quiver Hecke alg

$Q = (Q_0, Q_1)$: quiver

$\vec{v} \in \mathbb{Z}_{\geq 0}^{Q_0}$

$\mathbb{N}_Q(\vec{v}) \curvearrowleft GL_Q(\vec{v})$

$\vec{i}$: sequence $\quad \widetilde{\mathcal{F}}_{\vec{i}}$

$$Z(\vec{v}) := \coprod_{\vec{i}, \vec{i}'} \widetilde{\mathcal{F}}_{\vec{i}} \times_{\mathbb{N}_Q(\vec{v})} \widetilde{\mathcal{F}}_{\vec{i}'}$$

$R(\vec{v}) := H_*^{GL_Q(\vec{v})}(Z(\vec{v}))$ quiver Hecke algebra

☆ induction

$\vec{v} = \vec{v}_1 + \vec{v}_2$

$\vec{i}_1 \in I^{\vec{v}_1}$, $\vec{i}_2 \in I^{\vec{v}_2}$ $\rightsquigarrow (\vec{i}_1, \vec{i}_2) \in I^{\vec{v}}$

$\vec{i}_1 = (i_1, \cdots, i_{m_1})$, $\vec{i}_2 = (j_1, \cdots, j_{m_2})$, $(\vec{i}_1, \vec{i}_2) = (i_1, \cdots, i_{m_1}, j_1, \cdots, j_{m_2})$

Define an alg. hom. $R(\vec{v}_1) \otimes R(\vec{v}_2) \longrightarrow R(\vec{v})$

$$1_{\vec{i}_1} \otimes 1_{\vec{i}_2} \longmapsto 1_{(\vec{i}_1, \vec{i}_2)}$$

$$x(\ell) \otimes 1 \longmapsto x(\ell)$$

$$1 \otimes x(\ell') \longmapsto x(m_1 + \ell')$$

$$\sigma(\ell) \otimes 1 \longmapsto \sigma(\ell)$$

$$1 \otimes \sigma(\ell') \longmapsto \sigma(m_1 + \ell')$$

grading is preserved

Induced module

M_1, M_2 : $R(\vec{\nu}_1), R(\vec{\nu}_2)$-modules graded

$$M_1 \circ M_2 = \mathrm{Ind}_{R(\vec{\nu}_1)\otimes R(\vec{\nu}_2)}^{R(\vec{\nu})} (M_1 \otimes M_2) \leftarrow \text{graded}$$
$$= R(\vec{\nu}) \otimes_{R(\vec{\nu}_1)\otimes R(\vec{\nu}_2)} (M_1 \otimes M_2)$$

<u>Prop</u>. $R(\vec{\nu})$ is a free $R(\vec{\nu}_1)\otimes R(\vec{\nu}_2)$-module of finite rank

(base $\{\sigma_w\}$ $w \in \mathfrak{S}_m / \mathfrak{S}_{m_1} \times \mathfrak{S}_{m_2}$)

<u>Cor</u>. M_1, M_2 : finite dim (resp. projective)
$\Longrightarrow M_1 \circ M_2$: finite dim. (resp. proj.)

$R(\vec{\nu})$-gproj = additive category of finitely generated projective graded $R(\vec{\nu})$-module

$\langle \cdot \rangle$: grading shift functor

$K(R(\vec{\nu})\text{-gproj.})$: $\mathbb{Z}[q, q^{-1}]$-module
generator : $[P]$
relation : $P \cong P_1 \oplus P_2$
$\Longrightarrow [P] = [P_1] + [P_2]$

$q \longleftrightarrow \langle 1 \rangle$

free $\mathbb{Z}[q, q^{-1}]$-module with base
$\{ [P] \mid P$: indecomposable projective (grading is chosen) $\}$

induction $\rightsquigarrow \bigoplus_{\vec{v}} K(R(\vec{v})\text{-gproj.})$

algebra

Th [相原 8.1.3, 8.5.1, 8.5.5]

$$\bigoplus_{\vec{v}} K(R(\vec{v})\text{-gproj.}) \cong {}_{\mathcal{A}}U_q^-$$

integral form of the upper triangular part of the quantized enveloping algebra

s.t. (polynomial rep. of nil-Hecke algebra (at vertex i)) $\leftrightarrow F_i^{(n)}$ " $\frac{F_i^n}{[n]!}$ "

$Q = (Q_0, Q_1) \xrightarrow[\text{forget orientation}]{}$ (symmetric) Dynkin diagram $\rightsquigarrow \mathfrak{g} = \mathfrak{g}_Q$ Kac-Moody

"duality" $\longleftrightarrow$ bar involution

Th. [Varagnolo-Vasserot]

indecomposable projective graded module s.t. self-dual (grading is preserved by the duality) $\longleftrightarrow$ Lusztig's canonical base elements

under the above isom.

