

§ quiver Hecke algebras

References

Varagnolo - Vasserot : Canonical bases and
KLR algebras

J. reine angew Math 659 (2011), 67~100

Kashiwara

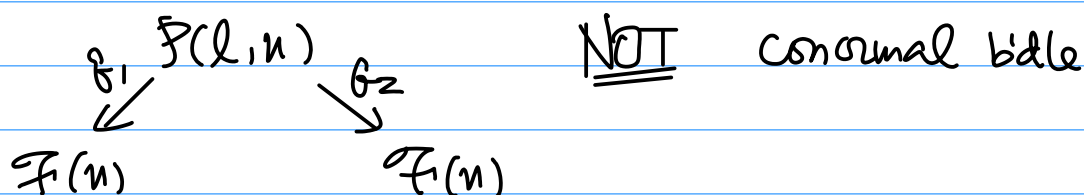
Khovanov-Lauda-Rouquier algebras
and categorification

Lecture Notes in Mathematical Sciences

The University of Tokyo, 2013

Recall nil Hecke algebras 20/9/202

$\mathcal{F}(n) = \text{flag variety } \{0 \subset S_1 \subset \dots \subset S_{n-1} \subset \mathbb{C}^n\}$



$$\beta_1 * \beta_2^* \curvearrowright H_*^{GL(n)}(\mathcal{F}(n)) \cong \mathbb{C}[x_1, \dots, x_n]$$

This is given by the divided difference operator

$$\partial_q f = \frac{f - S_q f}{x_q - x_{q+1}}$$

This was shown
via localization
thm.

(cf. degenerate affine
Hecke alg.)

$$\sigma_q = S_q - \hbar \partial_q$$

Demazure-Lusztig operator

relations $\partial_l \circ \partial_l = 0$, $\partial_l \circ \partial_{l+1} = \partial_{l+1} \circ \partial_l$ if $|l-l'| > 1$

$$\partial_l \circ \partial_{l+1} \circ \partial_l = \partial_{l+1} \circ \partial_l \circ \partial_{l+1}$$

$$x_l x_{l'} = x_{l'} x_l$$

$$\partial_l x_l - x_{s_0(l)} \partial_l = \begin{cases} 0 & |l-l'| > 1 \\ 1 & l=l \\ -1 & l=l+1 \end{cases}$$

nil Hecke alg $H_{nil} \stackrel{\text{def.}}{=} \text{generators } x_l, \partial_l$
relations as above

$$\mathbb{R} H_*^{GL(n)}(\mathbb{F}(n) \times \mathbb{F}(n)) \cong H_{nil}$$

$$[\mathbb{P}(l, n)] \mapsto \partial_l$$

$$-q(\mathbb{S}_l / \mathbb{S}_{l-1}) \mapsto x_l$$

H_{nil} will be the quiver Hecke algebra associated with the trivial graph \circ quiver type A_1

quiver $Q = (Q_0, Q_1)$
vertex oriented edge

$$d, i : Q_1 \rightarrow Q_0$$

$$\begin{array}{ccc} \circ & \xrightarrow{\quad} & \circ \\ d(l) & h & i(l) \end{array}$$

$$V = \bigoplus_{i \in Q_0} V_i$$

Q_0 -graded f.d. \mathbb{C} -v. sp.

$$\vec{V} = \dim V = \sum_{i \in Q_0} \dim V_i \cdot \alpha_i \in \mathbb{Z}_{\geq 0}^{Q_0}$$

dimension vector

$$N \equiv \text{IN}_{\mathbb{Q}}(V) := \bigoplus_{a \in Q_1} \text{Hom}(V_{o(a)}, V_{i(a)})$$

$$B = \bigoplus B_a$$

conjugation

$$V_{o(a)} \xrightarrow{B_a} V_{i(a)}$$

$$G = \text{GL}_{\mathbb{Q}}(V) := \prod_{i \in Q_0} \text{GL}(V_i)$$

B : representation of the quiver Q
($\alpha(B, V)$)

$$(B, V) \cong (B', V') \text{ isomorphic}$$

$$\Leftrightarrow \begin{array}{l} V \cong V' \text{ } Q_0\text{-graded v.sp.} \\ \text{det. } \rho \text{ intertwines } B \text{ \& } B' \end{array}$$

$\} G$ -orbits in $\text{IN}_{\mathbb{Q}}$ \Leftrightarrow isom. classes
bij. \mathcal{A} rep's of Q
with $\dim = \vec{V}$.

$$B: \text{indecomposable} \stackrel{\text{det.}}{\Leftrightarrow} B \cong B' \oplus B'' \Rightarrow B' \sim B'' = 0$$

Th [Gabriel, Bernstein-Gelfand-Ponomarev]

Q has only finitely many indecomposable rep. up to isom.

