○ admissible subcategory

D : triangulated category

$D_1 \xrightarrow[i_*]{} D$ strictly full subcategory

Def.

D_1 is right admissible $\Longleftrightarrow$ i_* has right adjoint $i^!$

left $\Longleftrightarrow$ left i^*

admissible $\Longleftrightarrow$ left and right admissible

We formally set $i_! = i_*$.

Ex. $Y \underset{i}{\subset} X$ closed subvariety $\quad D^b_c(\mathbb{C}_Y) \xrightarrow{i_*} D^b_c(\mathbb{C}_X)$ admissible

In fact,
$$\begin{tikzcd} Y \arrow[r, equal] \arrow[d, equal] & Y \arrow[d, "i"] \\ Y \arrow[r, "i"] & X \end{tikzcd}$$
$\therefore$ $i^* i_* = i^* i_! \cong id$

$\therefore$ full subcat

6/9 ↑

$$D_1^{\perp} \overset{\text{def}}{=} \{ E \in D \mid \mathrm{Hom}_D(F, E) = 0 \quad \forall F \in D_1 \}$$

$$^{\perp}D_1 = \{ E \in D \mid \mathrm{Hom}_D(E, F) = 0 \quad \forall F \in D_1 \}$$

Lemma.
$$\overset{\forall}{} \quad i_! i^! E \to E \to F \xrightarrow{+1} \qquad G \to \overset{\forall}{E} \to i_* i^* E \xrightarrow{+1}$$
$F \in D_1^{\perp}$, $\quad G \in {}^{\perp}D_1$

$\therefore$ $\begin{pmatrix} D^{\leq 0} := D_1 \\ D^{\geq 1} := D_1^{\perp} \end{pmatrix}$ and $\begin{pmatrix} D^{\leq 0} := {}^{\perp}D_1 \\ D^{\geq 0} := D_1 \end{pmatrix}$

are t-structures on D

Ex. (cont'd) $U = X \setminus Y \overset{j}{\underset{\text{open}}{\hookrightarrow}} X$ (recall $j^* = j^!$)

Recall

$$\begin{tikzcd} i_! i^! E \arrow[r] & E \arrow[r] & j_* j^* E \arrow[ll, bend left, "+1"] \end{tikzcd} \Rightarrow D_1^{\perp} \xleftarrow[\cong]{j_*} D_c^b(\mathbb{C}_U)$$

$$\begin{tikzcd} j_! j^! E \arrow[r] & E \arrow[r] & i_* i^* E \arrow[ll, bend left, "+1"] \end{tikzcd} \Rightarrow {}^{\perp}D_1 \xleftarrow[\cong]{j_!} D_c^b(\mathbb{C}_U)$$

In general, D_1: right (resp. left) admissible

$\Rightarrow$ $D_1^{\perp}$ (resp. ${}^{\perp}D_1$): left (resp. right) admissible given by above lemma

admissible $\Rightarrow$ $D_1^{\perp}$, ${}^{\perp}D_1$ also admissible

and ${}^{\perp}D_1 \xrightarrow{\text{"}j^* j_!\text{"}} D_1^{\perp}$ is an equivalence.

Inverse = "$j^! j_*$"

left adjoint of $j_*: D_1^{\perp} \to D$

- Recollement [BBDG]

We axiomize $D_c^b(\mathbb{C}_Y) \xrightarrow{i_*} D_c^b(\mathbb{C}_X) \xrightarrow{j^*} D_c^b(\mathbb{C}_U)$:

$$\begin{tikzcd} D_1 \arrow[r, "i_*"] & D \arrow[r, "j^*"] \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] & D_2 \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

(1) Let $i_! := i_*$, $j^! := j^*$.

$\Rightarrow$ (k^*, k_*), $(k_!, k^!)$ are adjoints for $k = i, j$

(2) $j^* i_* = 0$ $\therefore$ $\mathrm{Hom}(j_! G, i_* E) = 0$

$\mathrm{Hom}(i_* E, j_* G) = 0$

(3) $\forall E \in D$

$$\begin{tikzcd} j_!j^!E \arrow[r] & E \arrow[r] & i_*i^*E \arrow[ll, bend left, "+1"] \end{tikzcd}, \quad \begin{tikzcd} i_!i^!E \arrow[r] & E \arrow[r] & j_*j^*E \arrow[ll, bend left, "+1"] \end{tikzcd}$$

(4) $i_* = i_!$, j_*, $j_!$ are fully faithful.

