Recall we started to explain

(equivariant cohomology $H_T^*(M)$
" Borel-Moore homology $H_*^T(M)$

Th. $H_T^*(pt) \cong \mathbb{C}[\mathrm{Lie}\,T]$
polynomials on Lie T

ex. $H^n_{\mathbb{C}^\times}(pt) = H^n(\mathbb{C}P^N) \cong \mathbb{C}x^{n/2}$
$N \gg 0$

$H^*(\mathbb{C}P^N) = \mathbb{C}[x]/(x^{N+1})$

$\therefore H^*_{\mathbb{C}^\times}(pt) \cong \mathbb{C}[x]$

$x = c_1(\mathcal{O}(1))$
generator of $\mathbb{C}[\mathrm{Lie}\,\mathbb{C}^\times]$, $\mathrm{Lie}\,\mathbb{C}^\times = \mathbb{C}$

$H_T^*(pt) \xrightarrow{a^*} H_T^*(X) \curvearrowright H_*^T(X), H_T^*(X)$

$X \xrightarrow{a} pt$

☆ M: module over $\underbrace{\mathbb{C}[\mathrm{Lie}\,T]}_{=R}$

$\mathrm{Supp}\,M \overset{\text{def.}}{=} \{ P \in \mathrm{Spec}\,R \mid M_P \neq 0 \} \subset \mathrm{Spec}\,R$
localization closed (Zariski)

Easy fact

$0 \to M' \to M \to M'' \to 0$ exact

$\Rightarrow \mathrm{Supp}\,M = \mathrm{Supp}\,M' \cup \mathrm{Supp}\,M''$

$\underline{\text{Th}}$ [Borel] $\quad \operatorname{Supp} \underset{H^T_*(X)}{H^*_T(X)} \subset \bigcup_{x\in X} \operatorname{Lie}(\operatorname{Stab}_T(x))$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cap$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \operatorname{Lie} T$

$\underline{\text{e.g.}}$ ∘ $X = T \Rightarrow H^*_T(T) = H^*(ET \times_T T)$

$= H^*(ET) = \begin{cases} \mathbb{C} & * = 0 \\ 0 & \text{otherwise} \end{cases}$

$H^*_T(pt) = \mathbb{C}[\operatorname{Lie} T]$ evaluation at 0

$\therefore \operatorname{Supp} = \{0\}$

∘ $X \curvearrowleft T$ trivial $\Rightarrow H^*_T(X) = H^*_T(pt) \otimes H^*(X)$

$\operatorname{Supp} = \operatorname{Lie} T \qquad ET \times_T X = (ET/T) \times X$

∘ $X = T/T' \quad \Rightarrow H^*_T(X) = H^*(ET \times_T T/T')$

$\begin{pmatrix} T' \subset T \\ \text{subgrp} \end{pmatrix}$ $\quad = H^*(ET/T') = H^*(BT') = H^*_{T'}(pt)$

$= \mathbb{C}[\operatorname{Lie} T'] \qquad \operatorname{Supp} = \operatorname{Lie} T'$

X^T = fixed pt set $= \{x \in X \mid \operatorname{Stab}_T(x) = T\}$

$\overset{\curvearrowright}{X}$

$X \setminus X^T = \{x \in X \mid \operatorname{Stab}_T(x) \subsetneqq T\}$

type is finite as $X \subset \mathbb{R}^N$

$\therefore \quad \operatorname{Supp} \underset{H^T_*(X\setminus X^T)}{H^*_T(X \setminus X^T)} \subsetneqq \operatorname{Lie} T \qquad$ proper subset

$\therefore$ localization at generic pt $= 0$

$$\bullet \otimes_{H_T^*(pt)} \operatorname{Frac} H_T^*(pt)$$

$$H_T^*(pt) \otimes_{\mathbb{C}} H_*(X^T)$$
$$\parallel$$

Thm.

$$H^T_*(X^T) \xrightarrow{i_*} H^T_*(X) \quad \text{become} \quad \text{isom}$$

(localization thm of equivariant cohomology)

$$H_T^*(X) \xrightarrow{i^*} H_T^*(X^T)$$

after $\bullet \otimes_{H_T^*(pt)} \operatorname{Frac} H_T^*(pt)$.

