§ quiver Hecke algebras

References

Varagnolo - Vasserot : Canonical bases and KLR algebras
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Kashiwara
Khovanov - Lauda - Rouquier algebras and categorification
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The University of Tokyo, 2013

Recall nil Hecke algebras 20191202

$\mathcal{F}(n)$ = flag variety $\{0 \subset S_1 \subset \cdots \subset S_{n-1} \subset \mathbb{C}^n\}$

$$\begin{array}{ccc} & \mathcal{F}(l,n) & \\ {\scriptstyle f_1} \swarrow & & \searrow {\scriptstyle f_2} \\ \mathcal{F}(n) & & \mathcal{F}(n) \end{array}$$

$\underline{\underline{\text{NOT}}}$ conormal bdle

$f_{1*} f_2^* \curvearrowright H_*^{GL(n)}(\mathcal{F}(n)) \cong \mathbb{C}[x_1, \dots, x_n]$

This is given by the divided difference operator

$$\partial_l f = \frac{f - s_l f}{x_l - x_{l+1}}$$

This was shown via localization thm.

(cf. degenerate affine Hecke alg.

$\sigma_l = s_l - \hbar \partial_l$

Demazure-Lusztig operator

relations $\quad \partial_\ell \circ \partial_\ell = 0$, $\partial_\ell \circ \partial_{\ell'} = \partial_{\ell'} \circ \partial_\ell$ if $|\ell - \ell'| > 1$

$$\partial_\ell \circ \partial_{\ell+1} \circ \partial_\ell = \partial_{\ell+1} \circ \partial_\ell \circ \partial_{\ell+1}$$

$$x_k x_{k'} = x_{k'} x_k$$

$$\partial_\ell x_k - x_{s_\ell(k)} \partial_\ell = \begin{cases} 0 & |k-\ell| > 1 \\ 1 & k = \ell \\ -1 & k = \ell + 1 \end{cases}$$

nil Hecke alg $\quad H_{nil} \overset{\text{def.}}{=}$ generator x_k, ∂_ℓ
relations as above

Th $\quad H_*^{GL(n)}(\mathcal{F}(n) \times \mathcal{F}(n)) \cong H_{nil}$

$$[\mathcal{O}(\ell, n)] \mapsto \partial_\ell$$

$$-c_1(S_k / S_{k-1}) \longmapsto x_k$$

H_{nil} will be the quiver Hecke algebra associated with the trivial graph $\circ$ quiver type A_1

quiver $\quad Q = (Q_0, Q_1)$
vertex $\quad$ oriented edge

$o, i : Q_1 \longrightarrow Q_0 \qquad o(h) \xrightarrow{h} i(h)$

$$V = \bigoplus_{i \in Q_0} V_i \qquad Q_0\text{-graded f.d. } \mathbb{C}\text{-v.sp.}$$

$$\vec{V} = \dim V = \sum_{i\in Q_0} \dim V_i \cdot \alpha_i \in \mathbb{Z}_{\geq 0}^{Q_0}$$

dimension vector

$$\mathbb{N} \equiv \mathbb{N}_Q(V) := \bigoplus_{h\in Q_1} \mathrm{Hom}(V_{o(h)}, V_{i(h)})$$

$\mathbb{N} \ni B$ or $\vec{V}$

$$B = \bigoplus B_h \qquad V_{o(h)} \xrightarrow{B_h} V_{i(h)}$$

$\circlearrowleft$ conjugation

$$G = GL_Q(V) := \prod_{i\in Q_0} GL(V_i)$$

B : representation of the quiver Q
(or $(B, \vec{V})$)

$(B, V) \cong (B', V')$ isomorphic
$\underset{\text{def.}}{\Longleftrightarrow}$ $V \overset{\rho}{\cong} V'$ Q_0-graded v.sp.
ρ intertwines B & B'

$\{G\text{-orbits in } \mathbb{N}\} \underset{\text{bij.}}{\longleftrightarrow}$ isom. classes of rep.'s of Q with $\dim = \vec{V}$.

