§ flag variety and (degenerate) affine Hecke algebra

$\mathcal{F}l(n) = \{ 0 \subset S_1 \subset \cdots \subset S_{n-1} \subset \mathbb{C}^n \}$ flag variety

$S_0 = 0$, $S_n = \mathbb{C}^n$

$\circlearrowleft GL(n)$

$$M = T^*\mathcal{F}l(n) = \{ (S_\bullet, \xi) \mid \xi \in \mathrm{End}(\mathbb{C}^n),\ \xi(S_i) \subset S_{i-1} \}$$

$$\downarrow \pi$$

$$X = \mathcal{N} = \{ \xi \in \mathrm{End}(\mathbb{C}^n) \mid \xi^n = 0 \}$$

nilpotent cone

$\mathbb{C}^* \curvearrowright T^*\mathcal{F}l(n)$ multiplication on fibers.

$$T^*\mathcal{F}(n) \times_{\mathcal{N}} T^*\mathcal{F}(n) \curvearrowleft GL(n) \times \mathbb{C}^\times$$

$\| $

Z

Steinberg variety
(variety of triples)

<u>Th</u> [Lusztig]

$$H_*^{GL(n)\times\mathbb{C}^\times}(Z) \overset{\star}{\cong} H_{deg}$$ degenerate affine Hecke alg.

alg/ $H^*_{\mathbb{C}^\times}(pt) = \mathbb{C}[\hbar]$

generator $\sigma_i \; (1 \leq i \leq n-1)$
$x_i \; (1 \leq i \leq n)$

relations $\sigma_i^2 = 1$, $\sigma_i\sigma_j = \sigma_j\sigma_i$ if $|i-j|>1$
$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$
$\left(\langle \sigma_i \rangle \cong \mathbb{C}[\mathfrak{S}_n] \right.$

$x_i x_j = x_j x_i$ $\left(\langle x_i \rangle \cong \mathbb{C}[x_1 \cdots, x_n] \right.$

$$\sigma_i x_j - x_{s_i(j)}\sigma_i = \begin{cases} 0 & |i-j|>1 \\ -\hbar & j=i \\ +\hbar & j=i+1 \end{cases}$$

$\star$: $-c_1(\mathcal{S}_i/\mathcal{S}_{i-1}) \longleftarrow x_i$ as left

$\left(\begin{array}{l} H^*_{GL(n)}(pt) \longrightarrow \text{symmetric polynomial in } x_1 \cdots x_n \\ \| \\ (H^*_T(pt))^{\mathfrak{S}_n} \end{array} \right)$

$1 \leqq i \leqq n-1$

$$\mathcal{P}(i,n) = \left\{ 0 = S_0 \subset \cdots \subset S_{i-1} \begin{array}{c} \subset S_i^1 \subset \\ \subset S_i^2 \cup \end{array} S_{i+1} \subset \cdots \subset S_n = \mathbb{C}^n \right\}$$

$\cap$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\uparrow$

$\mathcal{F}(n) \times \mathcal{F}(n)$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ smooth closed submfd

$$\left(\begin{array}{l} \Delta GL(n) \curvearrowright \mathcal{F}(n) \times \mathcal{F}(n) \\ \quad \text{orbits} \longleftrightarrow \mathfrak{S}_n \\ \mathcal{P}(i,n) \setminus \Delta_{\mathcal{F}(n)} \longleftrightarrow s_i \end{array} \right.$$

$\overline{\mathcal{P}}(i,n) =$ conormal b'dle to $\mathcal{P}(i,n)$ $\subset T^*\mathcal{F}(n) \times T^*\mathcal{F}(n)$ $\quad$ (cf. $n=2$ $\quad Z = \Delta_{T^*\mathbb{P}^1} \cup \underbrace{\mathbb{P}^1 \times \mathbb{P}^1}$)

<u>Lemma</u> $\overline{\mathcal{P}}(i,n)$ is a closed submfd of $T^*\mathcal{F}(n) \times T^*\mathcal{F}(n)$