Above remark : admissible $\Leftrightarrow$ recollement

Th. Recollement
$$\begin{tikzcd} D_1 \arrow[r, "i_*"] & D \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & D_2 \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$
as above

Suppose t-structures $(D_1^{\leq 0}, D_1^{\geq 0})$, $(D_2^{\leq 0}, D_2^{\geq 0})$ are given.

$\Longrightarrow$ $D^{\leq 0} := \{E \in D \mid j^*E \in D_2^{\leq 0},\ i^*E \in D_1^{\leq 0}\}$

$D^{\geq 0} := \{E \in D \mid j^*E \in D_2^{\geq 0},\ i^!E \in D_1^{\geq 0}\}$

is a t-structure on D.

Moreover i_* (resp. j^*) are t-exact i.e. it sends $D_1^{\leq 0}, D_1^{\geq 0}$ (resp. $D^{\leq 0}, D^{\geq 0}$) to $D^{\leq 0}, D^{\geq 0}$ (resp. $D_2^{\leq 0}, D_2^{\geq 0}$)

(proof) Suppose $E \in D^{\leq 0}$, $F \in D^{\geq 0}$.

$$\mathrm{Hom}(i_*i^*E, F) \longrightarrow \mathrm{Hom}(E, F) \longrightarrow \mathrm{Hom}(j_!j^*E, F)$$

$$\mathrm{Hom}(i_*i^*E, F) = \mathrm{Hom}(\underset{\in D_1^{\leq 0}}{i^*E}, \underset{\in D_1^{\geq 0}}{i^!F}) = 0$$

$$\mathrm{Hom}(j_!j^*E, F) = \mathrm{Hom}(\underset{\in D_2^{\leq 0}}{j^*E}, \underset{\in D_2^{\geq 0}}{j^*F}) = 0$$

$\therefore \mathrm{Hom}(E, F) = 0$

Next suppose $E \in D$.

Define F by

$$\begin{tikzcd}
E \arrow[r, equal] \arrow[d] & E \arrow[d] \arrow[r] & {} \\
j_* j^* E \arrow[r] \arrow[d] & j_* \tau_{>0} j^* E \arrow[d] & \\
{} & F[1] \arrow[d, "+1"] & \\
 & {} &
\end{tikzcd}$$

Then define A by

$$\begin{tikzcd}
F \arrow[r, equal] \arrow[d] & F \arrow[d] \arrow[r] & {} \\
i_* i^* F \arrow[r] \arrow[d] & i_* \tau_{>0} i^* F \arrow[d] \arrow[r] & {} \\
{} & A[1] \arrow[d, "+1"] & \\
 & {} &
\end{tikzcd}$$

We then define B as

$$\begin{tikzcd}
A \arrow[r] \arrow[d, equal] & F \arrow[r] \arrow[d] & i_* \tau_{>0} i^* F \arrow[r, "+1"] \arrow[d] & {} \\
A \arrow[r] & E \arrow[r] \arrow[d] & B \arrow[r, "+1"] \arrow[d] & {} \\
 & j_* \tau_{>0} j^* E \arrow[r, equal] \arrow[d, "+1"'] & j_* \tau_{>0} j^* E \arrow[d, "+1"] & \\
 & {} & {} &
\end{tikzcd}$$

<u>Claim</u>. $A \to E \to B \xrightarrow{+1}$ has $A \in D^{\leq 0}$, $B \in D^{\geq 1}$.