☆ manifold case

Suppose $X = M$: oriented T-manifold

$$M^T = \coprod_\alpha F_\alpha \quad \text{connected components}$$

F_α $\leftarrow$ submanifold

$$\begin{tikzcd}
H^T_*(M^T) \arrow[r, "i_*"] \arrow[d, equal] & H^T_*(M) \arrow[d, "\cong\ \text{P.D.}"] & \\
\bigoplus_\alpha H^T_*(F_\alpha) \arrow[ddrr, dotted, bend right, "?"] & H_T^{\dim M - *}(M) \arrow[r, "i^*"] & H_T^{\dim M - *}(M^T) \arrow[d, "\cong"] \\
& & \bigoplus_\alpha H_T^{\dim M - *}(F_\alpha) \arrow[d, "\cong\ \text{P.D.}"] \\
& & \bigoplus_\alpha H^T_{* - \operatorname{codim} F_\alpha}(F_\alpha)
\end{tikzcd}$$

Th $\cdots\cdots\rightarrow$ preserves the decomposition, and

$$H_*^T(F_\alpha) \longrightarrow H^T_{*-\mathrm{codim} F_\alpha}(F_\alpha)$$

is given by $e(N_{F_\alpha/M}) \frown$

Euler class of the normal bdle.

$\because$ $F_\alpha \hookrightarrow M$ is locally T-equivariantly diffeomorphic to

$$F_\alpha \underset{\text{zero section}}{\hookrightarrow} \mathrm{Tot}(N_{F_\alpha/M})$$

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Cor [Atiyah–Bott, Berline–Vergne] (Bott's residue thm)

Suppose M: cpt

$p: M \longrightarrow pt$, $p_\alpha: F_\alpha \longrightarrow pt$

$c \in H_*^T(M)$

$$p_*(c) \underset{\substack{\uparrow \\ \text{equality in } \mathrm{Frac} H_T^*(pt)}}{=} \sum_\alpha p_{\alpha *}\left(\frac{1}{e(N_{F_\alpha/M})} \frown \underline{i_\alpha^* c}\right)$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \overset{\shortparallel}{P.D.(i_\alpha^* P.D.(c))}$

$\because$

$$\begin{tikzcd}
H_*^T(M) \arrow[d, "p_*"'] \arrow[dr, phantom, "\circlearrowright"] & \bigoplus_\alpha H_*^T(F_\alpha) \arrow[l, "i_*"'] \arrow[d, "p_{\alpha *}"] \\
H_*^T(pt) & \bigoplus_\alpha H_*^T(pt) \arrow[l, "\sum_\alpha"]
\end{tikzcd}$$

as $p \circ i = \sum \circ p_\alpha$

Now $i_*^{-1} = \underbrace{(i^* i_*)^{-1}}_{\sum e(N_{F_\alpha/M})^{-1}} i^*$

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equivariant (co)homology for more general groups

$G = GL(r)$

$S(r,N) = \{ \mathbb{C}^N \to \mathbb{C}^r \mid \text{surjective} \}$

$\circlearrowleft GL(r)$ $\quad$ $S(r,N)/GL(r)$ = Grassmannian of r-dim'l quotient of $\mathbb{C}^N$

<u>Fact</u> $\quad H^*(S(r,N)) = 0$ for $* = 1 \sim 2(N-r)$

$$\rightsquigarrow \quad H^*_{GL(r)}(X) := H^*(X \times_{GL(r)} S(r,N))$$

for suff. large N.

well-defined

G: cpx reductive group $\curvearrowright$ $\subset GL(r)$ subgrp

$$H^*_G(X) := H^*(X \times_G S(r,N))$$

Induction / restriction

$$H^*_G(X) \longrightarrow H^*_{G'}(X) \qquad EG \times_{G'} X \longrightarrow EG \times_G X$$