(Thm cont'd)

$$\star : -[\overline{\mathcal{P}}(i,n)] - [\Delta_{T^*\mathcal{F}(n)}] \longleftarrow \sigma_i$$

(sketch of proof) $\qquad\qquad\qquad$ $T \subset GL_n$ diagonal subgroup

$$(T^*\mathcal{F}(n))^T = \mathcal{F}(n)^T \cong \mathfrak{S}_n \ni w$$

$$\uparrow$$

$$0 \subset \underset{\langle e_{w(1)} \rangle}{\overset{\shortparallel}{S_1}} \subset \underset{\langle e_{w(1)}, e_{w(2)} \rangle}{\overset{\shortparallel}{S_2}} \subset \cdots$$

$Z^T = \mathfrak{S}_n \times \mathfrak{S}_n$

$$H^T_*(Z^T) = H^T_*(\mathfrak{S}_n \times \mathfrak{S}_n)$$
$$= H^*_T(pt) \otimes H_0(\mathfrak{S}_n \times \mathfrak{S}_n)$$
$$\circlearrowleft$$
$$H^T_*(T^*\mathcal{B}(n)^T)$$
$$\cong$$
$$H^*_T(pt) \otimes H_0(\mathfrak{S}_n)$$

described by (combinatorics) explicity.

$G = GL_n$

$$H^{G\times\mathbb{C}^*}_*(Z) \curvearrowright H^{G\times\mathbb{C}^*}_*(\mathcal{B}(n)) \overset{\text{induction}}{\cong} H^{T\times\mathbb{C}^*}_*(pt)$$

$\mathcal{B}(n) = G/B$, $B \simeq T$, $H^{T\times\mathbb{C}^*}_*(pt) = \mathbb{C}[x_j, \hbar]$, $j = 1, \cdots, n$

polynomial representation

See 2019.12.2 Note

Prop. This representation realises $\sigma_i = -[\overline{P(i,n)}] - [\Delta]$ as Demazure-Lusztig operator

$$f \in \mathbb{C}[x_j, \hbar] \qquad \sigma_i f = s_i f - \frac{\hbar}{x_i - x_{i+1}}(f - s_i f)$$

divided difference operator

Fact. $H_{deg} \curvearrowright \mathbb{C}[x_j, \hbar]_{1 \leq j \leq n}$ polynomial rep.

$H_{deg} = \langle x_j, \sigma_i \rangle$

x_j: mult. σ_i: Demazure–Lusztig operator

is a faithful representation.

Cor.

$$H_*^{G \times \mathbb{C}^\times}(Z) \xrightarrow{\star} \mathrm{End}(\mathbb{C}[x_j, \hbar])$$

$$\langle x_j, \sigma_i \rangle = H_{deg} \hookrightarrow \mathrm{End}(\mathbb{C}[x_j, \hbar])$$

image of $\star$ contains H_{deg}.

need to check: • $\star$ is injective

• $H_*^{G \times \mathbb{C}^\times}(Z)$ is generated by x_j, σ_i

$$\begin{array}{ccc} Z \subset & E_1 \times E_2 & \text{vector bundle} \curvearrowleft G \\ & \downarrow \quad \downarrow & \\ & M_1 \times M_2 & \\ \end{array}$$

M_1, M_2: compact oriented manifold

Assume $p : Z \longrightarrow E_1 \times M_2$ proper $\quad i : M_1 \times M_2 \hookrightarrow E_1 \times M_2$

Prop.

$$\begin{array}{ccccc} H_*^G(Z) & \xrightarrow{p_*} & H_*^G(E_1 \times M_2) & \xrightarrow{i^*} & H_*^G(M_1 \times M_2) \\ \text{convolution} \downarrow & & \circlearrowleft & & \downarrow \text{conv.} \\ \mathrm{Hom}(H_*^G(E_2), H_*^G(E_1)) & & \xrightarrow[\cong]{\text{Thom}} & & \mathrm{Hom}(H_*^G(M_2), H_*^G(M_1)) \end{array}$$

(proof) exercise

(cf. localization of convolution product $\frac{1}{1\otimes e(N_2)} i^*$

but we don't need to localize.