$\because$ Note $j^* i_* = 0 \Rightarrow i^* j_! = 0$, $i^! j_* = 0$ by adjoint

3rd CD $\Rightarrow j^* B = \tau_{>0} j^* E \in D_2^{\geq 1}$

2nd CD $\Rightarrow j^* A = j^* F \underset{1^{st}\,CD}{=} \tau_{\leq 0} j^* E \in D_2^{\leq 0}$

2nd CD $\Rightarrow i^* A = \tau_{\leq 0} i^* F \in D_1^{\leq 0}$

3rd CD $\Rightarrow i^! B = \tau_{>0} i^* F \in D_1^{\geq 1}$ //

○ perverse sheaves

$X = \coprod_{\alpha \in A} X_\alpha$ stratification $\quad d_\alpha := \dim_{\mathbb{C}} X_\alpha$

$D^b_A(X) = \{ E \in D^b_c(X) \mid H^i(i_\alpha^*(E))$ local system
$H^i(i_\alpha^!(E)) \quad \forall i, \alpha \}$

${}^p D_A(X)^{\leq 0} \overset{\text{def.}}{=} \{ E \in D^b_A(X) \mid i_\alpha^* E \in D^b_c(X_\alpha)^{\leq -d_\alpha} \}$

${}^p D_A(X)^{\geq 0} \overset{\text{def.}}{=} \{ E \in D^b_A(X) \mid i_\alpha^! E \in D^b_c(X_\alpha)^{\geq -d_\alpha} \}$

↑ std t-str.

<u>Note</u> Assume $X = X_\alpha$ single stratum, $H^i(E)$: loc. system

$E \in {}^p D_A(X)^{\leq 0} \cap {}^p D_A(X)^{\geq 0}$
$\iff H^i(E) = 0$ unless $i = -d_\alpha$
$\iff E = (\text{loc. system})[d_\alpha]$

Consider $\overline{X_\alpha} \setminus X_\alpha \overset{i}{\subset} \overline{X_\alpha} \overset{j}{\supset} X_\alpha \rightsquigarrow$ recollement

$({}^p D_A(\overline{X_\alpha})^{\leq 0}, {}^p D_A(\overline{X_\alpha})^{\geq 0})$ is given by the gluing of t-structures for $\overline{X_\alpha} \setminus X_\alpha$ and X_α

$({}^p D^b_c(X)^{\leq 0}, {}^p D^b_c(X)^{\geq 0}) \overset{\text{def}}{=}$ limit for stratifications A

<u>Def.</u> $\mathrm{Perv}(\mathbb{C}_X) :=$ heart ${}^p D^b_c(X)^{\leq 0} \cap {}^p D^b_c(X)^{\geq 0}$
abelian category.

<u>NB</u> $\mathbb{D}_X L[d_X] = L^*[d_X]$

☆ $\mathbb{D}_X$ exchanges ${}^p D_A(X)^{\leq 0} \longleftrightarrow {}^p D_A(X)^{\geq 0}$

In particular, $\mathbb{D}_X : \mathrm{Perv}(\mathbb{C}_X) \longrightarrow \mathrm{Perv}(\mathbb{C}_X)^{\mathrm{op}}$

$D_1 \leftrightarrows D \leftrightarrows D_2$ and $(D_1^{\leq 0}, D_1^{\geq 0}), (D_2^{\leq 0}, D_2^{\geq 0})$ as in Th. Define $(D^{\leq 0}, D^{\geq 0})$ as in Th.

$\mathcal{A}_1, \mathcal{A}, \mathcal{A}_2$ = hearts

$$\left.\begin{array}{l} {}^p j_! : {}^{(p)}H^0 \circ j_! \\ {}^p j_* : {}^{(p)}H^0 \circ j_* \end{array}\right\} : \mathcal{A}_2 \longrightarrow \mathcal{A}$$

$j^* j_! \simeq \mathrm{id}$ (as $\mathrm{Hom}(j_! A, j_! B) = \mathrm{Hom}(A, B)$)

We get $j_! \longrightarrow j_*$ from $\mathrm{Hom}(j^* j_! A, A) \ni \mathrm{id}$

$$\begin{array}{c} \mathrm{Hom}(j^* j_! A, A) \ni \mathrm{id} \\ \cong \\ \mathrm{Hom}(j_! A, j_* A) \end{array}$$