Rem. This is not true for non-symmetric Kac-Moody

§ derived categories of constructible sheaves

Hotta - Tanisaki - Takeuchi : D-modules, Perverse sheaves and Representation Theory

堀田 D加群

Kashiwara - Schapira Sheaves on Manifolds

Gelfand - Manin Methods of homological algebras

Beilinson - Bernstein - Deligne - Gabber Faisceaux Pervers

○ X : algebraic variety

- $\mathbb{C}_X$ = constant sheaf
- $\mathrm{Mod}(\mathbb{C}_X)$ = abelian category of sheaves of (left) $\mathbb{C}_X$-modules on X
- $D^b(\mathbb{C}_X) = D^b(\mathrm{Mod}(\mathbb{C}_X))$: bound derived category of $\mathrm{Mod}(\mathbb{C}_X)$
- $[\ \ ]$: shift functor
- global section functor

$$R\Gamma(X;\ -) : D^b(\mathbb{C}_X) \to D^b(\mathbb{C}\text{-Vect})$$

$$H^*(X) = H^*(R\Gamma(X, \mathbb{C}_X))$$

- $R\mathrm{Hom}_{\mathbb{C}_X}(-,-) : D^b(\mathbb{C}_X)^{op} \times D^b(\mathbb{C}_X) \longrightarrow D^b(\mathbb{C}\text{-Vect})$

$$H^0 R\mathrm{Hom}_{\mathbb{C}_X}(F,G) = \mathrm{Hom}_{D^b(\mathbb{C}_X)}(F,G)$$

- sheaf hom $R\mathcal{H}om_{\mathbb{C}_X}(-,-) : D^b(\mathbb{C}_X)^{op} \times D^b(\mathbb{C}_X) \to D^b(\mathbb{C}_X)$

$$R\Gamma(X, R\mathcal{H}om_{\mathbb{C}_X}(-,-)) = R\mathrm{Hom}_{\mathbb{C}_X}(-,-)$$

- $\otimes : D^b(\mathbb{C}_X) \times D^b(\mathbb{C}_X) \longrightarrow D^b(\mathbb{C}_X)$

$\otimes = \overset{L}{\otimes}$

$$F \otimes \mathbb{C}_X = F$$

○ $f: X \longrightarrow Y$ morphism

- direct image $\quad f_* : D^b(\mathbb{C}_X) \to D^b(\mathbb{C}_Y)$
- proper direct image $\quad f_! : D^b(\mathbb{C}_X) \to D^b(\mathbb{C}_Y)$

derived functors of left exact functors

$f_*, f_! : Mod(\mathbb{C}_X) \longrightarrow Mod(\mathbb{C}_Y)$

- inverse image $\quad f^* : D^b(\mathbb{C}_Y) \longrightarrow D^b(\mathbb{C}_X)$

($f^* : Mod(\mathbb{C}_Y) \to Mod(\mathbb{C}_X)$ exact)

- adjointness

$$RHom_{\mathbb{C}_Y}(F, f_* G) \cong RHom_{\mathbb{C}_X}(f^* F, G)$$

- base change

$$\begin{array}{ccc} Y \times_X X' & \xrightarrow{g'} & Y \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow[g]{} & X \end{array} \quad \Rightarrow g^* \circ f_! \simeq f'_! \circ g'^*$$

<u>Def</u> (1) $F \in Mod(\mathbb{C}_X)$ is constructible if

$X = \exists \coprod_{\alpha \in A} X_\alpha$ stratification (X_α : smooth, $\overline{X_\alpha} = \coprod_{\beta \in B} X_\beta$)

s.t. $i_\alpha^* F$ is a local system on X_α

($i_\alpha : X_\alpha \hookrightarrow X$)

(2) $D^b_c(\mathbb{C}_X) \subset D^b(\mathbb{C}_X)$ full subcategory

cohomology groups are constructible

○ right adjoint to $f_!$ $\qquad f : X \to Y$

- $f^! : D^b(\mathbb{C}_Y) \longrightarrow D^b(\mathbb{C}_X)$
  (Assume finite cohomological dimension)

$$R\mathrm{Hom}_{\mathbb{C}_X}(f_! F, G) = R\mathrm{Hom}_{\mathbb{C}_Y}(F, f^! G)$$

sheaf version $\quad R\mathcal{H}om_{\mathbb{C}_X}(f_! F, G) = f_* R\mathcal{H}om_{\mathbb{C}_Y}(F, f^! G)$