\Leftrightarrow underlying unoriented graph of Q
is a type ADE Dynkin diagram

Moreover indec. rep. up to isom. (B, V)

$\xrightarrow{1:1} \dim V$: positive roots of the Lie alg. corr. to the Dynkin diagram

Assume $X \circlearrowleft$ edge loop

$\vec{v} \in \mathbb{Z}_{\geq 0}^{Q_0}$ fix $V : Q_0$ -graded fix $V, Sp.$

$\vec{v} \in \mathbb{I} \vec{v}$ ($\mathbb{I} = Q_0$)
 $\vec{v} = (i_1, \dots, i_m) \mid \sum_l \alpha_{i_l} = \vec{v}$ } finite set

$\mathcal{F}_{\vec{v}} = \{ 0 = S^0 \subset S^1 \subset \dots \subset S^m = V \mid Q_0$ -graded subspaces

(i.e. $S^l = \bigoplus_{i \in Q_0} S_i^l$ $S_i^l \subset V_i$)

st. $\dim S^l / S^{l-1} = \alpha_{i_l}$ }

$\cong \prod_{i \in Q_0}$ flag variety for V_i

$G = GL_Q(V)$

compact, smooth projective variety

$$\widetilde{\mathcal{F}}_i = \{ (S, B) \in \mathcal{F}_i \times \mathbb{N}_Q(\vec{v}) \mid B \in (S_{0(h)}^l) \}$$

$$\star \subset S_{i(h)}^l$$

$\forall h, \forall l$

$$\begin{array}{ccc} V_{0(h)} & \xrightarrow{B} & V_{i(h)} \\ \cup & & \cup \\ S_{0(h)}^l & \longrightarrow & S_{i(h)}^l \end{array}$$

Rem
 condition \star automatically
 implies $B \in (S_{0(h)}^l) \subset S_{i(h)}^{l-1}$
 $\Rightarrow B$: nilpotent

$\widetilde{\mathcal{F}}_i \longrightarrow \mathcal{F}_i$ vector bundle
 In particular $\widetilde{\mathcal{F}}_i$ is smooth

$\pi_i \downarrow$
 $\mathbb{N}_Q(\vec{v})$
 forgetting S

proper $\odot \mathcal{F}_i$: cpt

$Q = \text{type } A_1 = 0$
 $\Rightarrow \mathbb{N}_Q(\vec{v}) = \text{pt}$
 $\Rightarrow \widetilde{\mathcal{F}}_i = \mathcal{F}_i = \text{flag variety}$

Def. $Z(\vec{v}) := \coprod_{\vec{v}, \vec{v}' } \widetilde{\mathcal{F}}_i \times_{\mathbb{N}_Q(\vec{v})} \widetilde{\mathcal{F}}_{i'}$

$Z \cong$

$$\coprod \tilde{\mathcal{F}}_{\vec{v}} = \mathcal{F}_{\vec{v}}$$

$$\mathcal{F}_{\vec{v}} \times_{\mathbb{N}_{\mathcal{Q}(\vec{v})}} \tilde{\mathcal{F}}_{\vec{v}} = \mathcal{Z}(\vec{v})$$

$$R(\vec{v}) := H_*^{GL_{\mathcal{Q}(\vec{v})}}(\mathcal{Z}(\vec{v})) \quad \text{convolution algebra}$$

↑ quiver Hecke algebra.

or Khovanov-Lauda-Rouquier algebra

defined by generators & relations
make sense for
symmetrizable Kac-Moody
Lie alg.

above construction

works only for symmetric KM Lie alg.

