2nd example

$\mathbb{C}^l$

$G(n,l) = \{ S \subset \mathbb{C}^l \mid \dim S = n \}$ subspace

$T^*G(n,l)$: cotangent b'dle

$$M = \coprod_{0 \le n \le l} \underbrace{T^*G(n,l)}_{} \cong \{ (x, S) \mid x : \mathbb{C}^l/S \to S \}$$

$$\pi \downarrow$$

$$X = \{ x \in \mathrm{End}(\mathbb{C}^l) \mid x^2 = 0 \}$$

$$Z = M \underset{X}{\times} M = \coprod_{0 \le n_1, n_2 \le l} Z(n_1, n_2; l) \subset T^*G(n_1, l) \times T^*G(n_2, l)$$

$$\underbrace{\coprod X_i}_{} \quad \text{"} \quad x \sim (2^i 1^{n-2i}) \quad \text{Jordan form}$$

$$x \in X_i \Rightarrow \overset{i}{\mathrm{Im}\, x} \subset S \subset \overset{l-i}{\mathrm{Ker}\, x}$$

$$\pi^{-1}(x) = \coprod_{0 \le n \le l} G(n-i, l-2i)$$

$$\swarrow \qquad \searrow$$

$$i \qquad l-i$$

Prop. $Z(n_1, n_2 ; l)$ is the union of conormal b'dles to ΔGL_l-orbits in $G(n_1, l) \times G(n_2, l)$ finite set

In particular, $Z(n_1, n_2 ; l)$ is lagrangian subvariety in $T^*G(n_1, l) \times T^*G(n_2, l)$

M : mfd $\supset S$ (loc. closed) submfd

T^*M : cotangent $\quad \omega$: sympl. form

$T^*_S M$: conormal b'dle

$\| \quad \subset \quad T^*M$

$\{ (q, \underbrace{p}_{\in T^*_q M}) \in T^*M = \bigsqcup_{q \in M} T^*_q M \mid \langle p, T_q S \rangle_\omega = 0 \}$

(proof) $G(n_1, l) \times G(n_2, l)$

$\quad\quad\quad \ni S_1 \quad\quad \ni S_2$

$T_{S_1} = \mathrm{Hom}(S_1, \mathbb{C}^l/S_1)$, $T_{S_2} = \quad \cdots$

tangent space
to $GL(l)$-orbit
through (S_1, S_2)

$\iota_{S_1} : S_1 \hookrightarrow \mathbb{C}^l$

$p_{S_1} : \mathbb{C}^l \twoheadrightarrow \mathbb{C}^l/S_1$

$$T_{S_1} \supset \{ p_{S_1} \xi \iota_{S_1} \mid \xi \in \mathrm{End}(\mathbb{C}^l) \}$$
$$: S_1 \to \mathbb{C}^l/S_1$$

$\therefore (X_1, X_2) \in T^*_{S_1} \times T^*_{S_2}$ is $\perp$ to $\Delta GL(l)$-orbit

$\iff 0 = \mathrm{tr}(X_1 p_{S_1} \xi \iota_{S_1}) - \mathrm{tr}(X_2 p_{S_2} \xi \iota_2)$

$$= \mathrm{tr}((\underbrace{\iota_{S_1} X_1 p_{S_1} - \iota_{S_2} X_2 p_{S_2}}_{\therefore \; = 0}) \xi) \quad \forall \xi$$

$X_1 = X_2$ as elements in $\mathrm{End}(\mathbb{C}^\ell)$ //

<u>Def.</u> $H_{[0]}(Z) \overset{\text{def.}}{=} \bigoplus_{n_1, n_2} H_{\dim_{\mathbb{C}} T^*G(n_1,\ell) \times T^*G(n_2,\ell)} (Z(n_1, n_2 : \ell))$

$\uparrow$ top

$\cap$ subalgebra

$H_*(Z)$

$$1 = \sum_n [\Delta_{T^*G(n,\ell)}] \in H_{[0]}(Z)$$

$T^*G(n,\ell) \times T^*G(n-1,\ell) \supset P(n,\ell) =$ conormal bdle to $\{S_1 \supset S_2\}$

closed submfd

$\cap$

$G(n,\ell) \times G(n-1,\ell)$

$\because$ closed

$$[P(n,\ell)] \in T_{[0]}(Z)$$

$$F \overset{\text{def.}}{=} \sum_n (-1)^{n-1} [P(n,\ell)]$$

$$E \overset{\text{def}}{=} \sum_n (-1)^{\ell-n} [\sigma(P(n,\ell))] \qquad \sigma : M \times_Z M \circlearrowleft$$

