standard modules

☆ $F = \bigoplus_{\alpha, i} L_{\alpha, i} \otimes IC_{X_\alpha}(\phi_\alpha)[i]$ semisimple cpx

$F \in D^b_c(X)$

$L_\alpha = \bigoplus_i L_{\alpha, i}$

(e.g. $\pi : M \to X$ proper, M smooth

$F = \pi_*(\mathbb{C}_M[\dim M])$

$X_\alpha \subset X$

ϕ_α : simple local system on X_α

Assume $H^i(i^!_{X_\alpha} F)$: local system $\forall \alpha, i$

$\Updownarrow$

rep. of fundamental group $\pi_1(X_\alpha, x_\alpha)$ $x_\alpha \in X_\alpha$

<u>Def.</u> (standard module)

$$M_\alpha = \mathrm{Hom}_{\pi_1(X_\alpha, x_\alpha)}(\phi_\alpha, \bigoplus_i H^i(i^!_{x_\alpha} F))$$

$\circlearrowleft$ $\mathrm{Ext}^*(F, F)$-module under Yoneda

Ex above $\Rightarrow$ $M_\alpha \cong \mathrm{Hom}_{\pi_1(X_\alpha, x_\alpha)}(\phi_\alpha, H_*(\pi^{-1}(X_\alpha)))$

base change BM homology

Substitute ☆

$$M_\alpha = \bigoplus_{\beta} \underbrace{L_\beta}_{\bigoplus_j L_{\beta,j}} \otimes \bigoplus_{p} \mathrm{Hom}(\phi_\alpha, H^p(i_\alpha^! IC_{X_\beta}(\phi_\beta)))$$

new grading

$$\mathrm{Ext}^*(F,F) = \bigoplus \mathrm{Hom}(L_\beta, L_\gamma) \otimes \mathrm{Ext}^p(IC_{X_\beta}(\phi_\beta), IC_{X_\gamma}(\phi_\gamma))$$

grading $\mathbb{Z}_{\geq 0}$

deg 0 $\quad \bigoplus \delta_{\beta\gamma}\, id_{IC_{X_\beta}(\phi_\beta)}$

$$F^p M_\alpha = \bigoplus_{g \geq p} \cdots\cdots \quad \cdots H^g(\quad)$$

Th [Ginzburg]

(1) $F^p M_\alpha$ is a submodule w.r.t. $\mathrm{Ext}^*(F,F)$

(2) $F^p M_\alpha / F^{p+1} M_\alpha \cong \bigoplus_\beta L_\beta \otimes \mathrm{Hom}(\phi_\alpha, H^p(i_\alpha^! IC_{X_\beta}(\phi_\beta)))$

$\mathrm{Ext}^*(F,F)$ acts trivially

$\therefore$ semisimple module

multiplicity of $L_\beta = \dim \mathrm{Hom}(\phi_\alpha, H^p(i_\alpha^! IC_{X_\beta}(\phi_\beta)))$

(3) $M_\alpha / F^{-d_\alpha+1} M_\alpha \cong L_\alpha$ (property of IC sheaves last week)

$d_\alpha = \dim X_\alpha$

Kazhdan–Lusztig type weight multiplicity formula

Grothendieck group of $\operatorname{Rep} \operatorname{Ext}^*(F,F)$

$$0 \to A \to B \to C \to 0 \overset{\text{def.}}{\Longleftrightarrow} [B] = [A] + [C]$$

$$[M_\alpha] = \sum_\beta [L_\beta] \cdot \sum_p \dim \operatorname{Hom}(\phi_\alpha, H^p(i_{X_\alpha}^! IC_{X_\beta}(\phi_\beta)))$$

Rem. We can formally consider (X_α, ϕ_α) s.t.

- $M_\alpha = \operatorname{Hom}(\phi_\alpha, H^*(i_{X_\alpha}^! F)) \neq 0$
- $L_\alpha = 0$

In many situation, such (X_α, ϕ_α) does <u>not</u> occur.

cf. affine Hecke algebra (degenerate) Kazhdan–Lusztig

If ε: roots of unity $\Rightarrow$ <u>does</u> occur.