- dualizing complex $\quad \omega_X = a^! \mathbb{C}_{pt} \qquad a : X \to pt$

- Verdier duality $\quad \mathbb{D}_X : D^b(\mathbb{C}_X)^{op} \longrightarrow D^b(\mathbb{C}_X)$

$$\mathbb{D}_X = R\mathcal{H}om_{\mathbb{C}_X}(-, \omega_X)$$

$$f : X \to Y \qquad f_* \mathbb{D}_X = \mathbb{D}_Y f_!$$

On $D^b_c(X)$, we have

$$\mathbb{D}_X^2 = id, \quad f^! = \mathbb{D}_X \circ f^* \circ \mathbb{D}_Y, \quad f_! = \mathbb{D}_Y \circ f_* \circ \mathbb{D}_X$$

$$R\mathcal{H}om_{\mathbb{C}_X}(F, G) \cong \mathbb{D}_X(\mathbb{D}_X G \otimes F)$$

$$F \overset{!}{\otimes} G := \mathbb{D}_X(\mathbb{D}_X F \otimes \mathbb{D}_X G)$$

$$\therefore R\mathcal{H}om_{\mathbb{C}_X}(F, G) = G \overset{!}{\otimes} \mathbb{D}_X F$$

$$F \overset{!}{\otimes} \omega_X = F \qquad \text{from} \qquad F \otimes \mathbb{C}_X = F$$

$\Delta : X \to X \times X$ diagonal emb.

$F \otimes G = \Delta^*(F \boxtimes G)$,

$F \overset{!}{\otimes} G = \Delta^!(F \boxtimes G)$

Ex. X : complex algebraic manifold of $\dim_{\mathbb{C}} = d_X$
Poincaré duality $\quad H_c^*(X) \cong H^{2d_X - *}(X)^*$

$$\rightsquigarrow \quad \omega_X = \mathbb{C}_X[2d_X]$$

Ex. $j : U \underset{\text{open}}{\hookrightarrow} X \qquad j_!F$: extension by 0

$$\Rightarrow \quad j^!G = j^*G$$

• $Z \overset{i}{\underset{\text{closed}}{\hookrightarrow}} X \overset{j}{\underset{\text{open}}{\hookleftarrow}} U = X \setminus Z$ $\qquad$ e.g. $\mathrm{Hom}(F, i_*i^*F) \cong \mathrm{Hom}(i^*F, i^*F) \ni \mathrm{id}$ (adjunction)

$\Rightarrow$ distinguished triangle

$$\begin{tikzcd} j_!j^!F \arrow[r] & F \arrow[r] & i_*i^*F \arrow[ll, bend left, "+1"] \end{tikzcd}$$

for $F \in D^b(\mathbb{C}_X)$

$\Rightarrow$ dual :

$$\begin{tikzcd} i_*i^!F \arrow[r] & F \arrow[r] & j_*j^*F \arrow[ll, bend left] \end{tikzcd}$$

as $i_! = i_*$

• Further suppose X : complex manifold
and $F = \omega_X = \mathbb{C}_X[2d_X]$

$$\begin{tikzcd} H^*(X, U) \arrow[r] & H^*(X) \arrow[r] & H^*(U) \arrow[ll, bend left, "+1"] \end{tikzcd}$$

$$\begin{tikzcd} Z \arrow[r, "a_Z"] \arrow[dr, "i"'] & pt \\ & X \arrow[u, "a_X"'] \end{tikzcd}$$

$\therefore \; i^!\omega_X = \omega_Z$ gives the relative cohomology

$\therefore \quad$ Borel-Moore homology $\quad H_*(Z) \cong H^{-*}(\omega_Z)$

Now we reformulate functors for BM homology by using sheaves

- proper pushforward

$f: X \to Y$ proper $\Rightarrow f_* = f_!$

$$H^*(X, \omega_X) = H^*(Y, f_*\omega_X) = H^*(Y, \underbrace{f_* f^! \omega_Y}_{f_! f^! \omega_Y})$$

$$\xrightarrow[\text{(adjunction)}]{} H^*(Y, \omega_Y)$$

- pull-back with support for BM homology

$$\begin{tikzcd} X \supset f^{-1}(Y) \arrow[r, hook, "i_X"] \arrow[d, "\tilde{f}"] & M \arrow[d, "f"] \\ Y \arrow[r, hook, "i_Y"] & N \end{tikzcd}$$

$f^{-1}(Y) \subset X$ closed

$\dim M = d_M$
$\dim N = d_N$

may assume $f^{-1}(Y) = X$

$$H^{-*}(Y, \omega_Y) = H^{-*}(Y, i_Y^! \mathbb{C}_N[2d_N])$$

$$\xrightarrow[\text{(adjunction)}]{} H^{-*}(Y, \underbrace{i_Y^! f_*}_{\tilde{f}_* i_X^!} f^* \mathbb{C}_N[2d_N])$$

base change

$$= H^{-*}(X, i_X^! \underbrace{f^* \mathbb{C}_N[2d_N]}_{\mathbb{C}_M[2d_M][2d_N - 2d_M]})$$

$$= H^{-*}(X, \omega_X[2d_N - 2d_M])$$

$$= H^{-(* + 2d_M - 2d_N)}(X, \omega_X)$$

- cap product

$X, Y \subset M$ closed $\uparrow$ cpx mfd

$$\begin{tikzcd} X\cap Y \arrow[r] \arrow[d] & M \arrow[d, "\delta_X"] \\ X\times Y \arrow[r, "i_X\times i_Y"'] & M\times M \end{tikzcd}$$