$(m_1, m_2) \mapsto (m_2, m_1)$

$$H \overset{\text{def}}{=} \sum_n (\ell - 2n) [\Delta_{T^*G(n,\ell)}]$$

$\uparrow$

$H_{[0]}(Z)$

Th [Ginzburg cf. Beilinson-Lusztig-MacPherson]

$$\mathcal{F}_1\Big(\coprod_{n_1,n_2} G(n_1,l)\times G(n_2,l)\Big)^{\Delta GL_l}$$

$k = \mathbb{F}_q$

$$U(\mathfrak{sl}_2) \twoheadrightarrow H_{[0]}(Z) \quad \text{surj. alg. hom}$$

$$E,F,H \mapsto E,F,H$$

$$[H,E]=2E$$
$$[H,F]=-2F$$
$$[E,F]=H$$

$$\omega : U(\mathfrak{sl}_2) \longrightarrow U(\mathfrak{sl}_2)^{op} \quad \text{involution}$$
$$E,F,H \mapsto F,E,H$$

$$\begin{array}{ccc} U(\mathfrak{sl}_2) & \xrightarrow{\ \omega\ } & U(\mathfrak{sl}_2)^{op} \\ \downarrow & \circlearrowright & \downarrow \\ H_{[0]}(Z) & \longrightarrow & H_{[0]}(Z)^{op} \end{array}$$

$\Sigma \frown \sigma$

$(-1)^{\frac{\dim M_1+\dim M_2}{2}}$

$Z_\alpha \subset M_1\times M_2$

$x\in X$

$$H_*(\pi^{-1}(x))_{left} \cong H_*(\pi^{-1}(x))_{right} \qquad (-1)^{*}\sigma$$

$H_{[0]}(Z)$ -module iso.

$$H_*(\pi^{-1}(x))_{left}^{\omega} \cong H_*(\pi^{-1}(x))_{right}$$

as $U(\mathfrak{sl}_2)$-module

dual representation

V : $\mathfrak{g}$-module $\quad$ ($U(\mathfrak{g})$-module)

$\rightsquigarrow$ V^* : dual space $\quad$ $\rho : \mathfrak{g} \to \mathrm{End}(V)$

$\uparrow$

$-\rho(X)^* \quad X \in \mathfrak{g}$

$$H_*(\pi^{-1}(x))^\omega = \left(\left(H_*(\pi^{-1}(x)) \right)^* \right)^{(-)\omega}$$

$$\omega : U(\mathfrak{sl}_2) \to U(\mathfrak{sl}_2)^{op} \xrightarrow{=} U(\mathfrak{sl}_2)$$

$$X \longmapsto -X$$

alg. isom.

(sketch of proof)

1°. relations

$$[H, E] = 2E, \quad [H, F] = -2F \quad \text{obvious}$$

Let us check $\quad [E, F] = H.$

$$U \underset{\text{open}}{\overset{i}{\subset}} Z \quad \text{complement of } \coprod_n \Delta_{T^*G(n,0)}$$

<u>Claim</u>. $\quad H_{[0]}(Z) \xrightarrow{i^*} H_{[0]}(U)$

$$i^*([E, F]) = 0$$

$$M \times_X M \times_X M \ni \{ (x, S_1, S_2, S_3) \}$$

Suppose $\cap\, U$, i.e. $S_1 \neq S_3$

$EF \ : \ (x, S_1^n \subset S_2^{n+1} \supset S_3^n)$

$FE \ : \ (x, S_1^n \supset S_2^{n-1} \subset S_3^n)$

$$S_1 \cap S_3 \subset S_2^{n+1}$$
$$\overset{\shortparallel}{S_2} \leftarrow (n-1)\text{-dim'l}$$
$$S_1 \supset S_2 \subset S_3$$

$$S_1 + S_3 \subset \mathbb{C}^\ell$$
$$\overset{\shortparallel}{S_2} \leftarrow (n+1)\text{-dim'l}$$
$$S_1 \subset S_2 \supset S_3$$

$n+1$?