- quiver variety ϕ_α: trivial local systems only occur.

$$M_\alpha \neq 0 \Rightarrow L_\alpha \neq 0$$

In fact, all L_α are classified by "ℓ-highest weight" (combinatorial objects)

• Kazhdan-Lusztig conjecture
new proof of

Verma modules

highest weights
classified by

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quiver Hecke algebra

Q : ADE type
$\Updownarrow$ Gabriel
$X = \mathbb{N}_Q(\vec{v}) \curvearrowleft GL_Q(\vec{v})$

finitely many orbits

$M_{\vec{v}} = \tilde{\mathcal{F}}_{\vec{v}} \xrightarrow{\pi} X$ $\leftarrow GL_Q(\vec{v})$-equivariant

$$F = \bigoplus_{\vec{v}} \pi_*(\mathbb{C}_{M_{\vec{v}}}[\dim M_{\vec{v}}])$$

$$= \bigoplus_{\alpha, i} L_{\alpha, i} \otimes \underbrace{IC_{X_\alpha}(\phi_\alpha)}[i]$$

$\uparrow$ $GL_Q(\vec{v})$-equivariant

$\Rightarrow$ · X_α : $\underset{\text{irr.}}{GL_Q(\vec{v})}$-orbit

· ϕ_α : $GL_Q(\vec{v})$-equivariant local system

$\longleftrightarrow$ irr. representation of

$X_\alpha \ni x_\alpha$ $\quad$ $\mathrm{Stab}_{GL_Q(\vec{v})}(x_\alpha) / \mathrm{Stab}^0_{GL_Q(\vec{v})}(x_\alpha)$

(component group)

connected component containing id

In our situation $x_\alpha = B \in \mathbf{N}_Q(\vec{v})$
quiver repr.

$$\underbrace{\mathrm{Stab}_{GL_Q(\vec{v})}}_{\text{connected}}(x_\alpha) \subset \underbrace{\mathrm{Lie}\ \mathrm{Stab}_{GL_Q(\vec{v})}(x_\alpha)}_{}$$

invertible elements

$\therefore \quad \Phi_\alpha =$ trivial local system

$$IC_{X_\alpha}(\Phi_\alpha) = IC_{\underbrace{X_\alpha}_{\text{orbit}}}$$

U_A^- canonical base

counting #

$\Longrightarrow$ # of canonical base elements

$= \dim U^-_{\vec{v}}$ ← (weight)

$=$ # of orbits

<u>Fact</u> IC_{X_α} for any $GL_Q(\vec{v})$-orbit X_α

occurs in $F = \pi_*(\cdots)$

proved by using "crystal structure"

PBW vs canonical base (cf. 20201207)

$L(\vec{c}, \vec{i})$ $b(\vec{c}, \vec{i})$

$\vec{c} = (c_1, \cdots, c_\nu) \in \mathbb{Z}_{\geq 0}^{\nu}$

$\vec{i}$ reduced expn for $w_0 = (i_1, \cdots, i_\nu)$ (adapted with Q)

$$\begin{cases} (a)\ b(\vec{c}, \vec{i}) = L(\vec{c}, \vec{i}) + \sum_{\vec{d} > \vec{c}} a_{\vec{d}} L(\vec{d}, \vec{i}) & a_{\vec{d}} \in q\mathbb{Z}[q] \\ (b)\ \overline{b(\vec{c}, \vec{i})} = b(\vec{c}, \vec{i}) \end{cases}$$

These two properties follows from properties of IC's

a) $i^!_{x_\beta} IC_{X_\alpha}$ ....

b) $\mathbb{D}\, IC_{X_\alpha} = IC_{X_\alpha}$

($q \leftrightarrow$ shift)

11.23 ~

Recall $\quad M = T^*\mathcal{F}l(n) \curvearrowleft GL(n)\times\mathbb{C}^\times \qquad \mathcal{F}l(n)$: flag variety of $GL(n)$

$$\begin{array}{c} M \\ \downarrow \\ X = \mathcal{N} \end{array}$$

$$Z = M \times_X M$$

$$H_*^{GL(n)\times\mathbb{C}^\times}(Z) \cong H_{deg} \quad \text{degenerate affine Hecke algebra}$$

<u>Fact</u>. center

$$H^*_{GL(n)\times\mathbb{C}^\times}(pt) = \mathbb{C}[\hbar, x_1, \cdots, x_n]^{\mathfrak{S}_n}$$
$$= \mathbb{C}[\mathrm{Lie}(T\times\mathbb{C}^\times)]^{\mathfrak{S}_n}$$

normalization

$$\zeta = (1, \lambda_1, \cdots, \lambda_n) \in \mathrm{Lie}(T\times\mathbb{C}^\times)$$

$$\chi_\zeta : H^*_{GL(n)\times\mathbb{C}^\times}(pt) \longrightarrow \mathbb{C} \quad \text{evaluation at } \zeta$$

$$H_*^{GL(n)\times\mathbb{C}^\times}(Z) \otimes_{H^*_{GL(n)\times\mathbb{C}^\times}(pt)} \mathbb{C} = H_{deg} \otimes_{\text{center}} \mathbb{C}$$

specialized deg. affine Hecke alg.