$\therefore\ {}^p j_! \longrightarrow {}^p j_*$

$j_{!*} F = \mathrm{Im}({}^p j_! F \longrightarrow {}^p j_* F)$

intermediate extension

<u>Th.</u> $\{$simple objects of $\mathcal{A}\}$

$= \{ {}^p i_* E : E :$ simple object of $\mathcal{A}_1 \}$

$\sqcup \{ j_{!*} F : F :$ simple object of $\mathcal{A}_2 \}$

(proof) [BBDG]

Return back to geometric context:

X : irreducible algebraic variety
$U \subset X_{reg}$. Zariski open dense subset
L : local system on U $\qquad j: U \hookrightarrow X$

$$IC_X(L) := j_{!*}(L[d_X]) \in Perv(\mathbb{C}_X)$$

This is also denoted by ${}^p j_{!*}$

More generally $Y \subset X$ closed subvariety
$U \subset Y_{reg}$, L : loc. system on U

$$IC_Y(L) \in Perv(\mathbb{C}_X)$$

$L :=$ trivial local system $\mathbb{C}_U$ $\qquad IC_X = IC_X(\mathbb{C}_U)$

__Proposition.__

$IC_Y(L)$ is characterised by the followings:

(1) $H^i(IC_Y(L)) = 0$ for $i < -d$ $\quad d = \dim Y$

(2) $H^{-d}(IC_Y(L))|_U = L$

(3) $\dim \operatorname{supp} H^i(IC_Y(L)) < -i$ if $i > -d$

(4) $\dim \operatorname{supp} H^i(\mathbb{D}_X IC_Y(L)) < -i$ if $i > -d$

__Rem.__ If $E \in D^b_A(X)$

- $\forall \alpha$ $\quad i_\alpha^* E \in D^b_C(X_\alpha)^{\leq -d_\alpha} \iff \dim \operatorname{supp} H^i(E) \leq -i$
- $\forall \alpha$ $\quad i_\alpha^! E \in D^b_C(X_\alpha)^{\geq -d_\alpha} \iff \dim \operatorname{supp} H^i(\mathbb{D}_X E) \leq -i$

The following result is very deep, and no elementary proof is known:

<u>Th</u>. (decomposition theorem [BBDG])

$f : X \longrightarrow Y$ proper

$$Rf_*(IC_X) \cong \bigoplus_{k} IC_{Y_k}(L_k)[l_k]$$

L_k: semisimple local system

$l_k \in \mathbb{Z}$

<u>Rem</u>. Mochizuki proved the same result for $IC_X(L)$ L: semisimple local system

<u>Def</u>. $\pi : M \longrightarrow X$ proper

M smooth

π is semismall $\overset{\text{def.}}{\Longleftrightarrow}$ $\exists$ $X = \coprod X_\alpha$ stratification

$\pi|_{\pi^{-1}(X_\alpha)} : \pi^{-1}(X_\alpha) \longrightarrow X_\alpha$ topological locally trivial fibration

s.t. $\dim X_\alpha + 2\dim \pi^{-1}(x_\alpha) \leqq \dim M$

$x_\alpha \in X_\alpha$

Note $Z = M \times_X M$ $\dim Z \leqq \dim M$

(all irreducible comp.)

$Z \supset \Delta_M$ ← $\dim = \dim M$

Th. [Goresky - MacPherson]

π : semismall $\Longrightarrow$ $\pi_*(\mathbb{C}_M[d_M])$ is perverse

$\because$ $\mathbb{D}_X \pi_*(\mathbb{C}_M[d_M]) = \pi_*(\mathbb{C}_M[d_M])$

$\therefore$ Enough to show that $\pi_*(\mathbb{C}_M[d_M]) \in {}^pD^b_c(X)^{\leq 0}$

$i_\alpha : X_\alpha \hookrightarrow X$

$$i_\alpha^* \underset{\underset{\pi_!}{\parallel}}{\pi_*}(\mathbb{C}_M[d_M]) = \pi|_{\pi^{-1}(X_\alpha)*}(\mathbb{C}_{\pi^{-1}(X_\alpha)}[d_M])$$

local system over X_α

fiber $H^{*+d_M}(\pi^{-1}(x_\alpha))$

$*+d_M > 2\dim \pi^{-1}(x_\alpha) \Longrightarrow$ vanishes

$\Updownarrow$

$* > \underbrace{2\dim \pi^{-1}(x_\alpha) - d_M}_{}$

$\wedge\!\|$ by semismallness //

$-d_{X_\alpha}$

Combined with the decomposition thm, we get

Th. [Borho - MacPherson]