Therefore it is enough to show

$$H_*^{G\times\mathbb{C}^\times}(Z) \xrightarrow{i^* p_*} H_*^{G\times\mathbb{C}^\times}(\mathcal{F}l(n)\times\mathcal{F}l(n)) \xrightarrow{\text{conv.}} \mathrm{End}(H_*^{G\times\mathbb{C}^\times}(\mathcal{F}l(n)))$$

injective.

Künneth $\Rightarrow$ faithful repr.

$$Z = T^*\mathcal{F}l(n) \underset{\mathcal{N}}{\times} T^*\mathcal{F}l(n) \curvearrowleft GL(n)$$

$$= \{ (S^1_\bullet, S^2_\bullet, \xi) \mid \cdots \}$$

normalize to the standard flag

$$S^{std}_\bullet = (\langle e_1\rangle \subset \langle e_1, e_2\rangle \subset \cdots \ )$$

$$= GL(n) \times_B \{ (S^1_\bullet, \xi) \mid \underbrace{\xi(S^1_i) \subset S^1_{i-1},\ \xi(S^{std}_i) \subset S^{std}_{i-1}}_{=\, R \,\subset\, T^*\mathcal{F}l(n)} \}$$

$$B = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$$

R is the union of conormal bundles to B-orbits in $\mathcal{F}l(n)$

Recall B-orbits in $\mathcal{F}\ell(n)$ $= \{ X_w \mid w \in \mathfrak{S}_n \}$

Schubert cell $\cong \mathbb{C}^{\ell(w)}$

$$R = \coprod_{w \in \mathfrak{S}_n} R_w$$

conormal b'dle to X_w

$R_w \longrightarrow X_w$ vector b'dle

$$R_{\leq w} = \coprod_{y \leq w} R_y$$

closed subvariety in R

$$\begin{array}{ccc}
H_*^{G\times\mathbb{C}^\times}(Z) & \xrightarrow{i^* p_*} & H_*^{G\times\mathbb{C}^\times}(\mathcal{F}\ell(n)\times\mathcal{F}\ell(n)) \\
\| \wr \text{ induction} & \circlearrowright & \| \wr \text{ induction} \\
H_*^{T\times\mathbb{C}^\times}(R) & \xrightarrow{\bar{i}^* \bar{p}_*} & H_*^{T\times\mathbb{C}^\times}(\mathcal{F}\ell(n))
\end{array}$$

$$R \xrightarrow{\bar{p}} T^*\mathcal{F}\ell(n) \xleftarrow{\bar{i}} \mathcal{F}\ell(n)$$

$$H_*^{T\times\mathbb{C}^\times}(R_w) \cong H_{*-\mathrm{rk}R_w}^{T\times\mathbb{C}^\times}(X_w) \cong H_{*-\mathrm{rk}R_w - 2\ell(w)}^{T\times\mathbb{C}^\times}(pt)$$

$\therefore$ $H_*^{T\times\mathbb{C}^\times}(R_w)$ is a free $H^*_{T\times\mathbb{C}^\times}(pt)$-module

$\longrightarrow$ $H_*^{T\times\mathbb{C}^*}(R)$ is a free $H^*_{T\times\mathbb{C}^\times}(pt)$-module with a base $[R_{\leq w}]$.