pull-back with support

$$H^{-k}(X\times Y, \omega_{X\times Y}) \longrightarrow H^{-(k-2d_M)}(X\cap Y, \omega_{X\cap Y})$$

$$\parallel$$

$$\bigoplus_{k_1+k_2=k} H^{-k_1}(X, \omega_X)\otimes H^{-k_2}(Y, \omega_Y)$$

| 11:30 ~

- Yoneda product vs convolution algebras (Chriss-Ginzburg Chapter 8)

$$\mathrm{Ext}^k(F,G) \overset{\text{def.}}{=} \mathrm{Hom}_{D^b(\mathbb{C}_X)}(F, G[k])$$

$$\mathrm{Ext}^k(F,G)\otimes \mathrm{Ext}^l(G,H) \longrightarrow \mathrm{Ext}^{k+l}(F,H)$$

$$\mathrm{Ext}^*(F,G) = H^*(X, R\mathcal{H}om_{\mathbb{C}_X}(F,G))$$

$$= H^*(X, G\overset{!}{\otimes} \mathbb{D}_X F)$$

Take $G=F$. $\mathrm{Ext}^*(F,F)$ is a graded algebra (ext algebra)

M_1, M_2 : mfds

$\pi_\alpha : M_\alpha \xrightarrow[\text{proper}]{} X$ $\qquad Z_{12} = M_1\times_X M_2 \underset{i}{\subset} M_1\times M_2$

$$\pi_{\alpha *}(\mathbb{C}_{M_\alpha}[\dim_{\mathbb{C}} M_\alpha]) = F_\alpha \in D^b_c(\mathbb{C}_X)$$

<u>Prop</u> $\exists$ a natural graded vector space isomorphism

$$H_{-*+\dim_{\mathbb{C}}M_1+\dim_{\mathbb{C}}M_2}(Z_{12}) \cong \mathrm{Ext}^*(F_1, F_2)$$

(proof) $\omega_{Z_{12}} = i^! \mathbb{C}_{M_1\times M_2}[2d_1+2d_2]$

$$\therefore\ H_{-k+d_1+d_2}(Z_{12}) = H^{k-d_1-d_2}(Z_{12}, i^!\mathbb{C}_{M_1\times M_2}[2d_1+2d_2])$$

$$= H^k(Z_{12}, i^!\mathbb{C}_{M_1\times M_2}[d_1+d_2])$$

$$\begin{tikzcd} Z_{12} \arrow[r, "i"] \arrow[d, "p"'] & M_1\times M_2 \arrow[d, "\pi_1\times\pi_2"] \\ X \arrow[r, "\delta_X"'] & X\times X \end{tikzcd}$$

$$= H^k(X, p_* i^! \mathbb{C}_{M_1\times M_2}[d_1+d_2])$$

$$\underset{\text{base change}}{\cong} H^k(X, \delta_X^! \underbrace{(\pi_1\times\pi_2)_* \mathbb{C}_{M_1\times M_2}[d_1+d_2]}_{\shortparallel\ F_1\boxtimes F_2})$$

$$\begin{pmatrix} \delta_X^*(F_1\boxtimes F_2) = F_1\otimes F_2 \\ \delta_X^!(F_1\boxtimes F_2) = F_1\overset{!}{\otimes} F_2 \end{pmatrix} \cong H^k(X, F_1\overset{!}{\otimes}F_2)$$

Now $F_1 = \pi_{1*}(\mathbb{C}_{M_1}[d_1])$

$$\mathbb{D}_X F_1 = \underbrace{\pi_{1!}}_{\shortparallel\ \pi_{1*}\text{ as proper}}(\underbrace{\mathbb{D}_{M_1}\mathbb{C}_{M_1}[d_1]}_{\shortparallel\ \mathbb{C}_{M_1}[d_1]\text{ as } M_1:\text{ mfd}}) = F_1$$

$$\therefore\ H^k(X, F_1\overset{!}{\otimes}F_2) = H^k(X, R\mathcal{H}om(F_1, F_2))$$

$$= \mathrm{Ext}^k(F_1, F_2) \quad //$$

Exercise Check the following is commutative.

$$\begin{tikzcd}
H_{-*+d_1+d_2}(\mathcal{Z}_{12}) \otimes H_{-*+d_2+d_3}(\mathcal{Z}_{23}) \arrow[r, "\text{convolution}"] \arrow[d, "\cong"'] \arrow[dr, phantom, "\circlearrowleft"] & H_{-*+d_1+d_3}(\mathcal{Z}_{13}) \arrow[d, "\cong"] \\
\mathrm{Ext}^*(F_1, F_2) \otimes \mathrm{Ext}^*(F_2, F_3) \arrow[r, "\text{Yoneda}"'] & \mathrm{Ext}^*(F_1, F_3)
\end{tikzcd}$$

Therefore convolution algebra can be analyzed as $\mathrm{Ext}^*(F, F)$ $\quad F = \pi_* \mathbb{C}_M[\dim M]$

Two extreme cases (trivial examples)