B : indecomposable $\overset{\text{def.}}{\Longleftrightarrow}$ $B \cong B' \oplus B''$
$\Rightarrow B' \text{ or } B'' = 0$

<u>Th</u> [Gabriel, Bernstein-Gelfand-Ponomarev]
Q has only finitely many indecomposable rep. up to isom.
$\Longleftrightarrow$ underlying unoriented graph of Q is a type ADE Dynkin diagram

Moreover indec. rep. (B,V) up to isom. $\overset{1:1}{\longleftrightarrow}$ $\dim V$: positive roots of the Lie alg. corr. to the Dynkin diagram

<u>Assume</u> $\times$ $\circ\circlearrowleft$ edge loop

$\vec{v} \in \mathbb{Z}_{\geq 0}^{Q_0}$ fix V : Q_0-graded v. sp. fix

$I^{\vec{v}}$ $(I = Q_0)$

$\vec{i} \in I^{\vec{v}} = \{ \vec{i} = (i_1, \dots, i_m) \mid \sum_\ell \alpha_{i_\ell} = \vec{v} \}$ finite set

$\mathcal{F}_{\vec{i}} = \{ 0 = S^0 \subset S^1 \subset \cdots \subset S^m = V \mid Q_0\text{-graded subspaces}$

$\left(\text{i.e. } S^\ell = \bigoplus_{i \in Q_0} S_i^\ell \quad S_i^\ell \subset V_i \right)$

s.t. $\dim S^\ell / S^{\ell-1} = \alpha_{i_\ell}$ $\}$

$\cong \prod_{i \in Q_0}$ flag variety for V_i

$G = GL_Q(V) \curvearrowright$

compact, smooth projective variety

$$\widetilde{\mathcal{F}}_{\vec{i}} = \{ (S^\bullet, B) \in \mathcal{F}_{\vec{i}} \times N_Q(\vec{v}) \mid B_h(S^l_{o(h)}) \subset S^l_{i(h)} \ \star \ \forall h, \forall l \}$$

$$\begin{array}{ccc} V_{o(h)} & \xrightarrow{B_h} & V_{i(h)} \\ \cup & & \cup \\ S^l_{o(h)} & \longrightarrow & S^l_{i(h)} \end{array}$$

<u>Rem</u>

condition $\star$ automatically implies $B_h(S^l_{o(h)}) \subset S^{l-1}_{i(h)}$

$\Rightarrow B$: nilpotent

$$\begin{array}{ccc} \widetilde{\mathcal{F}}_{\vec{i}} & \longrightarrow & \mathcal{F}_{\vec{i}} \\ & \searrow^{\pi_{\vec{i}}} & \\ & & N_Q(\vec{v}) \end{array}$$

vector b'dle

In particular $\widetilde{\mathcal{F}}_{\vec{i}}$ is smooth

forgetting $S^\bullet$

proper $\because \mathcal{F}_{\vec{i}}$: cpt

$Q =$ type $A_1 = \circ$

$\Rightarrow N_Q(\vec{v}) = \{0\}$

$\Rightarrow \widetilde{\mathcal{F}}_{\vec{i}} = \mathcal{F}_{\vec{i}} =$ flag variety

<u>Def.</u> $Z(\vec{v}) := \coprod_{\vec{i}, \vec{i}'} \widetilde{\mathcal{F}}_{\vec{i}} \times_{N_Q(\vec{v})} \widetilde{\mathcal{F}}_{\vec{i}'}$

$\|\!\!\|$

Z

$$\coprod \widetilde{\mathcal{F}}_{\vec{\nu}} = \mathcal{F}_{\vec{V}} \qquad \mathcal{F}_{\vec{V}} \times_{\mathbb{N}_Q(\vec{V})} \mathcal{F}_{\vec{V}} = Z(\vec{V})$$

$$R(\vec{V}) := H_*^{GL_Q(\vec{V})}(Z(\vec{V})) \quad \text{convolution algebra}$$

↑ quiver Hecke algebra.

or Khovanov - Lauda - Rouquier algebra

defined by generator & relation

make sense for symmetrizable Kac-Moody Lie alg.

above construction works only for <u>symmetric</u> KM Lie alg.

Cartan matrix is symmetric

VV's (repr. theory of KLR alg.) result is proved only for <u>symmetric</u> quiver Hecke alg.

