n} X_w$

<u>Ex</u>

$w = (426135)$

$$X_w = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{bmatrix} * & * & * & 1 & & \\ * & 1 & & & & \\ * & & * & & 1 & \\ 1 & & & & & \\ & & * & & & 1 \\ & & 1 & & & \end{bmatrix}$$

$\updownarrow$

$S_1 \subset S_2 \subset S_3 \cdots$

( rule

entries in $1 \Rightarrow$, $\Downarrow$ $= 0$

$* = 0$

$\leftrightarrow \langle e_{w(1)} \rangle \subset \langle e_{w(1)}, e_{w(2)} \rangle \subset \cdots$

$\therefore$ $BwB$ is above form.

Conversely given $S_1 \subset S_2 \subset \cdots$

$S_1 = \langle f_1 \rangle \longrightarrow$ normalize $f_1 = \begin{bmatrix} * \\ \vdots \\ * \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

last nonzero entry $= 1$

$$f_2 = \begin{bmatrix} * \\ 0 \\ * \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad w(2)$$

If $w(1) < w(2)$
normalize to $0$

**Ex.** $w = 1$ $\quad \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$ $\qquad w = w_0$ $\quad \begin{bmatrix} * & \cdots & * & 1 \\ \vdots & & ⋰ & \\ * & ⋰ & & \\ 1 & & & \end{bmatrix}$

$$\ell(w) = \#\text{ of } * = \#\{i<j \mid w(i) > w(j)\}$$

**Prop'** $\quad H(W) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{C} \cong \mathcal{F}[G/B \times G/B]^{\Delta G}$

$$T_w \longmapsto \text{characteristic function of } X_w$$

**preparatory Lemma** $\quad \mathcal{F}(G/B)^G = \{G\text{-invariant functions on } G/B\} \cong \mathbb{C}$
is a module of $\mathcal{F}[G/B \times G/B]^{\Delta G}$

$\therefore \quad \mathcal{F}[G/B \times G/B]^G \longrightarrow \mathbb{C}$ alg. hom.

$$K(x,y) \longmapsto \sum_{y \in G/B} K(x,y)$$

$$H(W)\big|_{q=|k|} \ni T_w \dashrightarrow \chi_{X_w} \longmapsto |X_w| = |k|^{\ell(w)}$$

$T_w \mapsto q^{\ell(w)}$ is indeed an algebra hom

- $(T_i+1)(T_i-q)=0$
- braid relation

We should check $T_i T_w = T_{s_i w}$.

if $\ell(s_i w) = \ell(w) + 1$

<u>Lemma</u>. $\mathcal{F}[H \backslash G / H] \ni \chi_{HxH}, \chi_{HyH}$

Hecke alg. cosets

$$\chi_{HxH} * \chi_{HyH} = \sum_{z \in H \backslash G / H} \mu_{xy}^{z} \chi_{HzH}$$

$$\mu_{xy}^{z} = \frac{|HyH \cap zHx^{-1}H|}{|H|}$$

considered as subsets of $G$

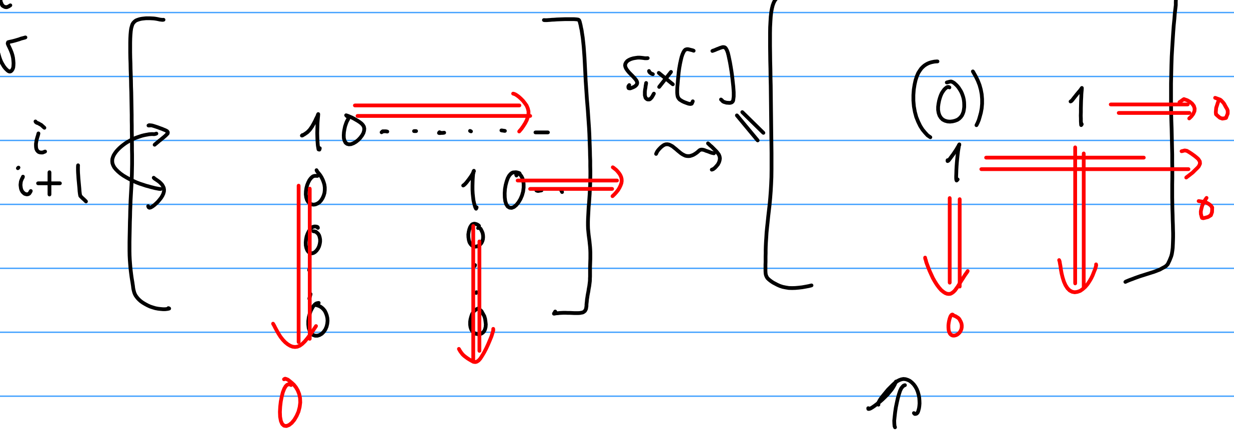
$$\begin{array}{c} i \\ i+1 \end{array} \begin{bmatrix} 0 & \ddots & & \\ & & 1 & \\ & 1 & & 0 \end{bmatrix}$$

$$\left( \begin{array}{l} y = s_i \\ x = w \end{array} \right.$$

$\ell(s_i w) = \ell(w) + 1$

$\Downarrow$

$w^{-1}(i) < w^{-1}(i+1)$




i.e. $Bs_iB \cdot BwB \subset Bs_iwB$

$\therefore \chi_{Bs_iB} * \chi_{BwB} \overset{\exists}{=} \mu \times \chi_{Bs_iwB}$

Lemma $\Longrightarrow \mu = 1$ //

# Kazhdan-Lusztig conjecture

representation theory of $\mathfrak{g}$ ($\mathbb{C}$)

"localization"

D-modules on $G/B(\mathbb{C})$

Riemann-Hilbert correspondence

functions associated with intersection cohomology

$G/B(\mathbb{F}_q)$

replace by $\underline{\mathbb{C}}$

(cf. HTT)

convolution algebra for $G/B$

KL polynomial (combinatorics)

$\underline{H(W)}$

$T_w, C_w$

$\overline{C_w} = C_w$

characteristic function

KL