① M : cpt , X : pt $\quad \mathcal{Z} = M \times M$

$\pi_* \mathbb{C}_M[\dim M] \cong H^{*+\dim M}(M) \otimes \mathbb{C}_{pt} = F$

$\mathrm{Ext}^*(F, F) \cong \mathrm{End}(H^*(M))$

② $M = X$ $\quad \pi_* \mathbb{C}_M[\dim M] = \mathbb{C}_M[\dim M]$

$\pi = \mathrm{id}$

In general, Beilinson-Bernstein-Deligne-Gabber decomposition theorem :

$F = \pi_* \mathbb{C}_M[\dim M]$ is a direct sum of shifts of <u>simple objects</u>

not in $\mathrm{Mod}(\mathbb{C}_X)$,
but $\mathrm{Perv}(\mathbb{C}_X)$
a different abelian category
in $D^b_c(X)$

Riemann-Hilbert correspondence $\quad$ D-modules
abelian cat. of

§ perverse sheaves

○ t-structure

D = triangular category e.g. $D^b_c(\mathbb{C}_X)$, $D^b_c(\mathcal{C}\ell_X)$

$D^{\leq 0}$, $D^{\geq 0}$ full subcategories of D

$D^{\leq n} := D^{\leq 0}[-n]$, $D^{\geq n} := D^{\geq 0}[-n]$

$(D^{\leq 0}, D^{\geq 0})$ is a t-structure on D

def. $\Longleftrightarrow$ (1) $D^{\leq -1} \subset D^{\leq 0}$, $D^{\geq 1} \subset D^{\geq 0}$

(2) $\mathrm{Hom}_D(X, Y) = 0$ for $X \in D^{\leq 0}$, $Y \in D^{\geq 1}$

(3) $\forall X \in D$ $\exists$ distinguished triangle $X_0 \to X \to X_1 \xrightarrow{+1}$ $X_0 \in D^{\leq 0}$, $X_1 \in D^{\geq 1}$

<u>Remark</u> $\exists$ $\tau^{\leq 0}$, $\tau^{\geq 1}$ functors $D \to D^{\leq 0}, D^{\geq 1}$

$X \mapsto X_0, X_1$

<u>Ex</u>. (standard t-structure)

$\mathcal{C}$: abelian category $D = D^b(\mathcal{C})$

$D^{\leq n} = \{ F \in D^b(\mathcal{C}) \mid H^i(F) = 0 \text{ if } i > n \}$

$D^{\geq n} = \{ \quad " \quad \mid \quad " \quad i < n \}$

$\tau^{\leq n}$, $\tau^{\geq n}$ are truncation functors

<u>Def</u>. $\mathcal{C} := D^{\geq 0} \cap D^{\leq 0}$ is called the heart of the t-structure $(D^{\leq 0}, D^{\geq 0})$

<u>Ex</u> std t-structure $\Longrightarrow$ $\mathcal{C} \cong$ the original abelian category

Def. $(D^{\le 0}, D^{\ge 0})$: t-structure

${}^tH^0(F) = H^0(F) := \tau^{\le 0}\tau^{\ge 0}(F) = \tau^{\ge 0}\tau^{\le 0}(F)$

${}^pH^0(F) \,\|\quad H^n(F) := H^0(F[n])$

Prop. H^0 is a cohomological functor, i.e. distinguished triangle $X \to Y \to Z \xrightarrow{+1}$ induces an exact sequence $H^0(X) \to H^0(Y) \to H^0(Z)$

Th. The heart $\mathcal{C} = D^{\ge 0} \cap D^{\le 0}$ is an abelian category

(sketch of proof)

$$F_1 \xrightarrow{f} F_2 \ (\text{in } \mathcal{C}) \implies F_1 \to F_2 \to \underbrace{\mathrm{Cone}(f)}_{=:G} \xrightarrow{+1}$$

$$\Rightarrow 0 \to H^{-1}(\mathrm{Cone} f) \to \underset{F_1}{\underset{\|}{H^0(F_1)}} \xrightarrow[\ f\ ]{H^0(f)} \underset{F_2}{\underset{\|}{H^0(F_2)}} \to H^0(\mathrm{Cone} f) \to 0$$

Check
- $H^{-1}(\mathrm{Cone} f)$ is $\mathrm{Ker} f$
- $H^0(\mathrm{Cone} f)$ is $\mathrm{Coker} f$
- $\mathrm{Coim} f := \mathrm{Coker}(\mathrm{Ker} f \to F_1) \cong \mathrm{Im} f := \mathrm{Ker}(F_2 \to \mathrm{Coker} f)$

octahedron axiom :

$$\begin{tikzcd}
\tau^{\ge 0}G \arrow[d, "+1"'] & & F_2 \arrow[ll] \arrow[dl] \\
& G \arrow[ul] \arrow[dr, "+1"] & \\
\tau^{<0}G \arrow[ur] \arrow[rr, "+1"'] & & F_1 \arrow[uu]
\end{tikzcd}
\longrightarrow
\begin{tikzcd}
\overset{\mathrm{Coker} f}{\tau^{\ge 0}G} \arrow[d, "+1"'] \arrow[dr, "+1"] & & F_2 \arrow[ll] \\
& I \arrow[ur] \arrow[dl] & \\
\underset{\mathrm{Ker} f}{\tau^{<0}G} \arrow[rr, "+1"'] & & F_1 \arrow[uu] \arrow[ul]
\end{tikzcd}$$