(1) $\pi_*(\mathbb{C}_M[d_M]) = \bigoplus_k IC_{Y_k}(L_k)$ $\leftarrow$ semisimple

(2) $H_{[0]}(Z) = \mathrm{End}(\bigoplus_k IC_{Y_k}(L_k))$

(proof) (2) : last week $\mathrm{Ext}^0 = \mathrm{End}$ //

Decompose $L_k = \bigoplus_{k'} \underbrace{M_{k k'}}_{\text{multiplicity}} \otimes \phi_{k'}$, irreducible local system

Cor. $H_{[0]}(Z) \cong \bigoplus_{k} \mathrm{End}(M_k)$

$\because$ $\mathrm{Hom}(IC_{Y_k}(\phi_k), IC_{Y_\ell}(\phi_\ell)) = \delta_{k\ell}\, \mathrm{id}$ //

$\therefore$ $H_{[0]}(Z)$: semisimple & $\{M_k\}$: irreducible rep.

Example $\pi : T^*\mathbb{P}^1 \longrightarrow N_{sl_2}$ $\cong \mathbb{C}^2/\mathbb{Z}_2$ semismall

$$\pi_*(\mathbb{C}_M[d_M]) = IC_N \oplus IC_{\{0\}}$$

Recall $H_{[0]}(Z) \cong \mathbb{C}[\mathfrak{S}_2]$ ⟵ two irreducible representations

$\mathfrak{S}_2 = \mathbb{Z}/2$

Next 11:31 ~

More general case : $\pi : M \longrightarrow X$ proper
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\uparrow$ smooth

$$Z = M \times_X M$$

Recall we have a graded algebra isomorphism (modulo Exercise)

$$H_{-*+2\dim_{\mathbb{C}} M}(Z) \cong \mathrm{Ext}^*(F,F) \quad (☆)$$

where $F = \pi_*(\mathbb{C}_M[\dim M])$

Rewrite the decomposion $= \bigoplus IC_{Y_k}(L_k)[d_k]$

as $$F \cong \bigoplus_{\alpha,i} L_{\alpha,i} \otimes_{\mathbb{C}} IC_{Y_\alpha}(\phi_\alpha)[i]$$

- ϕ_α : simple local system on $U_\alpha \subset (Y_\alpha)_{reg}$
- $IC_{Y_\alpha}(\phi_\alpha) \not\cong IC_{Y_\beta}(\phi_\beta)$ unless $\alpha = \beta$

Then

$$☆ \cong \bigoplus_{\substack{i,j,k\\ \alpha,\beta}} \mathrm{Hom}_{\mathbb{C}}(L_{\alpha,i}, L_{\beta,j}) \otimes_{\mathbb{C}} \mathrm{Ext}^{k+j-i}(IC_{Y_\alpha}(\phi_\alpha), IC_{Y_\beta}(\phi_\beta))$$

(graded by k)

$$\cong \bigoplus_{\substack{i,j,k'\\ \alpha,\beta}} \mathrm{Hom}(L_{\alpha,i}, L_{\beta,j}) \otimes_{\mathbb{C}} \mathrm{Ext}^{k'}(IC_{Y_\alpha}(\phi_\alpha), IC_{Y_\beta}(\phi_\beta))$$

grading is not preserved

graded by $k' = k+j-i$

Since $\{IC_{Y_\alpha}(\phi_\alpha)\}$ are pairwise non-isomorphic simple perverse sheaves,

- $\mathrm{Ext}^{<0}(IC_{Y_\alpha}(\phi_\alpha), IC_{Y_\beta}(\phi_\beta)) = 0$

$\mathrm{Hom}(D^{\leq 0}, D^{\geq 1}) = 0$

- $\mathrm{Ext}^0(IC_{Y_\alpha}(\phi_\alpha), IC_{Y_\beta}(\phi_\beta)) = \begin{cases} \mathbb{C}\,\mathrm{id} & \alpha = \beta \\ 0 & \text{otherwise} \end{cases}$

