§ Borel-Moore homology (preparation)

Fulton Young tableaux, Appendix B

$$\underset{\text{closed}}{X \subset \mathbb{R}^n} \rightsquigarrow H_*(X) \overset{\text{def}}{=} H^{n-*}(\mathbb{R}^n, \mathbb{R}^n \setminus X)$$

↑ $\mathbb{C}$-coeff.

Properties

(1) independent of the choice of embedding

(2) $X \underset{\text{closed}}{\subset} M$ oriented mfd $\quad H_*(X) \cong H^{\dim M - *}(M, M \setminus X)$

(3) (pull-back with support)

$f: M^m \longrightarrow N^n$ conti. map between oriented mfd

$$\begin{array}{ccc} M^m & \longrightarrow & N^n \\ \cup & & \cup \\ X & & Y \end{array}$$

$$\Rightarrow f^*: \begin{array}{ccc} H_*(Y) & \xrightarrow{f^{-1}(Y) \subset X} & H_{*-n+m}(X) \\ \text{\rotatebox{90}{$\cong$}} & & \text{\rotatebox{90}{$\cong$}} \\ H^{n-*}(N, N \setminus Y) & & H^{n-*}(M, M \setminus X) \end{array}$$

(4) $f: X \to Y$ proper (i.e. $f^{-1}(\text{cpt})$: cpt)

$\rightsquigarrow f_*: H_*(X) \to H_*(Y)$

<u>sketch</u>

$$\begin{array}{ccccc} X & \xrightarrow{(f,i)} & Y \times (0,1)^n & \longrightarrow & Y \times [0,1]^n \\ & & & & \cap \\ & \searrow_{\text{closed}} & & & \mathbb{R}^m \times \mathbb{R}^n \end{array}$$

$X \overset{i}{\subset} \mathbb{R}^n \cong (0,1)^n$

$$\rightsquigarrow \quad H_*(X) \longrightarrow H_*(Y \times [0,1]^n)$$

$$H_*(Y \times [0,1]^n) \cong H_*(Y) \quad \text{Künneth} + H_*([0,1]^n) \cong \begin{cases} \mathbb{C} & * = 0 \\ 0 & \text{else} \end{cases}$$

(5) $i : Y \hookrightarrow X$ closed

$j : U = X \setminus Y \hookrightarrow X$ open

$$\rightarrow H_k(Y) \xrightarrow{i_*} H_k(X) \xrightarrow{j^*} H_k(U) \xrightarrow{\delta} H_{k-1}(Y) \rightarrow$$

(exact) defined

(6) intersection product w.r.t. M

$X, Y \subset M$

$$H_k(X) \otimes H_l(Y) \xrightarrow{\cap} H_{k+l-\dim M}(X \cap Y)$$

$\cap_M$

(depends on M)

$\cup$ product for relative cohomology

(7) Alexander duality

If X is cpt and locally contractible

$$\Longrightarrow \quad H_i(X) \cong H_i^{ord}(X)$$

(8) $H^i(X) \otimes H_j(X) \longrightarrow H_{j-i}(X)$

$\cup$-product module

$X \subset \mathbb{R}^n$ $\quad H_*(X) = H^{n-*}(\mathbb{R}^n, \mathbb{R}^n \setminus X)$

$\subset\ \subset$
U open

$\downarrow \quad \downarrow$
$U \quad U$

$H^*(U)$ product

$$\varinjlim_{U} H^*(U) = H^*(X)$$

assume this holds by either assume X is a reasonable space

or H^* is appropriately defined

(9) fundamental classes of $\mathbb{C}$-analytic varieties

X : C^∞-mfd oriented $\quad H_{\dim_{\mathbb{R}} X}(X) \cong H^0(X)$

$$[X] \longleftarrow 1$$

fund. class

X : $\mathbb{C}$-analytic variety $\quad \{X_\alpha\}_\alpha$: irr. component of $\dim = n$

$\dim_{\mathbb{C}} X = n$

$$\Longrightarrow \quad H_i(X) = 0 \quad \text{for } i > 2n$$

$$H_{2n}(X) \cong \bigoplus_\alpha H^0(X_\alpha)$$

$$[X_\alpha] \longleftrightarrow 1$$

(∵) $X \supset Z$: singular set, irr. components ~~of~~ with $\dim \leq n-1$

$X \setminus Z$: C^∞-mfd $\dim = 2n$

$$H_i(X\setminus Z) = H^{2n-i}(X\setminus Z) = 0 \quad \text{if } i > 2n$$

$$H_{2n}(\underbrace{X\setminus Z}_{\coprod X_\alpha \setminus Z}) \cong \bigoplus_\alpha H^0(X_\alpha \setminus Z)$$

connected components

$$H_{2n}(Z) \longrightarrow H_{2n}(X) \longrightarrow H_{2n}(X\setminus Z) \longrightarrow H_{2n-1}(Z)$$

$H_{2n}(Z) = 0$ ($\dim Z \leq n-1$), $H_{2n-1}(Z) = 0$ exact.