$$\begin{tikzcd}
U \cap p_1^{-1}(P(n,\ell)) \cap p_2^{-1}(\sigma(P(n,\ell))) \arrow[d, "\cong"] \arrow[dd, bend right, "p_{1,3}"'] \\
U \cap p_1^{-1}(\sigma(P(n,\ell))) \cap p_2^{-1}(P(n,\ell)) \arrow[d, "p_{1,3}"] \\
Z \cap U
\end{tikzcd}$$

$\therefore\ i^*[E,F] = 0$ //

$$\xrightarrow{\ 2^0\ } (0\to) H_{[0]}(\Delta) \longrightarrow H_{[0]}(Z) \longrightarrow H_{[0]}(U)$$

$[E,F] \in H_{[0]}(Z)$

exact

$$\therefore\ [E,F] = \sum_n {}^{\exists} \underset{\sim\sim}{C(n,\ell)} [\Delta_{T^*G(n,\ell)}]$$

$\uparrow$ compute this!

We assume $2n \leqq \ell$. ($2n \geqq \ell$ case is similar)

$T^*G(n,\ell) \supset U := \{ (x, S) \mid x : \underset{\ell-n}{\mathbb{C}^\ell/S} \to \underset{n}{S} \text{ surjective} \}$ ($\cap \operatorname{Im} j$)

nonempty open subset

Consider $Z(n,n;\ell) \cap (T^*G(n,\ell) \times U)$

$FE : (x, S_1 \overset{1}{\supset} S_2^{n-1} \overset{1}{\subset} S_3)$

$x : \mathbb{C}^\ell/S_3 \twoheadrightarrow S_3$

$\not\ni /S_2$ i.e. $FE|_{T^*G(n,\ell)\times U} = 0$

enough to compute $EF|_{T^*G(n,\ell)\times U}$

$(x, S_1 \subset S_2 \supset S_3)$

$S_2 = \operatorname{Im} x$ $\therefore S_1 = \operatorname{Im} x$

$\therefore S_1 = S_3$

Possibility of S_2 ?

$\underset{\text{1 dim subspace}}{S_2/\operatorname{Im} x} \underset{\text{sub space}}{\subset} \underbrace{\operatorname{Ker} x/\operatorname{Im} x}_{(\ell-2n)\text{ dim}}$

since $(x, S_3) \in U$

$$\begin{array}{ccc} p_{12}^{-1}(P) \cap p_{23}^{-1}(\sigma P) & \cap & T^*G(n,l) \times T^*G(n+1,l) \times U \\ & & \| \wr \\ & & \mathbb{P}(\operatorname{Ker} x / \operatorname{Im} x) \end{array}$$

$$\begin{tikzcd} p_{12}^{-1}(P) \cap p_{23}^{-1}(\sigma P) \arrow[d, "p_{13}"'] \arrow[dr, phantom, "\circlearrowright"] & \mathbb{P}(\underbrace{\operatorname{Ker} x / \operatorname{Im} x}) \arrow[d] \\ \Delta_U & \cong U \end{tikzcd}$$

projective bdle

↑ vector bdle over U

☆ $p_{12}^{-1}(P) \cap p_{23}^{-1}(\sigma P)$ $(\cap \quad \cdots \quad)$

is <u>not</u> transversal intersection

<u>Lemma</u> $S_1, S_2 \subset M$ clean intersection

(i.e. $S_1 \cap S_2$: submfd

tangent sp. of $S_1 \cap S_2 \ni m$

$= T_m S_1 \cap T_m S_2$)

$$[S_1] \cap [S_2] = e(TM / TS_1 + TS_2) \cap [S_1 \cap S_2]$$

↑ Euler class

☹ exercise

In our situation $P_{12} \subset M_1 \times M_2$ w.r.t. ω

$P_{23} \subset M_2 \times M_3$

clean intersection

lagrangian $TP_{12} \cong (TP_{12})^{\perp}$

$$\left(T(M_1 \times M_2 \times M_3) \big/ T(p_{12}^{-1} P_{12}) + T(p_{23}^{-1} P_{23}) \right)^*$$

$$\underset{\omega}{\cong} \left(T p_{12}^{-1} P_{12} + T p_{23}^{-1} P_{23} \right)^{\perp}$$

$$= T p_{12}^{-1} P_{12}^{\perp} \cap T p_{23}^{-1} P_{23}^{\perp}$$

$$= \Big(T P_{12} \times \underset{\substack{\uparrow \\ M_3 \\ \text{comp.}}}{\{0\}} \Big) \cap \Big(\underset{\substack{\uparrow \\ M_1 \\ \text{comp.}}}{\{0\}} \times T P_{23} \Big)$$

$$= T(p_{12}^{-1} P_{12} \cap p_{23}^{-1} P_{23}) \cap \underbrace{\operatorname{Ker} dp_{13}}_{\substack{\uparrow \\ T_f}}$$