$G' \subset G$

$$H^*_G(G \times_{G'} X) \cong H^*_{G'}(X)$$

e.g. $G \supset B \supset T$ (Borel subgrp)

$$H^*_G(G/B) \cong H^*_B(pt) \cong H^*_T(pt) \cong \mathbb{C}[\mathrm{Lie}\,T]$$

$G/B = \mathcal{F}l(r)$ flag mfd; homotopy equiv

$$GL(r) \supset \left\{ \begin{bmatrix} * & * \\ & * \end{bmatrix} \right\} \supset \left\{ \begin{bmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{bmatrix} \right\}$$

$$\mathcal{F}l(r) = \{ 0 \subset S_1 \subset S_2 \subset \cdots \subset S_{r-1} \subset \mathbb{C}^r \mid \dim S_i = i \}$$

$$\mathcal{S}_i \to \mathcal{F}l(r) \qquad \mathcal{S}_i \big|_{S_\bullet} = S_i \quad \text{tautological b'dle}$$

$$-c_1(\mathcal{S}_i / \mathcal{S}_{i-1}) =: x_i$$

$$H^*_G(G/B) \cong \mathbb{C}[x_1, \ldots, x_r] \cong \mathbb{C}[\mathrm{Lie}\,T]$$

<u>Th.</u> $T \subset G$ as above $\quad W = $ Weyl group $= N(T)/T \quad \leftarrow$ finite group

$$H^*_G(X) \cong H^*_T(X)^W$$

(sketch of proof)

$$H^*_{N(T)}(X) = H^*(S(r,N) \times_{N(T)} X) = H^*(S(r,N) \times_T X)^W, \quad \text{invariant part}$$

$$\begin{array}{ccc} S(r,N) \times_{N(T)} X & \longleftarrow & G/N(T) \\ \downarrow & & \text{fiber b'dle} \\ S(r,N) \times_G X & & \end{array}$$

$$H^*(G/N(T)) = H^*(\underbrace{G/T}_{\substack{\text{IS} \\ G/B}})^W \quad \text{homotopy equiv.}$$

$$H^*(G/B) \cong \mathbb{C}[x_1, \cdots, x_r] \Big/ \left(\begin{matrix}\text{nonconstant} \\ \text{symmetric} \\ \text{polynomial}\end{matrix} = 0\right)$$

$G = GL(r)$

$$\therefore H^*(G/B)^{W = \mathfrak{S}_r} = \mathbb{C}[x_1 \cdots x_r]^{\mathfrak{S}_r} \Big/ (\quad = 0)$$

$$= \begin{cases} \mathbb{C} & * = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\therefore H^*_{N(T)}(X) \cong H^*_G(X) \quad //$$

11:27 ~

§ convolution algebra and localization

$\pi : M \longrightarrow X$ proper

$\uparrow$ mfd

Assume T-equivariant

$Z = M \times_X M \curvearrowleft T \qquad Z^T = M^T \times_{X^T} M^T$

fixed pt set

$N =$ normal b'dle of M^T in M

<u>Prop.</u> $i : M^T \times M^T \longrightarrow M \times M$ inclusion

$(i^{-1}(Z) \subset Z^T)$

$$\underset{\substack{\uparrow \\ \left(\text{2nd factor only}\right)}}{\frac{1}{1 \otimes e(N)}} \, i^* : H^T_*(Z) \longrightarrow \underset{(\text{shift})}{H^T_*(Z^T)} \otimes_{H_T^*(pt)} \mathrm{Frac}\, H_T^*(pt)$$

$$\cong \quad H_*(Z^T) \otimes_{\mathbb{C}} \mathrm{Frac}\, H_T^*(pt)$$

is an algebra hom.