- upper $\triangle \Rightarrow I \in D^{\ge 0}$
- lower $\triangle \Rightarrow I \in D^{\le 0}$

$\therefore I \in \mathcal{C}$

Also
- $I = \mathrm{Im} f$
- $I = \mathrm{Coim} f$ //

Ex. (torsion pair)

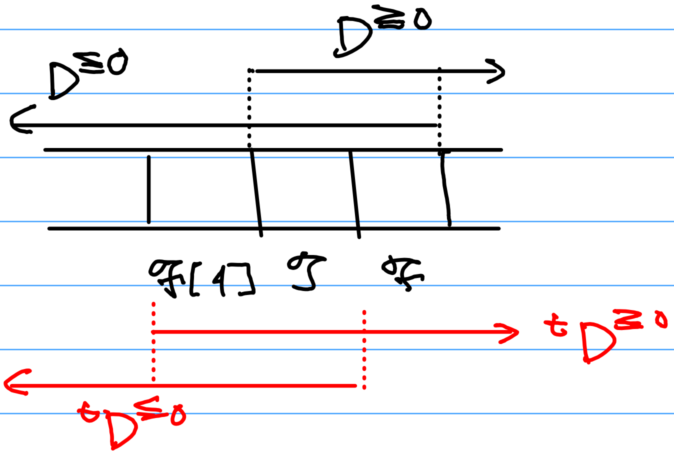
$\mathcal{C}$: abelian category $\quad (\mathcal{T}, \mathcal{F})$: torsion pair

i.e. (1) $\mathrm{Hom}_{\mathcal{C}}(T, F) = 0 \quad \forall T \in \mathcal{T} \ \forall F \in \mathcal{F}$

(2) $\forall E \in \mathcal{C}$ $\exists$ a short exact sequence $0 \to \underset{\mathcal{T}}{T} \to E \to \underset{\mathcal{F}}{F} \to 0$

$D = D^b(\mathcal{C})$ $\quad {}^tD^{\leq 0} := \{ E \in D^{\leq 0} \mid H^0(E) \in \mathcal{T} \}$

$\quad {}^tD^{\geq 0} := \{ E \in D^{\geq -1} \mid H^{-1}(E) \in \mathcal{F} \}$



$${}^tD^{\leq 0} \cap {}^tD^{\geq 0} = \left\{ E \in D \;\middle|\; \begin{array}{l} H^i(E) = 0 \text{ if } i \neq 0, -1 \\ H^0(E) \in \mathcal{T} \\ H^{-1}(E) \in \mathcal{F} \end{array} \right\}$$

○ admissible subcategory

$\mathcal{D}$: triangulated category

$\mathcal{D}_1 \xrightarrow[i_*]{} \mathcal{D}$ strictly full subcategory

Def.

$\mathcal{D}_1$ is right admissible $\iff$ i_* has right adjoint $i^!$

left $\iff$ left i^*

admissible $\iff$ left and right admissible

We formally set $i_! = i_*$.

Ex. $Y \overset{i}{\hookrightarrow} X$ closed subvariety $\quad D^b_c(\mathbb{C}_Y) \xrightarrow{i_*} D^b_c(\mathbb{C}_X)$ admissible

In fact,
$$\begin{tikzcd} Y \arrow[r, equal] \arrow[d, equal] & Y \arrow[d, "i"] \\ Y \arrow[r, "i"] & X \end{tikzcd}$$
$\therefore$ $i^* i_* = i^* i_! \cong \mathrm{id}$

$\therefore$ full subcat

6/9 ↑

$$\mathcal{D}_1^\perp \overset{\text{def}}{=} \{ E \in \mathcal{D} \mid \mathrm{Hom}_{\mathcal{D}}(F, E) = 0 \quad \forall F \in \mathcal{D}_1 \}$$

$$^\perp\mathcal{D}_1 = \{ E \in \mathcal{D} \mid \mathrm{Hom}_{\mathcal{D}}(E, F) = 0 \quad \forall F \in \mathcal{D}_1 \}$$

Lemma.

$$\begin{tikzcd} i_! i^! E \arrow[r] & \overset{\forall}{E} \arrow[r] & F \arrow[ll, bend left, "+1"] \\ & & \mathcal{D}_1^\perp \arrow[u, dotted] \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r] & \overset{\forall}{E} \arrow[r] & i_* i^* E \arrow[ll, bend left, "+1"] \\ {}^\perp\mathcal{D}_1 \arrow[u, dotted] & & \end{tikzcd}$$