Moreover

$$\begin{tikzcd}
H_*^{T\times\mathbb{C}^\times}(R) \arrow[r, hook, "\text{injective}"'] \arrow[d, "\bar{i}^*\bar{p}_*"] & H_*^{T\times\mathbb{C}^\times}(R) \otimes_{H^*_{T\times\mathbb{C}^\times}(pt)} \mathrm{Frac}(\ ) \cong H_*(R^{T\times\mathbb{C}^\times}) \otimes_{\mathbb{C}} \mathrm{Frac} \arrow[d, "\text{localization}\ \ \ \star"] \\
H_*^{T\times\mathbb{C}^\times}(\mathcal{F}(n)) \arrow[r, hook, "\text{inj.}"'] & H_*^{T\times\mathbb{C}^\times}(\mathcal{F}(n)) \otimes_{H^*_{T\times\mathbb{C}^\times}(pt)} \mathrm{Frac}(\ ) \cong H_*(\mathcal{F}(n)^{T\times\mathbb{C}^\times}) \otimes_{\mathbb{C}} \mathrm{Frac}
\end{tikzcd}$$

Since $R^{T\times\mathbb{C}^\times} \cong \mathcal{F}(n)^{T\times\mathbb{C}^\times}$, $\star$ is isom.

$\therefore$ $\bar{i}^*\bar{p}_*$ is injective

H_{deg} has a filtration $H_{\leq w}$ corresponding to $H_*^{T\times\mathbb{C}^\times}(R_{\leq w})$

defined algebraically.

Compatible with polynomial rep.

$$H_*^{T\times\mathbb{C}^\times}(R_{\leq w}) / H_*^{T\times\mathbb{C}^\times}(R_{<w}) \cong \underset{\uparrow \text{ free}}{H_*^{T\times\mathbb{C}^\times}(R_w)}$$

$\rightsquigarrow$ $H_*^{T\times\mathbb{C}^\times}(R)$ is generated by $\langle \sigma_i, x_j \rangle$

(explained later)

again

Springer representation

Th. $H_{[0]}(Z) \cong \mathbb{C}[\mathfrak{S}_n]$

middle degree

This holds for flag varieties of arbitrary type

W : Weyl group

type A $\quad Z = T^*\mathcal{F}(n) \times_{\mathcal{N}} T^*\mathcal{F}(n)$

$\mathcal{N} = \coprod_{\lambda\text{: partition of } n} \mathcal{N}_\lambda$ (GL(n)-orbits) (Jordan type)

$Z = \coprod_\lambda Z_\lambda$

$x \in \mathcal{N}$

$\pi : T^*\mathcal{F}(n) \longrightarrow \mathcal{N}$

$\pi^{-1}(x) \subset T^*\mathcal{F}(n)$

projective variety (singular in general)

called Springer fiber.

e.g. $x = \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{bmatrix}$

$\mathbb{C}^n \supset \operatorname{Im} x \supset \operatorname{Im} x^2 \supset \cdots$

$\pi^{-1}(x) = pt.$

Fact. (semismallness)

$\pi^{-1}(x)$ is equi-dimensional and

$x \in \mathcal{N}_\lambda$ $\quad 2\dim \pi^{-1}(x) + \dim \mathcal{N}_\lambda = \dim Z = \dim T^*\mathcal{F}(n)$

$$Z_\lambda \longrightarrow N_\lambda \qquad \text{fiber bundle}$$
$$N_\lambda = GL(n)\cdot x \qquad \text{with fiber} = \pi^{-1}(x)\times\pi^{-1}(x)$$

$$\therefore \dim Z_\lambda = \dim N_\lambda + 2\dim \pi^{-1}(x)$$
$$\underset{\text{Fact}}{=} \dim Z$$

$\therefore$ $\overline{Z_\lambda}$: union of irreducible components of Z

$$\updownarrow 1:1$$

$$(\text{irr. comp. of } \pi^{-1}(x))^2$$

$$\begin{array}{ccc} \operatorname{Irr} Z & \overset{\text{bij}}{\longleftrightarrow} & \mathfrak{S}_n \\ \text{bij} \updownarrow & & \\ \coprod_\lambda (\operatorname{Irr}.\pi^{-1}(x))^2 & & \end{array}$$

(Robinson – Schensted correspondence

2nd part starts at 11:40 〜.

<u>お知らせ</u>　今日はKavli IPMUのコロキウムに11:30まで出ています.

そのため、前半約1時間分はオンデマンド型式の講義とします.

ITC-LMSのリンクから録画されたビデオをご視聴下さい。

後半は11:40から始めます.