$\therefore \cong$

by induction on dimension

(10) paving by affine spaces

$$X = X_s \supset X_{s-1} \supset \cdots \supset X_0 = \phi$$

$\mathbb{C}$-analy sequence of closed ~~subsets~~ subvarieties

s.t. $X_k \setminus X_{k-1} = \coprod_l \mathbb{C}^{n(k,l)}$

$$\Longrightarrow \quad H_*(X) = \bigoplus_{k,l} \mathbb{C}[\overline{\mathbb{C}^{n(k,l)}}]$$

fund. class of closures

e.g. $X = \mathcal{F}(n)$: flag variety $/\mathbb{C}$

$= \coprod \underbrace{BwB}_{X_w \cong \mathbb{C}^{l(w)}}$ Bruhat decomposition (proof) Induction on k

$$\to H_i(X_{k-1}) \to H_i(X_k) \to \bigoplus_{l} \underbrace{H_i(\mathbb{C}^{n(k,l)})}_{\substack{\| \\ \begin{cases} \mathbb{C} & \text{if } i = 2n(k,l) \\ 0 & \text{otherwise} \end{cases}}} \to$$

$H_{odd}(X_{k-1}) = 0$ by induction hypothesis

$\therefore$ short exact sequence for i = even

$$0 \to \underset{\substack{\| \\ \bigoplus_{k' \leq k-1} \mathbb{C}[\overline{\mathbb{C}^{n(k',l)}}]}}{H_i(X_{k-1})} \to \underset{\substack{\cup \\ [\overline{\mathbb{C}^{n(k,l)}}]}}{H_i(X_k)} \to \bigoplus_{l} \underbrace{H_i(\mathbb{C}^{n(k,l)})}_{\substack{\cup \\ [\mathbb{C}^{n(k,l)}]}} \to 0$$

$$[\overline{\mathbb{C}^{n(k,l)}}] \mapsto [\mathbb{C}^{n(k,l)}]$$

$\therefore$ OK for k //

§ Convolution algebra by Borel-Moore homology

⊗ trivial examples

① X : oriented, compact mfd

$H_*(X \times X)$ has a convolution product:

$$\begin{tikzcd}[column sep=small]
& X \times X \times X \arrow[dl, "p_{12}"'] \arrow[d, "p_{23}"] \arrow[dr, "p_{13}"] & \\
X \times X & X \times X & X \times X
\end{tikzcd}$$

$\alpha, \beta \in H_*(X \times X)$

$$\alpha * \beta = p_{13*}(p_{12}^* \alpha \cap p_{23}^* \beta)$$

$H_*(X \times X)$ is an algebra (forget grading)

with unit $= [\Delta_X]$ ← diagonal in $X \times X$

$$H_*(X \times X) \cong \mathrm{End}(H_*(X))$$

matrix algebra $\left(H_*(X) \cong H^*(X) \right)$ by P.D.

$H_*(X) =$ vector representation

② X : oriented, mfd, not necessarily cpt

$$H_*(X \underset{X}{\times} X) = H_*(X)$$

$$\left(\begin{array}{c} \cap \\ X \times X \end{array} \right)$$

$$\begin{array}{ccccc} & & X\underset{X}{\times}X\underset{X}{\times}X = X & & \\ & \swarrow^{p_{12}} & \downarrow p_{23} & \searrow^{p_{13}} & \\ X\underset{X}{\times}X & & X\underset{X}{\times}X & & X\underset{X}{\times}X = X \\ \parallel & & \parallel & & \\ X & & X & & \end{array}$$

all maps are identity

$$\alpha * \beta = p_{13*}(p_{12}^*\alpha \cap p_{23}^*\beta) = \alpha \cap \beta$$

$\therefore$ $H_*(X \underset{X}{\times} X)$ + convolution product $\cong (H_*(X), \cap)$

(super)commutative algebra

◎ We consider convolution algebras defined as follows:

<u>Assumptions</u> $\pi : M \longrightarrow X$ proper

M: oriented smooth mfd (not nec. cpt)

$$Z = M \underset{X}{\times} M = \{(m_1, m_2) \mid \pi(m_1) = \pi(m_2)\}$$

$$\begin{array}{ccc} & M\times M\times M & \\ \swarrow & \downarrow & \searrow \\ M\times M & & \end{array}$$

$H_*(Z) \ni \alpha, \beta$

$$\alpha * \beta \overset{\text{def.}}{=} p_{13*}(p_{12}^*\alpha \cap p_{23}^*\beta)$$

<u>well-defined</u>?