Cartan
matrix is symmetric

$UV_{\mathbb{Z}}$ (repr. theory of KLR alg.)
result is proved

only for symmetric
quiver Hecke alg.

grading (recall 0421)

$$\bigoplus_{\vec{i}, \vec{i}'} H_{-* + \underbrace{\dim_{\mathbb{C}} \tilde{\mathcal{F}}_{\vec{i}} + \dim_{\mathbb{C}} \tilde{\mathcal{F}}_{\vec{i}'}}_{\text{grading}}}(\tilde{\mathcal{F}}_{\vec{i}} \times_{\mathbb{N}} \tilde{\mathcal{F}}_{\vec{i}'})$$

polynomial representation

$$\begin{aligned}
 H_*^{GL_Q(\vec{v})}(\mathbb{Z}(\vec{v})) &\xrightarrow{\cong} H_*^{GL_Q(\vec{v})}(\mathbb{F}_{\vec{v}}) = \bigoplus_{\vec{i}} H_*^{GL_Q(\vec{v})}(\mathbb{F}_{\vec{i}}) \\
 \parallel & \\
 R(\vec{v}) &\cong \bigoplus_{\vec{i}} \mathbb{C}[\chi_{\vec{i}}(1), \dots, \chi_{\vec{i}}(m)]
 \end{aligned}$$

$$\chi_{\vec{i}}(l) = -c_1 \binom{Q}{l} / g^{l-1}$$

\uparrow
 concentrated at vertex i_l

$$||: 2| \sim$$

Recall 0512

$$\begin{aligned}
 \text{cf) } H_*^T(M_1 \times M_2) &\hookrightarrow H_*^T(\pi^{-1}(x)) & \pi^{-1}(x) \subset M_2 \\
 &\uparrow i_1 \times i_2 & \uparrow i_2 \\
 H_*^T(M_1^T \times_{X^T} M_2^T) &\hookrightarrow H_*^T(\pi^{-1}(x)^T)
 \end{aligned}$$

$$\text{T-fixed pt} \quad \frac{1}{e(N_2)} (i_1 \times i_2)^* \alpha \circ i_2^* \beta = i_1^* (\alpha \circ \beta)$$

$N_2 =$ normal bundle of M_2^T in M_2

In our situation

$$\mathcal{F}_{\vec{i}}^{TQ(\vec{v})} \cong \prod_{i \in Q_0} \mathbb{C}_{v_i}$$

$$TQ^{(\vec{v})} \subset GL_Q(\vec{v})$$

diagonal

★ generators of $R(\vec{v})$ as operators on

$$\bigoplus_{\vec{i}} \mathbb{C}[x_{\vec{i}}(1), \dots]$$

• $\vec{i} \in I^{\vec{v}}$ fix $1_{\vec{i}} = [\Delta_{\vec{i}}]$ $\deg = 0$

$$\widehat{\vec{i}}_{\vec{i}} \times_{\mathbb{N}} \widehat{\vec{i}}_{\vec{i}}$$

$$1_{\vec{i}} \cdot 1_{\vec{i}'} = \delta_{\vec{i}, \vec{i}'} 1_{\vec{i}}$$

$$\sum_{\vec{i}} 1_{\vec{i}} = \text{unit of } R(\vec{v})$$

• $x_{\vec{i}}(l) = -q(S^l / q^{l-1})$ as above

$$\deg = 2$$

• analog of \mathcal{D}_l and $P(l, n)$

$$\bigcap_{\vec{i}, \vec{i}'} Z_{\vec{i}, \vec{i}'}^{S^l} = \{ (0 = S^0 \subset \dots \subset S^{l-1} \subset S^l \subset S^{l+1} \subset \dots \subset S^m = V, B) \}$$

two flags of Q_0 -graded v.sp.

$$\widehat{\vec{i}}_{\vec{i}'} \times_{\mathbb{N}} \widehat{\vec{i}}_{\vec{i}} \Rightarrow \vec{i}' = S_Q(\vec{i}) \quad \vec{i} = (i_1 \dots i_l i_{l+1} \dots i_m)$$