grading (recall 0421)

$$\bigoplus_{\vec{\imath}, \vec{\imath}'} H^{GL_Q(\vec{V})}_{\underbrace{-* + \dim_{\mathbb{C}} \widetilde{\mathcal{F}}_{\vec{\imath}} + \dim_{\mathbb{C}} \widetilde{\mathcal{F}}_{\vec{\imath}'}}} (\widetilde{\mathcal{F}}_{\vec{\imath}} \times_{\mathbb{N}} \widetilde{\mathcal{F}}_{\vec{\imath}'})$$

polynomial representation

$$H_*^{GL_Q(\vec{v})}(Z(\vec{v})) \curvearrowright H_*^{GL_Q(\vec{v})}(\mathcal{F}_{\vec{v}}) = \bigoplus_{\vec{i}} H_*^{GL_Q(\vec{v})}(\mathcal{F}_{\vec{i}})$$

$$\| \qquad \cong \bigoplus_{\vec{i}} \mathbb{C}[x_{\vec{i}}(1), \cdots, x_{\vec{i}}(m)]$$

$$R(\vec{v})$$

$$x_{\vec{i}}(l) = -c_1(\mathcal{V}^l / \mathcal{V}^{l-1})$$

concentrated at vertex i_l

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~~Recall~~ 0512

cf)
$$H_*^T(M_1 \times M_2) \curvearrowright H_*^T(\pi^{-1}(x)) \qquad \pi^{-1}(x) \subset M_2$$

$$X$$

$$\uparrow i_1 \times i_2 \qquad \uparrow i_2$$

$$H_*^T(M_1^T \times_{X^T} M_2^T) \curvearrowright H_*^T(\pi^{-1}(x)^T)$$

T-fixed pt

$$\frac{1}{e(N_2)} (i_1 \times i_2)^* \alpha \circ i_2^* \beta = i_1^*(\alpha \circ \beta)$$

N_2 = normal bdle of M_2^T in M_2

In our situation

$$\mathcal{F}_{\vec{i}}^{T_Q(\vec{v})} \cong \prod_{i \in Q_0} \mathfrak{S}_{v_i}$$

$T_Q(\vec{v}) \subset GL_Q(\vec{v})$

diagonal

✫ generators of $R(\vec{v})$ as operators on $\bigoplus_{\vec{i}} \mathbb{C}[x_{\vec{i}}(1), \dots]$

- $\vec{i} \in I^{\vec{v}}$ fix $\qquad 1_{\vec{i}} = [\Delta_{\widetilde{\mathcal{F}}_{\vec{i}}}]$ $\qquad \deg = 0$

$$\widetilde{\mathcal{F}}_{\vec{i}} \times_{\mathcal{N}} \widetilde{\mathcal{F}}_{\vec{i}}$$

$$1_{\vec{i}} \cdot 1_{\vec{i}'} = \delta_{\vec{i},\vec{i}'} 1_{\vec{i}}$$

$$\sum_{\vec{i}} 1_{\vec{i}} = \text{unit of } R(\vec{v})$$

- $x_{\vec{i}}(\ell) = -c_1(\mathcal{S}^\ell / \mathcal{S}^{\ell-1})$ as above

$$\deg = 2$$

- analog of ∂_ℓ and $P(\ell, n)$

$$Z^{s_\ell}_{\vec{i}',\vec{i}} = \left\{ \left(0 = S^0 \subset \cdots \subset S^{\ell-1} \begin{smallmatrix} \subset S^\ell \subset \\ \subset {}'S^\ell \subset \end{smallmatrix} S^{\ell+1} \subset \cdots \subset S^m = V, B\right) \right\}$$

two flags of Q_0-graded v.sp.

$$Z_{\vec{i}',\vec{i}} = \widetilde{\mathcal{F}}_{\vec{i}'} \times_{\mathcal{N}} \widetilde{\mathcal{F}}_{\vec{i}}$$

$\Rightarrow \vec{i}' = s_\ell(\vec{i})$ $\qquad \vec{i} = (i_1 \cdots i_\ell \, i_{\ell+1} \cdots i_m)$

$\qquad s_\ell(\vec{i}) = (\;\; " \;\; i_{\ell+1} \, i_\ell \;\; " \;\;)$

$$\sigma_{\vec{i}',\vec{i}}(\ell) = \sigma_{s_\ell(\vec{i}),\vec{i}}(\ell) \overset{\text{def.}}{=} \text{fundamental class of } Z^{s_\ell}_{\vec{i}',\vec{i}}$$

<u>Case 1°</u> $i_\ell = i_{\ell+1}$ $\quad (\Leftrightarrow \; s_\ell(\vec{i}) = \vec{i} \,)$