$\downarrow$ fiber b'dle

$p_{13}(\quad " \quad)$

T_f tangent space along fiber

$$p_{12}^*[P_{12}] \wedge p_{23}^*[P_{23}] = e(T_f^*) \cap [p_{12}^{-1} P_{12} \cap p_{23}^{-1} P_{23}]$$

$$p_{13*}(\qquad\qquad) = (-1)^{\dim(\text{fiber})} \text{Euler}(\text{fiber.}) \underbrace{p_{13*}[p_{12}^{-1} P_{12} \cap p_{23}^{-1} P_{23}]}_{\substack{\| \\ [\text{base}]}}$$

$$\text{fiber} = \mathbb{P}^{n-2l-1}$$

$$\therefore c_l = n-2l$$

11:35~

$\underline{3^\circ}$ $x \in X_i$ $\qquad \pi^{-1}(x) = \coprod_{i \leq n \leq l-i} G(n-i, l-2i)$

$S \subset \ker x / \operatorname{Im} x$

$$H_{[0]}(\pi^{-1}(x)) \overset{\text{def}}{=} \bigoplus_n H_{top}(G(n-i, l-2i))$$

$$= \bigoplus_n \mathbb{C}[\underbrace{G(n-i, l-2i)}_{\cong \, \pi^{-1}(x) \cap T^* G(n, l)}]$$

$H_{[0]}(Z)$-module

$\therefore$ $\mathfrak{sl}_2$-module

weight (i.e. H eigenvalue)

$$n = i \longrightarrow l - 2i$$
$$n = i+1 \longrightarrow l - 2i - 2$$
$$\vdots$$
$$n = l - i \longrightarrow 2i - l$$

$$E[G(\underbrace{n-i}_{\substack{= \\ i}}, l-2i)] = 0$$

$\underbrace{}_{= 0}$

$\therefore$ highest weight $(= l-2i)$ vector is $[G(0, l-2i)]$.

$\therefore$ $H_{[0]}(\pi^{-1}(x)) \underset{\text{left}}{\cong}$ irr. representation of $\mathfrak{sl}_2$ with h.w $= l-2i$.

as $\mathfrak{sl}_2$-module

$$H_{[0]}(\pi^{-1}(x))_{\text{right}} \cong \text{dual rep.}$$

(up to twist by auto $(-)\,\omega$)

$$\begin{array}{c} Z \longrightarrow X = \coprod_{0 \leq i \leq \lfloor \ell/2 \rfloor} X_i \\ \| \\ M \times_X M \end{array}$$

$$X_{\leq i} := \bigcup_{j \leq i} X_j \subset X$$

closed

$Z_{\leq i} \overset{\text{def}}{=}$ inverse image of $X_{\leq i}$

$$0 \to H_{[0]}(Z_{\leq i-1}) \longrightarrow H_{[0]}(Z_{\leq i}) \longrightarrow H_{[0]}(Z_i)$$

two sided ideal ! (exercise)

$$\sum_{k,k'} \mathbb{C}\, F^{k} [\Delta_{T^*G(i,\ell)}] E^{k'} \cong \underbrace{V(\ell-2i)}_{\text{irr. h.w rep}} \otimes V(\ell-2i)^*$$

mod $H_{[0]}(Z_{\leq i-1})$

$$\| \wr$$

$$H_{[0]}(Z_{\leq i}) / H_{[0]}(Z_{\leq i-1})$$

by above analysis

$\Rightarrow U(\mathfrak{sl}_2) \longrightarrow H_{[0]}(Z)$ is surjective by induction on i