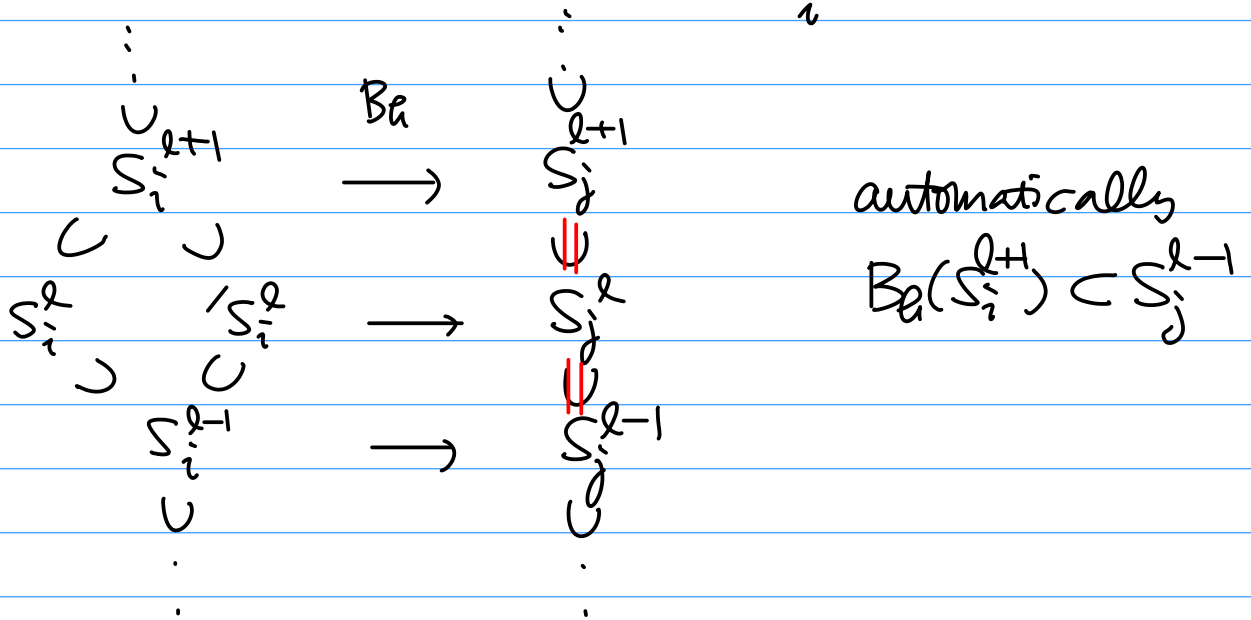
$$S_Q(\vec{i}) = (\dots i_{l+1} i_l \dots)$$

$$\sigma_{\vec{i}', \vec{i}}(l) \equiv \sigma_{S_Q(\vec{i}), \vec{i}}(l) \stackrel{\text{def.}}{=} \text{fundamental class of}$$

$$Z_{\vec{i}, \vec{i}'}^{S^l}$$

Case 1^o $i_l = i_{l+1}$ ($\Leftrightarrow S_l(\vec{i}) = \vec{i}$)

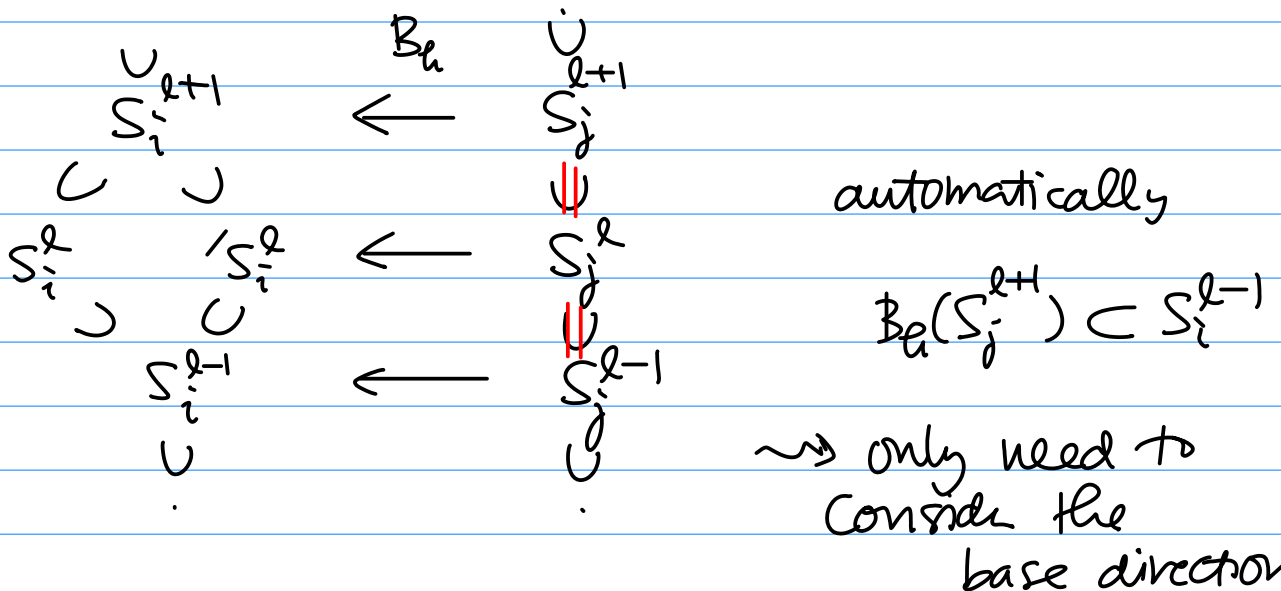
1.1^o Consider $\text{Ann } B_a$ $\alpha : i_l \rightarrow j$



$\vec{F}_{\vec{i}}$, $\vec{F}_{S_l(\vec{i})}$, $\sum_{\vec{i}, \vec{i}}$ \leftarrow B_a -components are all the same

Only base directions are different.