1.1° Consider form B_ℓ $\quad f_\ell : i_\ell \longrightarrow j$, $\quad i_\ell = i_{\ell+1} = i$

$$
\begin{tikzcd}[column sep=small]
 & \vdots & & & \vdots \\
 & S_i^{\ell+1} \arrow[rrr, "B_\ell"] & & & S_j^{\ell+1} \arrow[d, phantom, "\Downarrow"] \\
S_i^{\ell} \arrow[ur, phantom, "\subset"] & & {}'S_i^{\ell} \arrow[ul, phantom, "\supset"] \arrow[rr] & & S_j^{\ell} \arrow[d, phantom, "\Downarrow"] \\
 & S_i^{\ell-1} \arrow[ul, phantom, "\supset"] \arrow[ur, phantom, "\subset"] \arrow[rrr] & & & S_j^{\ell-1} \\
 & \vdots & & & \vdots
\end{tikzcd}
$$

automatically $B_\ell(S_i^{\ell+1}) \subset S_j^{\ell-1}$

$\tilde{\mathcal{F}}_{\vec{i}}$, $\tilde{\mathcal{F}}_{s_\ell(\vec{i})}$, $Z_{\vec{i},\vec{i}}$ $\longleftarrow$ B_ℓ-components are <u>all the same</u>
$\qquad\quad s_\ell(\vec{i}) = \vec{i}$

Only <u>base directions</u> are different.

1.2° $\quad f_\ell : j \longrightarrow i_\ell$

$$
\begin{tikzcd}[column sep=small]
 & S_i^{\ell+1} & & & S_j^{\ell+1} \arrow[lll, "B_\ell"'] \arrow[d, phantom, "\Downarrow"] \\
S_i^{\ell} \arrow[ur, phantom, "\subset"] & & {}'S_i^{\ell} \arrow[ul, phantom, "\supset"] & & S_j^{\ell} \arrow[ll] \arrow[d, phantom, "\Downarrow"] \\
 & S_i^{\ell-1} \arrow[ul, phantom, "\supset"] \arrow[ur, phantom, "\subset"] & & & S_j^{\ell-1} \arrow[lll] \\
 & \vdots & & & \vdots
\end{tikzcd}
$$

automatically

$B_\ell(S_j^{\ell+1}) \subset S_i^{\ell-1}$

$\rightsquigarrow$ only need to consider the base direction

$\therefore$ computation is the same as for H_{nil} $(\widetilde{\mathcal{F}} = \mathcal{F})$

$\therefore$ $\sigma_{\vec{i},\vec{i}}(l) \curvearrowright \mathbb{C}[x_{\vec{i}}(1), \cdots, x_{\vec{i}}(m)]$ is given by divided diff. op.

$$\frac{f - s_l f}{x_{\vec{i}}(l) - x_{\vec{i}}(l+1)}$$

$$\deg = \underbrace{-2}_{\dim_{\mathbb{R}} \mathbb{P}^1(\underbrace{S_{\vec{i}}^{l+1}/S_{\vec{i}}^{l-1}}_{\cong\, \mathbb{C}^2})}$$

<u>Case 2°</u> $i_l \neq i_{l+1} \iff \underbrace{s_l(\vec{i})}_{\vec{i}'} \neq \vec{i}$

<u>Claim</u>. (base space) $\mathcal{F}_{\vec{i}} \cong \mathcal{F}_{\vec{i}'}$

$S^\bullet \longmapsto {}'S^\bullet$

given by ${}'S_{i_l}^{l} = S_{i_l}^{l-1}$

${}'S_{i_{l+1}}^{l} = S_{i_{l+1}}^{l+1}$

$$\begin{array}{ccc} S^\bullet & i_\ell & i_{\ell+1} \\ & S^{\ell+1}_{i_\ell} & S^{\ell+1}_{i_{\ell+1}} \\ & \| & \uplus \\ & S^{\ell}_{i_\ell} & S^{\ell}_{i_{\ell+1}} \\ & \uplus & \| \\ & S^{\ell-1}_{i_\ell} & S^{\ell-1}_{i_{\ell+1}} \end{array} \qquad \begin{array}{ccc} {}'S^\bullet & i_\ell & i_{\ell+1} \\ & S^{\ell+1}_{i_\ell} & S^{\ell+1}_{i_{\ell+1}} \\ & \uplus & \| \\ & {}'S^{\ell}_{i_\ell} & {}'S^{\ell}_{i_{\ell+1}} \\ & \| & \uplus \\ & S^{\ell-1}_{i_\ell} & S^{\ell-1}_{i_{\ell+1}} \end{array}$$

$$Z^{s_\ell}_{s_\ell(\vec{\tau}),\,\vec{\tau}} = \{ ({}'S^\bullet, S^\bullet, B) \mid \cdots \}$$