$\therefore$ $\begin{cases} \mathcal{D}^{\leq 0} := \mathcal{D}_1 \\ \mathcal{D}^{\geq 1} := \mathcal{D}_1^\perp \end{cases}$ and $\begin{cases} \mathcal{D}^{\leq 0} := {}^\perp\mathcal{D}_1 \\ \mathcal{D}^{\geq 0} := \mathcal{D}_1 \end{cases}$

are t-structures on $\mathcal{D}$

<u>Ex</u> (cont'd) $U = X \setminus Y \underset{\text{open}}{\overset{j}{\hookrightarrow}} X$ (recall $j^* = j^!$)

Recall $i_! i^! E \to E \to j_* j^* E \xrightarrow{+1}$ $\Longrightarrow$ $D_1^\perp \underset{\cong}{\xleftarrow{j_*}} D^b_c(\mathbb{C}_U)$

$j_! j^! E \to E \to i_* i^* E \xrightarrow{+1}$ $\Longrightarrow$ ${}^\perp D_1 \underset{\cong}{\xleftarrow{j_!}} D^b_c(\mathbb{C}_U)$

In general, D_1 : left (resp. right) admissible
$\Rightarrow {}^\perp D_1$ (resp. $D_1^\perp$) : right (resp. left) admissible
given by above lemma

admissible $\Rightarrow$ ${}^\perp D_1$, $D_1^\perp$ also admissible
and ${}^\perp D_1 \longrightarrow D_1^\perp$ is an equivalence.
"$j^* j_!$" Inverse = "$j^! j_*$"
left adjoint of $j_* : D_1^\perp \to D$

- Recollement

We axiomize $D^b_c(\mathbb{C}_Y) \xrightarrow{i_*} D^b_c(\mathbb{C}_X) \xrightarrow{j^*} D^b_c(\mathbb{C}_U)$:

$$\begin{tikzcd} D_1 \arrow[r, "i_*"] & D \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & D_2 \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

(1) Let $i_! := i_*$, $j^! := j^*$.
$\Longrightarrow (k^*, k_*), (k_!, k^!)$ are adjoints for $k = i, j$

(2) $j^* i_* = 0$ $\therefore$ $\mathrm{Hom}(j_! G, i_* E) = 0$
$\mathrm{Hom}(i_* E, j_* G) = 0$

(3) $\forall E \in D$

$$j_! j^! E \to E \to i_* i^* E \xrightarrow{+1}, \quad i_! i^! E \to E \to j_* j^* E \xrightarrow{+1}$$

(4) $i_* = i_!$, j_*, $j_!$ are fully faithful.

Above remark : admissible $\Leftrightarrow$ recollement $\leftarrow$ [BBDG]

Th. Recollement
$$\begin{tikzcd} D_1 \arrow[r, "i_*"] & D \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & D_2 \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$
as above

Suppose t-structures $(D_1^{\leq 0}, D_1^{\geq 0})$, $(D_2^{\leq 0}, D_2^{\geq 0})$ are given.

$\Longrightarrow$
$$D^{\leq 0} := \{E \in D \mid j^* E \in D_2^{\leq 0},\ i^* E \in D_1^{\leq 0}\}$$
$$D^{\geq 0} := \{E \in D \mid j^* E \in D_2^{\geq 0},\ i^! E \in D_1^{\geq 0}\}$$
is a t-structure on D.

Moreover i_* (resp. j^*) are t-exact i.e. it sends $D_1^{\leq 0}, D_1^{\geq 0}$ (resp. $D^{\leq 0}, D^{\geq 0}$) to $D^{\leq 0}, D^{\geq 0}$ (resp. $D_2^{\leq 0}, D_2^{\geq 0}$)

(proof) Suppose $E \in D^{\leq 0}$, $F \in D^{\geq 0}$.

$$\mathrm{Hom}(i_* i^* E, F) \to \mathrm{Hom}(E, F) \to \mathrm{Hom}(j_! j^* E, F)$$

$\mathrm{Hom}(i_* i^* E, F) = \mathrm{Hom}(i^* E, i^! F) = 0$

$\mathrm{Hom}(j_! j^* E, F) = \mathrm{Hom}(j^* E, j^* F) = 0$

$\therefore \mathrm{Hom}(E, F) = 0$

Next suppose $E \in D$.