$\left(\sqrt{e(N) \otimes e(N)}^{\,-1} i^* \text{ also work} \right)$

(proof) Use notation Z_{12}, Z_{23}, Z_{13}

$$\begin{tikzcd} & M \times M \times M \arrow[dl, "p_{12}"'] \arrow[d, "p_{23}"] \arrow[dr, "p_{13}"] & \\ M \times M & \times & \times \end{tikzcd}$$

$\overline{p}_{\alpha\beta}$: proj for <u>fixed pt set</u>

$i^* p_{12}^* = \overline{p}_{12}^* i^*$ etc

$\therefore$ $\underline{i^*(p_{12}^* c \cap p_{23}^* c') = \overline{p}_{12}^* i^* c \cap \overline{p}_{23}^* i^* c'}$ ☆

pushforward p_{13*} vs $\overline{p}_{13*}$

$$\begin{tikzcd} M^T \times M^T \times M^T \arrow[r, "i"] & M \times M \times M \\ \overline{p}_{12}^{-1}(Z_{12}^T) \cap \overline{p}_{23}^{-1}(Z_{23}^T) \arrow[u, phantom, "\cup"] \arrow[r, "i"] \arrow[d, "\overline{p}_{13}"] \arrow[dr, phantom, "\circlearrowright"] & p_{12}^{-1}(Z_{12}) \cap p_{23}^{-1}(Z_{23}) \arrow[u, phantom, "\cup"] \arrow[d, "p_{13}"] \\ Z_{13}^T \arrow[r, "i"] \arrow[d, phantom, "\subset"] & Z_{13} \arrow[d, phantom, "\subset"] \\ M^T \times M^T \arrow[r, "i"] & M \times M \end{tikzcd}$$

$$\therefore\ p_{13*}\underbrace{i_*}_{(i^*)^{-1}e(N_1\boxplus \underset{\text{difference}}{N_2}\boxplus N_3)} = \underbrace{i_*}_{(i^*)^{-1}e(N_1\boxplus N_3)}\bar{p}_{13*}$$

N_α $\quad \alpha=1,2,3$
$\parallel$
normal b'dle of M^T in M
αth factor

$$\therefore\ \underbrace{i^* p_{13*}(i^*)^{-1}e(N_2) = \bar{p}_{13*}}_{☆☆}$$

$$\bar{p}_{13*}\left(\bar{p}_{12}^*\left(\frac{1}{e(N_2)}i^*c\right)\cap p_{23}^*\left(\frac{1}{e(N_3)}i^*c'\right)\right)$$

$$\underset{☆}{=}\ \bar{p}_{13*}\left(\frac{1}{e(N_2)e(N_3)}i^*\left(p_{12}^*c\cap p_{23}^*c'\right)\right)$$

$$\underset{☆☆}{=}\ \frac{1}{e(N_3)}i^*p_{13*}\left(p_{12}^*c\cap p_{23}^*c'\right)$$

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§ flag variety and (degenerate) affine Hecke algebra

$\mathcal{F}l(n) = \{\underset{S_0}{0}\subset S_1\subset\cdots\subset S_{n-1}\subset \underset{S_n}{\mathbb{C}^n}\}$ flag variety

$GL(n) \curvearrowright$

$M = T^*\mathcal{F}l(n) = \{(S_\bullet,\xi) \mid \xi\in \mathrm{End}(\mathbb{C}^n),\ \xi(S_i)\subset S_{i-1}\}$

$\downarrow$

$X = \mathcal{N} = \{\xi\in\mathrm{End}(\mathbb{C}^n) \mid \xi^n = 0\}$ nilpotent cone

$\mathbb{C}^*\curvearrowright T^*\mathcal{F}l(n)$ multiplication on fibers.

$$T^*\mathcal{F}(n) \times_{\mathcal{N}} T^*\mathcal{F}(n) \curvearrowleft GL(n)\times\mathbb{C}^\times$$

$\parallel$

Z

Steinberg variety
(variety of triples)