$p_{12}^*\alpha \in H_*(p_{12}^{-1}Z)$

pull-back support for
$$\begin{array}{ccc} M\times M\times M & \longrightarrow & M\times M \\ \cup & & \cup \\ p_{12}^{-1}Z & & Z \end{array}$$

$$p_{23}^{*}\beta \in H_*(p_{23}^{-1}Z)$$

$$p_{12}^{*}\alpha \cap p_{23}^{*}\beta \in H_*(\underbrace{p_{12}^{-1}Z \cap p_{23}^{-1}Z}_{\shortparallel})$$

$\cap_{M\times M\times M}$

$$M\times_X M\times_X M \xrightarrow{p_{13}} M\times_X M$$

proper

$\therefore$ $p_{13*} : H_*(p_{12}^{-1}Z \cap p_{23}^{-1}Z) \to H_*(Z)$ is well-defined

$x \in X$ $\quad M_x = \pi^{-1}(x)$

$$\begin{tikzcd} & M\times M \arrow[dl, "id"'] \arrow[d, "p_1"] \arrow[dr, "p_2"] & \\ M\times M & M & M \end{tikzcd}$$

$$id^{-1}(\underbrace{M\times_X M}_{\underset{Z}{\shortparallel}}) \cap p_1^{-1}(M_x) = M_x \times M_x \xrightarrow{p_2} M_x$$

$\therefore$ $H_*(M_x)$ is a right module of $H_*(Z)$

$1 \leftrightarrow 2$ $\quad H_*(M_x)$ is a left module of $H_*(Z)$.

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<u>grading</u> $M = \coprod \underbrace{M_\alpha}$

possibly disconnected

connected component

possibly different dimensions

$Z = \coprod Z_{\alpha\beta}$ $\quad Z_{\alpha\beta} = M_\alpha \underset{\times}{\times} M_\beta$

index set for α could be possibly an ∞ set.

$$H_*(Z) = \bigoplus H_*(Z_{\alpha\beta})$$ ← without unit

or $\prod H_*(Z_{\alpha\beta})$ $\quad \sum_{\alpha \in A} [\Delta_{M_\alpha}]$

$*$ may not be well-defined

$$c * c' = \sum_{\alpha,\beta,\gamma} c_{\alpha\beta} * c'_{\beta\gamma}$$

<u>Assume</u> index set is finite

(not essential)

$$H_i(Z_{\alpha\beta}) \otimes H_j(Z_{\beta\gamma}) \longrightarrow H_{i+j-\dim M_\beta}(Z_{\alpha\gamma})$$

$$\underbrace{i + \dim M_\gamma}_{p_{12}^*} + \underbrace{j + \dim M_\alpha}_{p_{23}^*} - (\dim M_\alpha + \dim M_\beta + \dim M_\gamma)$$

$\bigoplus H_{\underset{\sim}{*} + \frac{\dim M_\alpha + \dim M_\beta}{2}}(Z_{\alpha\beta})$ is a graded algebra

e.g. $[\Delta_{M_\alpha}]$ is degree 0
$(\alpha = \beta)$

<u>Def.</u> $H_{[0]}(Z) \overset{\text{def.}}{=\!=} \bigoplus H_{\frac{\dim M_\alpha + \dim M_\beta}{2}}(Z_{\alpha\beta})$

is a subalgebra (grading $= 0$)

Similarly $\bigoplus_{\alpha} H_{i + \frac{1}{2}\dim M_\alpha}(\pi^{-1}(x) \cap M_\alpha)$

(i: fix) is a module of $H_{[0]}(Z)$

$H_{[0]}(\pi^{-1}(x))$ is defined later

should be defined from the information of $\underset{\sim}{x}$ (and M_α)

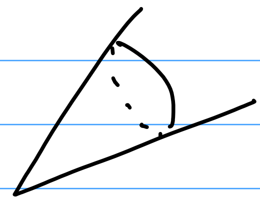
1^{st} example

$$\mathbb{P}^1 = \mathbb{CP}^1 = \{ S \subset \mathbb{C}^2 \mid 1 \text{ dim subspace} \}$$

$$\begin{array}{l} M = T^*\mathbb{P}^1 = \{ (x, S) \mid x : \mathbb{C}^2/S \to S \} \\ \qquad\qquad\qquad\qquad\quad \mathbb{C}^2 \twoheadrightarrow \mathbb{C}^2/S \qquad S \hookrightarrow \mathbb{C}^2 \\ \quad \downarrow \pi \\ X = \{ x \in \mathrm{End}(\mathbb{C}^2) \mid x^2 = 0 \} = \left\{ \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \,\middle|\, a^2 + bc = 0 \right\} \end{array}$$