Define F by

$$\begin{tikzcd} E \arrow[r, equal] \arrow[d] & E \arrow[d] \arrow[r] & {} \\ j_* j^* E \arrow[r] \arrow[d] & j_* \tau_{>0} j^* E \arrow[d] \\ {} & F[1] \arrow[d, "+1"] \\ & {} \end{tikzcd}$$

Then define A by

$$\begin{tikzcd} F \arrow[r, equal] \arrow[d] & F \arrow[d] \arrow[r] & {} \\ i_* i^* F \arrow[r] \arrow[d] & i_* \tau_{>0} i^* F \arrow[d] \arrow[r] & {} \\ {} & A[1] \arrow[d, "+1"] \\ & {} \end{tikzcd}$$

We then define B as

$$\begin{tikzcd} A \arrow[r] \arrow[d, equal] & F \arrow[r] \arrow[d] & i_* \tau_{>0} i^* F \arrow[r, "+1"] \arrow[d] & {} \\ A \arrow[r] & E \arrow[r] \arrow[d] & B \arrow[r, "+1"] \arrow[d] & {} \\ & j_* \tau_{>0} j^* E \arrow[r, equal] \arrow[d, "+1"'] & j_* \tau_{>0} j^* E \arrow[d, "+1"] \\ & {} & {} \end{tikzcd}$$

<u>Claim</u>. $\begin{tikzcd} A \arrow[r] & E \arrow[r] & B \arrow[ll, bend left, "+1"] \end{tikzcd}$ has $A \in D^{\leq 0}$, $B \in D^{\geq 1}$.

$\because$ Note $j^* i_* = 0 \Rightarrow i^* j_! = 0$, $i^! j_* = 0$ by adjoint

3rd CD $\Rightarrow j^* B = \tau_{>0} j^* E \in D_2^{\geq 1}$

2nd CD $\Rightarrow j^* A = j^* F \overset{\text{1st CD}}{=} \tau_{\leq 0} j^* E \in D_2^{\leq 0}$

2nd CD $\Rightarrow i^* A = \tau_{\leq 0} i^* F \in D_1^{\leq 0}$

3rd CD $\Rightarrow i^! B = \tau_{>0} i^* F \in D_1^{\geq 1}$ //

○ perverse sheaves

$X = \coprod_{\alpha \in A} X_\alpha$ stratification $\quad d_\alpha := \dim_{\mathbb{C}} X_\alpha$

$D^b_A(X) = \{ E \in D^b_c(X) \mid H^i(i_\alpha^*(E)),\ H^i(i_\alpha^!(E))$ local system $\forall i, \alpha \}$

${}^pD_A(X)^{\leq 0} \overset{\text{def.}}{=} \{ E \in D^b_A(X) \mid i_\alpha^* E \in D^b_c(X_\alpha)^{\leq -d_\alpha} \}$

${}^pD_A(X)^{\geq 0} \overset{\text{def.}}{=} \{ E \in D^b_A(X) \mid i_\alpha^! E \in D^b_c(X_\alpha)^{\geq -d_\alpha} \}$

Consider $\overline{X_\alpha} \setminus X_\alpha \overset{i}{\subset} \overline{X_\alpha} \overset{j}{\supset} X_\alpha \rightsquigarrow$ recollements

$({}^pD_A(\overline{X_\alpha})^{\leq 0}, {}^pD_A(\overline{X_\alpha})^{\geq 0})$ is given by the gluing of t-structures for $\overline{X_\alpha} \setminus X_\alpha$ and X_α

$({}^pD^b_c(X)^{\leq 0}, {}^pD^b_c(X)^{\geq 0}) \overset{\text{def}}{=}$ limit for stratifications A

<u>Def.</u> $\mathrm{Perv}(\mathbb{C}_X) :=$ heart ${}^pD^b_c(X)^{\leq 0} \cap {}^pD^b_c(X)^{\geq 0}$

abelian category.

☆ $\mathbb{D}_X$ exchanges ${}^pD_A(X)^{\leq 0} \longleftrightarrow {}^pD_A(X)^{\geq 0}$

In particular, $\mathbb{D}_X : \mathrm{Perv}(\mathbb{C}_X) \longrightarrow \mathrm{Perv}(\mathbb{C}_X)^{op}$

$D_1 \leftrightarrows D \leftrightarrows D_2$ and $(D_1^{\leq 0}, D_1^{\geq 0})$, $(D_2^{\leq 0}, D_2^{\geq 0})$ as in Th. Define $(D^{\leq 0}, D^{\geq 0})$ as in Th

$\mathcal{A}_1, \mathcal{A}, \mathcal{A}_2$ = hearts

$${}^p j_! : H^0 \circ j_! : \mathcal{A}_2 \longrightarrow \mathcal{A}$$
$${}^p j_* : H^0 \circ j_* :$$

$j^* j_! \cong id$ (as $Hom(j_! A, j_! B) = Hom(A, B)$)

We get $j_! \longrightarrow j_*$ from $Hom(j^* j_! A, A) \ni id$

$$\begin{array}{c} Hom(j^*j_! A, A) \\ \cong \\ Hom(j_! A, j_* A) \end{array}$$

$\therefore {}^p j_! \longrightarrow {}^p j_*$

$j_{!*} F = Im({}^p j_! F \longrightarrow {}^p j_* F)$

intermediate extension

Th. {simple objects of $\mathcal{A}$}

$= \{ {}^p i_* E : E$: simple object of $\mathcal{A}_1 \}$

$\sqcup \{ j_{!*} F : F$: simple object of $\mathcal{A}_2 \}$