$\therefore \quad \bigoplus_{i,j,\alpha,\beta} \mathrm{Hom}(L_{\alpha,i}, L_{\beta,j}) \otimes \mathrm{Ext}^{>0}(IC_{Y_\alpha}(\phi_\alpha), IC_{Y_\beta}(\phi_\beta))$

forms a nilpotent ideal of $\star$,
hence acts by 0 on simple modules

Moreover $\star/\text{the ideal} \cong \bigoplus_\alpha \mathrm{End}(\bigoplus_i L_{\alpha,i}, \bigoplus_i L_{\alpha,i})$

semisimple algebra

Hence we get

<u>Th</u> [Ginzburg]
simple modules of $H_*(Z)$ (up to isom) are

$\bigoplus_i L_{\alpha,i}$ where α as above, i.e.
$IC_{Y_\alpha}(\phi_\alpha)[i]$ appearing
in $\pi_*(\mathbb{C}_M)$ for some i

☆ Proof of Varagnolo–Vasserot

$$Z(\vec{v}) = \coprod_{\vec{\nu},\vec{\nu}'} \tilde{\mathcal{F}}_{\vec{\nu}} \times_{N_Q(\vec{v})} \tilde{\mathcal{F}}_{\vec{\nu}'}, \qquad N_Q(\vec{v}) \curvearrowleft GL_Q(\vec{v})$$

$$F := \bigoplus_{\vec{\nu}} \pi_* \left(\mathbb{C}_{\tilde{\mathcal{F}}_{\vec{\nu}}} [\dim \tilde{\mathcal{F}}_{\vec{\nu}}] \right)$$

regarded as an object of the equivariant derived category $D^b_{GL_Q(\vec{v}),c}(N_Q(\vec{v}))$ [Bernstein–Lunts, LNM]

$$H^{GL_Q(\vec{v})}_{-[*]}(Z(\vec{v})) \cong \underbrace{\mathrm{Ext}^*_{D^b_{GL_Q(\vec{v}),c}(N_Q(\vec{v}))}}_{= \mathrm{Ext}^*_{GL_Q(\vec{v})}}(F, F)$$

$Q_{\vec{v}}$:= additive full subcategory of $D^b_{GL_Q(\vec{v}),c}(N_Q(\vec{v}))$ consisting of finite direct sums of shifts of simple perverse sheaves appearing $F[i]$ $\exists i$

$K(Q_{\vec{v}})$: Grothendieck group

- $\underline{\mathbb{Z}}[q, q^{-1}]$-module ($= A$) by shift
- : bar involution by $\mathbb{D}$ $\quad \bar{q} = q^{-1}$

<u>Note</u> $\mathbb{D}F = F$ $\rightsquigarrow$ canonical base elements $\leftarrow$ bar invariant

<u>Fact</u> [Lusztig]

$$\bigoplus_{\vec{v}} K(Q_{\vec{v}}) \cong {}_A U_q^-$$ (algebra)

canonical base $= \{$ simple perverse sheaves $\}$

$$E \in \mathcal{Q}_{\vec{V}} \longmapsto \operatorname{Ext}^*_{GL_{\mathbb{Q}}(\vec{V})}(F, E)$$

$$\in R(\vec{V})\text{-gproj} \quad \text{graded projective modules}$$

(Lemma projective $\Longleftrightarrow$ direct summand of a free module

This functor induces

$$\begin{tikzcd}
\bigoplus K(\mathcal{Q}_{\vec{V}}) \arrow[r] \arrow[dr, "\cong"'] & \bigoplus K(R(\vec{V})\text{-gproj.}) \arrow[d, "\cong"] \arrow[dl, phantom, "\circlearrowright"] \\
& {}_{\mathcal{A}}U_v^-
\end{tikzcd}$$

s.t. [Lusztig] $\qquad$ ↖ last week

∵ check for generators

∴ We only need to check that

Claim $P_E := \operatorname{Ext}^*_{GL_{\mathbb{Q}}(\vec{V})}(F, E)$ for $E = IC_{V_\alpha}(\phi_\alpha)$ ← simple perverse sheaf in $\mathcal{Q}_V$

is indecomposable.