graph of the above isom. given by Claim

<u>Lem.</u>

2.1° $i_\ell \xrightarrow{h} i_{\ell+1}$

$$(B, S^\bullet) \quad \begin{tikzcd}[row sep=small] S^{\ell+1}_{i_\ell} \arrow[dddr] & S^{\ell+1}_{i_{\ell+1}} \\ \| & \uplus \\ S^{\ell}_{i_\ell} & S^{\ell}_{i_{\ell+1}} \\ \uplus & \| \\ S^{\ell-1}_{i_\ell} & S^{\ell-1}_{i_{\ell+1}} \end{tikzcd} \qquad (B, {}'S^\bullet) \quad \begin{tikzcd}[row sep=small] S^{\ell+1}_{i_\ell} \arrow[r] & S^{\ell+1}_{i_{\ell+1}} \\ \uplus & \| \\ {}'S^{\ell}_{i_\ell} & {}'S^{\ell}_{i_{\ell+1}} \\ \| & \uplus \\ S^{\ell-1}_{i_\ell} & S^{\ell-1}_{i_{\ell+1}} \end{tikzcd}$$

$$\Uparrow$$

$$Z$$

2.2° $\quad i_\ell \xleftarrow{h} i_{\ell+1}$

$$\begin{array}{ccc} S^{\ell+1}_{i_\ell} & \longleftarrow & S^{\ell+1}_{i_{\ell+1}} \\ \| & & \cup\!\!\!\!\!\!\;\# \\ S^{\ell}_{i_\ell} & & S^{\ell}_{i_{\ell+1}} \\ \cup\!\!\!\!\!\!\;\# & & \| \\ S^{\ell-1}_{i_\ell} & & S^{\ell-1}_{i_{\ell+1}} \end{array} \qquad\qquad \begin{array}{ccc} S^{\ell+1}_{i_\ell} & & S^{\ell+1}_{i_{\ell+1}} \\ \cup\!\!\!\!\!\!\;\# & & \| \\ {}'S^{\ell}_{i_\ell} & \swarrow & {}'S^{\ell}_{i_{\ell+1}} \\ \| & & \cup\!\!\!\!\!\!\;\# \\ S^{\ell-1}_{i_\ell} & & S^{\ell-1}_{i_{\ell+1}} \end{array}$$

fiber directions are different
for $\widetilde{\mathcal{F}}_{\vec{i}}$, $\widetilde{\mathcal{F}}_{\vec{i}'}$, $Z^{s}_{\vec{i}',\vec{i}}$

factors $\mathrm{Hom}\left(S^{\ell+1}_{i_{\ell+1}}/S^{\ell}_{i_{\ell+1}},\ S^{\ell}_{i_\ell}/S^{\ell-1}_{i_\ell}\right)$ appear

$$\sigma_{s_\ell(\vec{i}),\vec{i}}(\ell)\cdot f = \left(x_{s_\ell(\vec{i})}(\ell+1) - x_{s_\ell(\vec{i})}(\ell)\right)^{h_{i_\ell,i_{\ell+1}}} s_\ell(f)$$

$$h_{i,j} = \#\{h : i \xrightarrow{h} j\}$$

$$\deg = h_{i_\ell,i_{\ell+1}} + h_{i_{\ell+1},i_\ell}$$

Case 1°, 2°

Th. $\quad R(\vec{\nu}) \cong \left\langle 1_{\vec{i}},\ x_{\vec{i}}(\ell),\ \sigma_{\vec{i}',\vec{i}}(\ell) \right\rangle$

faithful $\quad \cap$

$$\mathrm{End}\Big(\bigoplus_{\vec{i}} \mathbb{C}[x_{\vec{i}}(1),\dots,x_{\vec{i}}(m)]\Big)$$

Fact: [Kashiwara Th 2.3.1]

Choose a reduced expr. $w = s_{\ell_1} \cdots s_{\ell_m}$

for each $w \in \mathfrak{S}_m$

$R(\nu)$ is a free $\bigoplus_{\vec{i}} \mathbb{C}[x_{\vec{i}}(1), \ldots, x_{\vec{i}}(m)]$-module

with base $\{ \underbrace{\sigma(\ell_1)}_{= \sum_{\vec{i}', \vec{i}} \sigma_{\vec{i}', \vec{i}}(\ell_1)} \cdots \sigma(\ell_m) \}_{w \in \mathfrak{S}_m}$