1.2^o $\alpha : j \rightarrow i_l$



\therefore computation is the same as
for $H_{\text{nil}} (\hat{\mathcal{F}} = \mathcal{F})$

$\therefore \sigma_{\vec{i}, \vec{i}}(l) \sim \mathbb{C}[x_{\vec{i}}(1), \dots, x_{\vec{i}}(l)]$
is given by divided diff. op.

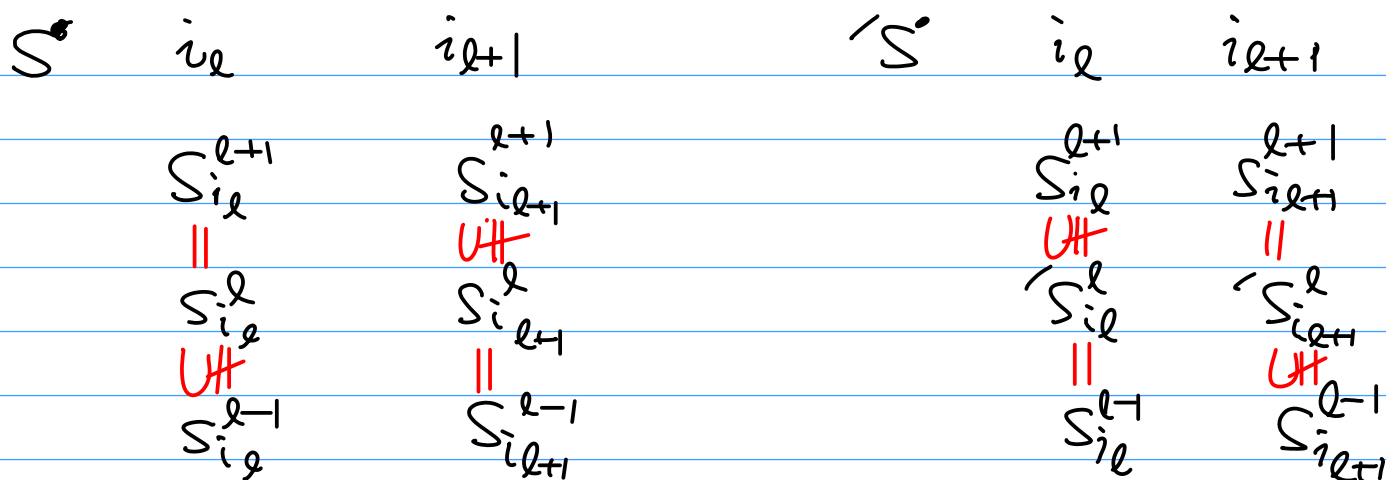
$$\frac{f - S_{\vec{i}} f}{x_{\vec{i}}(l) - x_{\vec{i}}(l+1)}$$

$$\deg = -2 \sim \dim_{\mathbb{R}} \mathbb{P}^1 \left(\begin{array}{c} S_{\vec{i}}^{l+1} \\ \downarrow \\ S_{\vec{i}}^l \\ \downarrow \\ \mathbb{C}^2 \end{array} \right)$$

Case 2° $i_l \neq i_{l+1} \iff S_{\vec{i}}(\vec{i}) \neq \vec{i}$
 \Downarrow
 \vec{i}'

Claim (base space) $\mathcal{F}_{\vec{i}} \cong \mathcal{F}_{\vec{i}'}$
 $S \longmapsto S'$

given by $S_{i_l}^l = S_{i_l}^{l-1}$
 $S_{i_{l+1}}^l = S_{i_{l+1}}^{l+1}$



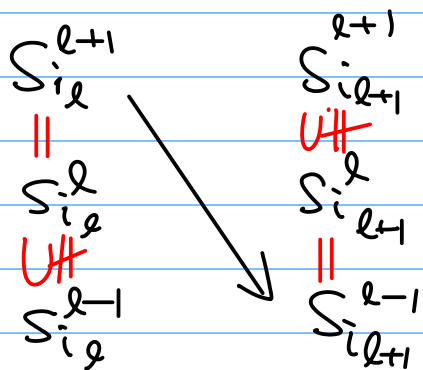
$$\sum_{S_2(\vec{v}), \vec{v}}^{S_2} = \{ (S^\circ, S^\circ, B) \mid \dots \}$$