$x \in N_\lambda$

$\pi^{-1}(x)$

$$H_{[0]}(\pi^{-1}(x)) = H_{2\dim \pi^{-1}(x)}(\pi^{-1}(x))$$
$$\uparrow$$
module of $H_{[0]}(Z) \cong \mathbb{C}[\mathfrak{S}_n]$

<u>Th</u>. (Springer correspondence in type A)

(1) $H_{[0]}(\pi^{-1}(x))$ is an irreducible representation of $\mathfrak{S}_n$

(2) $\{\lambda : \text{partition of } n\} \longleftrightarrow \{N_\lambda : \text{nilpotent orbit}\}$ $\quad N_\lambda \ni x$

Specht module $\qquad\qquad H_{[0]}(\pi^{-1}(x))$

standard tableaux

$\{\text{irreducible repr. of } \mathfrak{S}_n \text{ (over } \mathbb{C})\}/\text{isom}.$

This is a bijection.

more general case (not type A)

- semismallness is true.
- $H_{[0]}(\pi^{-1}(x))$ is <u>not</u> necessary irreducible.

$$\bigsqcup_x H_{[0]}(\pi^{-1}(x)) \longrightarrow N_\lambda \quad \text{local system over } N_\lambda$$

is given by a monodromy representation of $C(x) = \mathrm{Stab}_G(x)/\mathrm{Stab}^0_G(x)$

finite group

e.g. $\mathfrak{sl}_2$ ($\mathfrak{gl}_2$) $\quad \mathcal{N} = \{0\} \sqcup \underline{\mathcal{N}_{reg.}}$

$$\underline{\mathcal{N}_{reg.}} \cong S^3/\pm 1$$

$$\bigcup_x H_{[0]}(\pi^{-1}(x)) \cong \bigoplus \underbrace{V_\rho}_{\text{irreducible rep. of } W = \text{Weyl group}} \otimes \rho$$

ρ : irreducible rep. of $C(x)$

$$\left\{ \begin{array}{l} \text{Irr rep.} \\ \text{of } W \end{array} \middle/ \text{isom} \right\} \underset{\text{injective}}{\hookrightarrow} \left\{ \begin{array}{l} N_x : \text{nilpotent orbit through } x \\ \rho : \text{irr. rep of } C(x) \end{array} \right\} \Big/ \text{conj.}$$

surjectivity is achived by generalised Springer correspondence.

Yangian (cf. 2019.11.18, 11.25)

$$M = \bigsqcup_{0 \le n \le \ell} T^* G(n, \ell)$$

$$\downarrow$$

$$X = \{ x \in \mathrm{End}(\mathbb{C}^\ell) \mid x^2 = 0 \}$$

$$Z = M \times_X M \qquad U(\mathfrak{sl}_2) \xrightarrow{\exists} H_{[0]}(Z)$$

alg. hom.

E, F, H

This can be also proved by localization as above.
☆

$GL_\ell \curvearrowright G(n,\ell)$

$\cup$

T

$\mathbb{C}^\times \curvearrowright T^*G(n,\ell)$

mult on fiber

$T^*G(n,\ell)^{T\times\mathbb{C}^\times} \overset{\text{bijective}}{\Longleftrightarrow} \{ I \overset{\text{subset}}{\subset} \{1,\dots,\ell\} \mid |I| = n \}$

<u>Th</u> (Ginzburg)

$$\begin{array}{ccc} Y(\mathfrak{sl}_2) & \xrightarrow{\exists} & H_*^{G\times\mathbb{C}^\times}(Z) \quad \text{algebra hom.} \\ \| & & \\ \langle E_r, F_r, H_r \mid r = 0,1,2,\dots \rangle & & \\ \uparrow r=0 & & \\ U(\mathfrak{sl}_2) & & \end{array}$$

§ quiver Hecke algebra (Khovanov-Lauda-Rouquier algebra

Varagnolo-Vasserot : canonical bases and KLR-algebras (2011)

Lusztig 1990 quiver rep.