By a standard argument (see [柏原, 系5.4.7])

$$P_E \twoheadrightarrow \operatorname{hd}(P_E) := P_E / \underbrace{\operatorname{rad} P_E}$$

$\operatorname{rad} P_E$ =: intersection of all maximal graded submodules

indecomposable $\Longleftrightarrow$ simple

$$\mathrm{Ext}^*_{GL_Q(\vec{v})}(F,F) = H^{GL_Q(\vec{v})}_*(Z(\vec{v}))$$

$$\nearrow$$

$$H^*_{GL_Q(\vec{v})}(pt) = \bigotimes_{i \in Q_0} H^*_{GL(V_i)}(pt)$$

$$\cong \left(\bigoplus_{\vec{i}} \mathbb{C}[x_{\vec{i}}(1), \dots, x_{\vec{i}}(m)] \right)^{\mathfrak{S}_m}$$

↑ also permute $\vec{i}$

This is contained in the center. (Fact. This is the center) [柏原, 補題2.6.4]

$$H^*(pt) \;/\!/\; H^0_{GL_Q(\vec{v})}(pt)$$

Graded maximal ideal $= \mathrm{Ker}\left(H^*_{GL_Q(\vec{v})}(pt) \longrightarrow \mathbb{C} = H^0_{GL_Q(\vec{v})}(pt) \right)$

$\therefore$ any graded simple quotient L of P_E factors through

$$\begin{array}{ccc} P_E = \mathrm{Ext}^*_{GL_Q(\vec{v})}(F,E) & \longrightarrow\!\!\!\!\rightarrow & L \\ \searrow & & \nearrow \\ & \mathrm{Ext}^*_{GL_Q(\vec{v})}(F,E) \otimes_{H^*_{GL_Q(\vec{v})}(pt)} \mathbb{C} & \end{array}$$

Prop.

(1) $\mathrm{Ext}^*_{GL_Q(\vec{v})}(F,E) \otimes_{H^*_{GL_Q(\vec{v})}(pt)} \mathbb{C} \cong \mathrm{Ext}^*(F,E)$ non-equivariant Ext.

(2) $\mathrm{Ext}^*_{GL_Q(\vec{v})}(F,F) \otimes_{H^*_{GL_Q(\vec{v})}(pt)} \mathbb{C} \cong \mathrm{Ext}^*(F,F)$

$\because$ (2) : Use stratification for $Z(\vec{v})$ to show

$$H^{GL_Q(\vec{v})}_*(Z(\vec{v})) \otimes_{H^*_{GL_Q(\vec{v})}(pt)} \mathbb{C} \cong H_*(Z(\vec{v}))$$

(1) (2) respects the direct sum decomposition

Now $\mathrm{Ext}^*(F,E)$ non-equivariant Ext.

$$\begin{aligned}&\| \\ &\bigoplus_{\beta,i,k} L^*_{\beta,i} \otimes \mathrm{Ext}^k(\underbrace{IC_{Y_\beta}(\phi_\beta), E}_{\substack{k>0 \\ \text{or } k=0 \text{ \& } \alpha=\beta}} = IC_{Y_\alpha}(\phi_\alpha))\end{aligned}$$

$$\begin{aligned}&\cup \\ &\bigoplus_i L^*_{\alpha,i} = \bigoplus_i L^*_{\alpha,i} \otimes \mathrm{Ext}^0(IC_{Y_\alpha}(\phi_\alpha), IC_{Y_\alpha}(\phi_\alpha))\end{aligned}$$

This subspace generates $\mathrm{Ext}^*(F,E)$ under

$$\mathrm{Ext}^*(F,F) \supset \bigoplus \underset{\substack{\| \\ \mathrm{Hom}(L^*_{\alpha,i}, L^*_{\beta,i})}}{\mathrm{Hom}(L_{\beta,i}, L_{\alpha,i})} \otimes \mathrm{Ext}^*(IC_{Y_\beta}(\phi_\beta), IC_{Y_\alpha}(\phi_\alpha))$$

$\therefore$ $\bigoplus_i L^*_{\alpha,i}$ is the simple quotient //