\uparrow
 \uparrow
 graph of the above
 isom. given by Claim

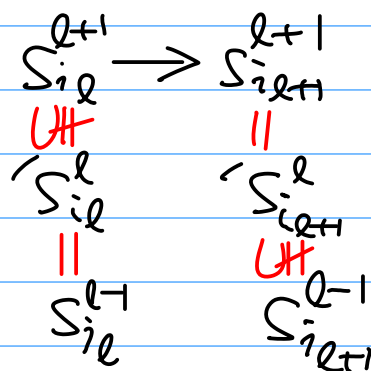
hom.

$$2.1^\circ \quad i_l \xrightarrow{h} i_{l+1}$$

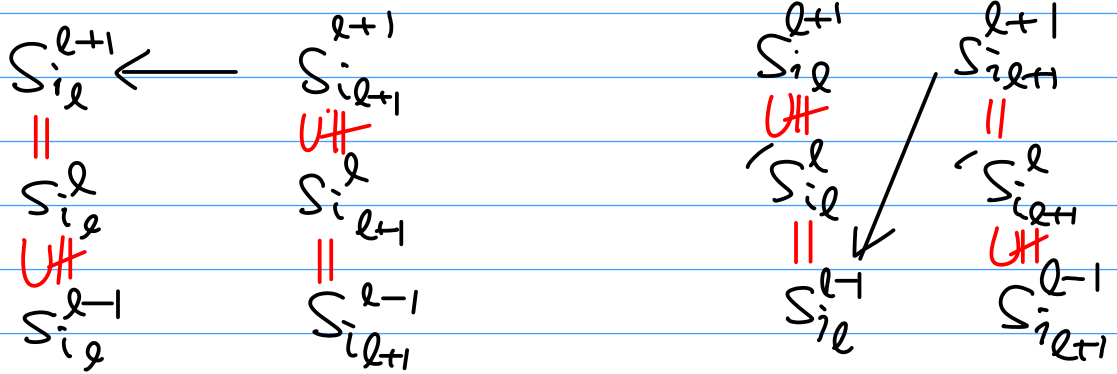
(B, S°)



(B', S°)



$$2.2^0 \quad i_l \xleftarrow{h} i_{l+1}$$



fiber directions are different
for $\vec{F}_1, \vec{F}_2, \vec{Z}_{1,2}$

facts $\text{Hom}(S_{i_{l+1}}^{l+1}/S_{i_{l+1}}^l, S_{i_l}^l/S_{i_l}^{l-1})$ appear

$$\sigma_{S_{i_l}(\vec{v}), \vec{v}}(l) \cdot f = \left(x_{S_{i_l}(\vec{v})}(l+1) - x_{S_{i_l}(\vec{v})}(l) \right)_{S_{i_l}(\vec{v})}^{i_l, i_{l+1}}$$

$$h_{i,j} = \#\{h: i \xrightarrow{h} j\}$$

$$\text{deg} = h_{i_l, i_{l+1}} + h_{i_{l+1}, i_l}$$

Th. $R(\vec{v}) \cong \langle 1_{\vec{v}}, x_{\vec{v}}(l), \sigma_{\vec{v}, \vec{v}}(l) \rangle$ Case 1⁰, 2⁰

faithful \cap
 $\text{End}(\bigoplus_{\vec{v}} \mathbb{C}[x_{\vec{v}}(1), \dots, x_{\vec{v}}(m)])$

Fact: [Kashiwara Th 2.3.1]

Choose a reduced expr. $w = s_{l_1} \cdots s_{l_m}$

for each $w \in \mathfrak{S}_m$

$R(\mathcal{V})$ is a free $\bigoplus_{\vec{i}} \mathbb{C}[\mathcal{X}_{\vec{i}}(1), \dots, \mathcal{X}_{\vec{i}}(m)]$ -module

with base $\{ \underbrace{\sigma(l_1) \cdots \sigma(l_m)}_{\sum_{\vec{i}, \vec{i}'} \sigma_{\vec{i}, \vec{i}'}(l_1)} \}_{w \in \mathfrak{S}_m}$