$\mathfrak{gl}_2$ (a $\mathfrak{sl}_2$) nilpotent cone $\subset \mathbb{C}^3$

X is a cpx algebraic variety
$\ni 0$ singular point



fiber of π $\qquad \pi^{-1}(x) = \left\{ S \subset \mathbb{C}^2 \,\middle|\, \begin{array}{l} \mathrm{Im}\, x \subset S \\ S \subset \mathrm{Ker}\, x \end{array} \right\}$

$$\begin{cases} \pi^{-1}(0) = \mathbb{P}^1 \\ \pi^{-1}(x) = pt \qquad S = \mathrm{Im}\, x = \mathrm{Ker}\, x \\ {\scriptstyle x \neq 0} \end{cases}$$

$$\therefore \ Z = M \times_X M = \underbrace{\mathbb{P}^1 \times \mathbb{P}^1}_{\substack{\uparrow \\ x = 0}} \cup \underbrace{\Delta_M}_{\substack{\uparrow \\ \text{inv. im of } x \neq 0 \\ \text{closure of}}}$$

intersect along $\Delta_{\mathbb{P}^1}$

$$H_{[0]}(Z) = H_4(Z) = \mathbb{C}[\mathbb{P}^1 \times \mathbb{P}^1] \oplus \mathbb{C}[\Delta_M]$$

$x \neq 0$ $\quad H_{[0]}(\pi^{-1}(x)) = H_0(\pi^{-1}(x)) = \mathbb{C}[\pi^{-1}(x)]$

"natural" degree

$$H_{[0]}(\underbrace{\pi^{-1}(0)}_{=\mathbb{P}^1}) = H_2(\mathbb{P}^1) = \mathbb{C}[\pi^{-1}(0)]$$

<u>Th</u>. [$\mathfrak{sl}_2$-case of Kashiwara-Tanisaki
Ginzburg

cf) Springer representation of Weyl groups]

$$H_{[0]}(Z) \cong F(\mathfrak{S}_2) \quad \text{group ring of } \mathfrak{S}_2 = \langle \pm 1 \rangle$$

$$[\Delta_M] \longmapsto 1$$

$$[\mathbb{P}^1 \times \mathbb{P}^1] \longmapsto -(1+s)$$

$$(-(1+s))^2 = 1 + 2s + s^2 = \underset{\sim}{-2}(-(1+s))$$

$$\underbrace{[\mathbb{P}^1\times\mathbb{P}^1]}_{\alpha} * \underbrace{[\mathbb{P}^1\times\mathbb{P}^1]}_{\beta} = ?$$

$$p_{12}^*\alpha \cap p_{23}^*\beta = [\mathbb{P}^1\times\mathbb{P}^1\times M]\cap[M\times\mathbb{P}^1\times\mathbb{P}^1]$$

$$= [\mathbb{P}^1]\times\big(([\mathbb{P}^1]\cap[\mathbb{P}^1]\big)\times[\mathbb{P}^1]$$

self-intersection of $\mathbb{P}^1$ in $T^*\mathbb{P}^1$

$= -2$

$$\xrightarrow{p_{13*}} -2[\mathbb{P}^1\times\mathbb{P}^1]$$

<u>modules</u>

$x\neq 0$ $\quad H_0(\pi^{-1}(x))$ $\quad [\mathbb{P}^1\times\mathbb{P}^1]$ acts by 0

$[\mathbb{P}^1\times\mathbb{P}^1] = -(1+s)$

$\therefore$ s acts by -1

i.e. sign representation

$H_2(\pi^{-1}(0))$ $\quad [\mathbb{P}^1\times\mathbb{P}^1]$ acts by -2

$\therefore$ s acts by 1

i.e. trivial repr.

$Fl(\mathfrak{S}_2)$ $\quad$ irr. rep of $\mathfrak{S}_2$ $= \{$triv, sign$\}$

convolution algebra realize irreducible repr.

2^{nd} example

W = l-dim cpx vector space ($l \in \mathbb{Z}_{>0}$)

$G(n, l) = G(n, W)$ = Grassman variety of n-dim'l subspaces of W

$$M = \bigsqcup_{0 \leqq n \leqq l} T^* G(n, l) = \{ (x, S) \mid x : W/S \to S \}$$

$$\downarrow$$

$$X = \{ x \in \mathrm{End}(W) \mid x^2 = 0 \} \subset \text{nilpotent cone for } \mathfrak{gl}(W)$$

$$Z = M \times_X M = \bigsqcup_{0 \leqq n_1, n_2 \leqq l} Z(n_1, n_2 ; l)$$

$$X = \bigsqcup \{ x \mid \text{Jordan form } 2^i 1^{l-2i} \}$$

i times