//

$$\text{Kernel} = \{x \in U(\mathfrak{sl}_2) \mid x \text{ acts on } V(\ell-2i) \text{ by } 0\}$$
$$i = 0, 1, \cdots, \lfloor \ell/2 \rfloor$$

$$\begin{array}{c} U(\mathfrak{sl}_2)/\text{Kernel} \\ \parallel \\ \bigoplus_{i=0,\cdots,\lfloor \ell/2 \rfloor} \mathrm{End}(V(\ell-2i)) \end{array} \xrightarrow{\simeq} H_{[r]}(\delta)$$

§ equivariant Borel-Moore homology

$$N \gg 0 \qquad \begin{array}{ccc} \mathbb{C}^N \setminus 0 & \xrightarrow{\mathbb{C}^\times} & \mathbb{C}P^{N-1} \\ \| & & \| \\ E\mathbb{C}^*_N & & B\mathbb{C}^\times_N \end{array} \qquad\qquad (B\times P)^r = B^r \times P^r$$

$$T = (\mathbb{C}^\times)^r \qquad \begin{array}{ccc} ET_N & \longrightarrow & BT_N \\ \| & & \| \\ (E\mathbb{C}^\times_N)^r & & (B\mathbb{C}^\times_N)^r \end{array}$$

$X \curvearrowleft T$ s.t. $X \overset{\text{closed}}{\subset} \mathbb{R}^n$ T-equivariant emb.

$$H^n_T(X) = H^n(ET_N \times_T X) \quad \text{for } N \gg n$$

$$H^T_n(X) = H_{n+\dim BT_N}(ET_N \times_T X) \quad \left(\begin{array}{c}\text{Borel} \\ \text{-Moore}\end{array}\right)$$

- independent of choices (of N)
- $X \to ET_N \times_T X$ fiber bdle $\rightsquigarrow$ Leray spectral sequence play important role
$$\downarrow$$
$$BT_N$$
- $\underset{\text{algebra}}{H^*_T(X)} \curvearrowright \underset{\text{module}}{H^T_*(X)}$
- fundamental class $[X] \in H^T_{\dim_{\mathbb{R}} X}(X)$ $\quad [ET_N \times_T X]$

  X: T-mfd oriented
- Poincaré duality $\quad H^*_T(X) \cong H^T_{\dim X - *}(X)$

  X: T-mfd $\qquad \alpha \mapsto \alpha \cap [X]$
- pull-back with support

— proper pushforward

— $\to H^T_{lf}(Y) \to H^T_{lf}(X) \to H^T_{lf}(U) \to$ (exact)

— intersection product $\quad X, Y \subset M$ ($T \circlearrowright X, Y$, $\circlearrowright M$)

— equivariant Chern class $\rightsquigarrow c_i(E)$

$E \to X$ T-equiv. vector b'dle $\in H^{2i}_T(X)$

<u>Th</u>. $H^*_T(pt) \cong \mathbb{C}[\mathrm{Lie}\,T]$

polynomials on Lie T

<u>ex</u>. $H^n_{\mathbb{C}^\times}(pt) = H^n(\mathbb{C}P^N) \overset{\sim}{\leftarrow} \mathbb{C}x^{n/2}$

$N \gg 0$

$H^*(\mathbb{C}P^N) = \mathbb{C}[x]/(x^{N+1})$

$H^*_{\mathbb{C}^\times}(pt) \cong \mathbb{C}[x]$

$x = c_1(\mathcal{O}(1))$

generator of $\mathbb{C}[\mathrm{Lie}\,\mathbb{C}^\times]$, $\mathrm{Lie}\,\mathbb{C}^\times = \mathbb{C}$

$H^*_T(pt) \xrightarrow{a^*} H^*_T(X) \curvearrowright H^T_*(X)$

$X \xrightarrow{a} pt$

☆ M: module over $\underbrace{\mathbb{C}[\mathrm{Lie}\,T]}_{=R}$

$\mathrm{Supp}\,M \overset{\mathrm{def}}{=} \{ P \in \mathrm{Spec}\,R \mid M_P \neq 0 \} \subset \mathrm{Spec}\,R$

localization closed (Zariski)

$0 \to M' \to M \to M'' \to 0$ exact

$\Rightarrow \mathrm{Supp}\,M = \mathrm{Supp}\,M' \cup \mathrm{Supp}\,M''$

Th [Borel] $\mathrm{Supp}\, H^*_T(X) \subset \bigcup_{x\in X} \mathrm{Lie}(\mathrm{Stab}_T(x))$