<u>Th</u> [Lusztig]

$$H_*^{GL(n)\times\mathbb{C}^\times}(Z) \overset{\star}{\cong} H_{deg} \quad \text{degenerate affine Hecke alg.}$$

alg/ $H^*_{\mathbb{C}^\times}(pt) = \mathbb{C}[\hbar]$

generator σ_i $(1\leq i\leq n-1)$
x_i $(1\leq i\leq n)$

relations $\sigma_i^2=1$, $\sigma_i\sigma_j=\sigma_j\sigma_i$ if $|i-j|>1$
$\sigma_i\sigma_{i+1}\sigma_i=\sigma_{i+1}\sigma_i\sigma_{i+1}$

$\left(\langle\sigma_i\rangle \cong \mathbb{C}[\mathfrak{S}_n]\right.$

$x_ix_j=x_jx_i$ $\left(\langle x_i\rangle \cong \mathbb{C}[x_1,\dots,x_n]\right.$

$$\sigma_i x_j - x_{s_i(j)}\sigma_i = \begin{cases} 0 & |i-j|>1 \\ -\hbar & j=i \\ +\hbar & j=i+1 \end{cases}$$

$\star$: $-c_1(\mathcal{S}_i/\mathcal{S}_{i-1}) \longmapsfrom x_i$ as Chern

$\left(\begin{array}{l} H^*_{GL(n)}(pt) \longrightarrow \text{symmetric polynomial in } x_1\dots x_n \\ \parallel \\ (H^*_T(pt))^{\mathfrak{S}_n} \end{array} \right)$

$1 \leq i \leq n-1$

$$\mathcal{P}(i,n) = \left\{ 0 = S_0 \subset \cdots \subset S_{i-1} \begin{array}{c} \subset S_i^1 \subset \\ \subset S_i^2 \subset \end{array} S_{i+1} \subset \cdots \subset S_n = \mathbb{C}^n \right\}$$

$\cap$

$\mathcal{F}(n) \times \mathcal{F}(n)$

$$\left(\begin{array}{l} \Delta GL(n) \curvearrowright \mathcal{F}(n) \times \mathcal{F}(n) \\ \quad \text{orbits} \longleftrightarrow \mathfrak{S}_n \\ \quad \mathcal{P}(i,n) \setminus \Delta_{\mathcal{F}(n)} \longleftrightarrow s_i \end{array} \right.$$

$\overline{\mathcal{P}}(i,n)$ = conormal b'dle to $\mathcal{P}(i,n)$ $\subset T^*\mathcal{F}(n) \times T^*\mathcal{F}(n)$

$\left(\text{cf. } n=2 \quad Z = \Delta_{T^*\mathbb{P}^1} \cup \underbrace{\mathbb{P}^1 \times \mathbb{P}^1} \right)$

<u>Lemma</u> $\overline{\mathcal{P}}(i,n)$ is a closed submfd of $T^*\mathcal{F}(n) \times T^*\mathcal{F}(n)$

(Thm cont'd)

$$\star : -[\overline{\mathcal{P}}(i,n)] - [\Delta_{T^*\mathcal{F}(n)}] \longmapsto \sigma_i$$

(sketch of proof)

$$\left(T^*\mathcal{F}(n)\right)^T = \mathcal{F}(n)^T = \mathfrak{S}_n \ni w$$

$$\uparrow$$

$$0 \subset \underset{\langle e_{w(1)} \rangle}{\overset{}{S_1}} \subset \underset{\langle e_{w(1)}, e_{w(2)} \rangle}{S_2} \subset \cdots$$

$$Z^T = \mathfrak{S}_n \times \mathfrak{S}_n$$

$$H^T_*(Z^T) = H^T_*(\mathfrak{S}_n \times \mathfrak{S}_n)$$

$$= H^*_T(pt) \otimes H_0(\mathfrak{S}_n \times \mathfrak{S}_n)$$

$$\circlearrowright$$

$$H^T_*(T^*\mathcal{F}l(n)^T)$$

$$\cong$$

$$H^*_T(pt) \otimes H_0(\mathfrak{S}_n)$$

described by (combinatorics) explicitly.