$H^T_*(X)$

e.g. ◦ $X = T \Rightarrow H^*_T(T) = H^*(ET \times_T T)$

$= H^*(ET) = \begin{cases} \mathbb{C} & * = 0 \\ 0 & \text{otherwise} \end{cases}$

$H^*_T(pt)$ evaluation at 0

$\| $

$\mathbb{C}[\mathrm{Lie}\, T]$

$\therefore \mathrm{Supp} = \{0\}$

◦ $X \curvearrowleft T$ trivial $\Rightarrow H^*_T(X) = H^*_T(pt) \otimes H^*(X)$

$\mathrm{Supp} = \mathrm{Lie}\, T$

◦ $X = T/T' \Rightarrow H^*_T(X) = H^*(ET \times_T T/T')$

$= H^*(ET/T') = H^*(BT')$

$= \mathbb{C}[\mathrm{Lie}\, T']$

X^T = fixed pt set $= \{x \in X \mid \mathrm{Stab}_T(x) = T\}$

$\circlearrowright$

X

$X \setminus X^T = \{x \in X \mid \mathrm{Stab}_T(x) \subsetneqq T\}$

type is finite as $X \subset \mathbb{R}^N$

$\therefore \mathrm{Supp}\, H^*_T(X \setminus X^T) \subsetneqq \mathrm{Lie}\, T$ proper subset

$H^T_*(X \setminus X^T)$

$\therefore$ localization at generic pt $= 0$

$$\bullet \otimes_{H_T^*(pt)} \mathrm{Frac}\, H_T^*(pt)$$

<u>Th</u>. $H_*^T(X^T) \xrightarrow{i_*} H_*^T(X)$ become isom

$H_T^*(X) \xrightarrow{i^*} H_T^*(X^T)$

after $\bullet \otimes_{H_T^*(pt)} \mathrm{Frac}\, H_T^*(pt)$.

☆ manifold case

Suppose $X = M$: T-manifold

$M^T = \coprod_\alpha F_\alpha$ connected components

F_α ← submanifold

$$\begin{tikzcd}
H_*^T(M^T) \arrow[r, "i_*"] \arrow[d, equal] & H_*^T(M) \arrow[d, "\cong"] & \\
\bigoplus_\alpha H_*^T(F_\alpha) \arrow[ddrr, dotted, bend right] & H_T^{\dim M - *}(M) \arrow[r, "i^*"] & H_T^{\dim M - *}(M^T) \arrow[d, "\cong"] \\
& & \bigoplus_\alpha H_T^{\dim M - *}(F_\alpha) \arrow[d, "\cong"] \\
& & \bigoplus_\alpha H^T_{*-\mathrm{codim} F_\alpha}(F_\alpha)
\end{tikzcd}$$

Thm ⇢ preserves the decomposition, and

$$H^T_*(F_\alpha) \longrightarrow H^T_{*-\operatorname{codim} F_\alpha}(F_\alpha)$$

is given by $e(N_{F_\alpha/M}) \frown$

Euler class of the normal b'dle.

$\because$ $F_\alpha \hookrightarrow M$ is locally T-equivariantly diffeomorphic to

$$\underset{\text{zero section}}{F_\alpha \hookrightarrow} \operatorname{Tot}(N_{F_\alpha/M})$$

//

Cor. [Atiyah-Bott, Berline-Vergne]
Suppose M : cpt
$p : M \longrightarrow pt$, $p_\alpha : F_\alpha \longrightarrow pt$

$$p_*(c) = \sum_\alpha p_{\alpha *}\left(\frac{1}{e(N_{F_\alpha/M})} \frown i_\alpha^* c \right)$$

$\because$

$$\begin{tikzcd}
H^T_*(M) \arrow[d, "p_*"'] & \bigoplus_\alpha H^T_*(F_\alpha) \arrow[l, "i_*"'] \arrow[d, "p_{\alpha *}"] \arrow[dl, phantom, "\circlearrowright"] \\
H^T_*(pt) & \bigoplus_\alpha H^T_*(pt) \arrow[l, "\sum_\alpha"]
\end{tikzcd}$$

as $p \circ i = \sum \circ p_\alpha$

Now $i_*^{-1} = \underbrace{(i^* i_*)^{-1}}_{= \sum e(N_{F_\alpha/M})^{-1}} i^*$ //