$$T_\zeta := \overline{\exp \mathbb{R}\zeta} \subset T\times\mathbb{C}^\times$$

$$H_*^{T_\zeta}(Z) \otimes_{H^*_{T_\zeta}(pt)} \mathbb{C} \cong H_*(Z^{T_\zeta}) \curvearrowleft$$

5.12 $\qquad \dfrac{1}{1\otimes e(N)} \; \nu^*$

alg. hom.

$$Z^{T_3} = M^{T_3} \times_{X^{T_3}} M^{T_3}$$

X^{T_3} : (essentially) $\overbrace{\text{representations}}^{\text{space of}}$ of type A-quiver

$\|$

$\left\{ x \in \mathcal{N} \;\middle|\; \left[\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}, x \right] + \underbrace{x}_{(x_{ij})} = 0 \right\}$

$(\lambda_i - \lambda_j + 1) x_{ij} = 0$

$$(\lambda_i\text{-eigensp}) \to \begin{pmatrix} \lambda_i - 1 \\ \text{eigensp} \end{pmatrix} \to \begin{pmatrix} \lambda_i - 2 \\ \text{eigensp.} \end{pmatrix} \to \cdots$$

M^{T_3} = space of pairs of quiver. reps + preserved flags

$\overline{\mathcal{F}}_{\vec{v}}$

simple modules $\leftrightarrow$ canonical base elements (type A)

equivariant K-theory

$$K^{GL(n)\times\mathbb{C}^\times}(Z) \cong \text{Haff affine Hecke algebra}$$

center

$$K^{GL(n)\times\mathbb{C}^\times}(pt) = R(GL(n)\times\mathbb{C}^\times)$$

$$= \mathbb{C}[T\times\mathbb{C}^\times]^{S_n} \text{ repr. ring}$$

$$\xi = \big((\underset{\substack{\Uparrow\\ T}}{\lambda_1,\dots,\lambda_n}),\ \underset{\in\, \mathbb{C}^\times}{q}\big)$$

$q \notin \sqrt{1}$ $\rightsquigarrow$ repr. of type A quiver

$$\begin{tikzcd} \lambda_i \arrow[r] & \lambda_i q^{-1} \arrow[r] & \lambda_i q^{-2} \arrow[r] & \cdots \arrow[lll, dashed, bend left, "q \in \sqrt{1}"'] \end{tikzcd}$$

$\rightsquigarrow$ repr. of affine type A quiver (cf. Ariki)

Yangian $Y(\mathfrak{sl}_2)$ $\quad \coprod_{0 \le n \le l} T^* Gr(n, l)$

$$\downarrow$$

$$X = \{x \in \mathrm{End}(\mathbb{C}^l) \mid x^2 = 0\}$$

$\rightsquigarrow$ $X^{T_\xi} \subset$ space of quiver rep. of type A

$$\rotatebox{90}{$\in$}$$

$$\to V_i \xrightarrow{B_i} V_{i+1} \xrightarrow{B_{i+1}} \qquad B_{i+1} B_i = 0$$

indecomposable $\leftrightarrow$ positive roots

Only $(\cdots 0\ 1\ 1\ 0 \cdots)$ appear.

§ Hyperbolic localization

ref · Braden : Transformation Groups 8 (2003)
· Drinfeld–Gaitsgory : arXiv 1308.3786

$\mathbb{C}^\times \curvearrowright X$: complex algebraic variety
(possibly singular)

$X^{\mathbb{C}^\times}$: fixed pt set
$\shortparallel$
$\bigsqcup_k F_k$: connected components

$$X_k^\pm := \{ x \in X \mid \lim_{t \to 0/\infty} t \cdot x \in F_k \}$$

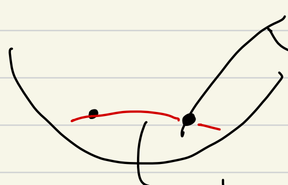
($+ \to 0$, $- \to \infty$)

$$X^\pm = \bigsqcup_k X_k^\pm \quad \text{(disjoint union)}$$

not considered as a subvariety of X

<u>Ex</u>.

$$\begin{array}{c} T^*\mathbb{P}^1 \\ \downarrow \\ \mathbb{C}^2/\pm \\ (t, t^{-1}) \end{array}$$



strict transform of the x-axis

$\mathbb{P}^1$, pt $\cong \mathbb{C}$

$\pi_\pm : X^\pm \longrightarrow X^{\mathbb{C}^\times}$ (limit)

$g_\pm : X^\pm \longrightarrow X$ (induced from the inclusion)

$F \in D^b_{\mathbb{C}^\times}(X)$ $\mathbb{C}^\times$-equivariant cpx

$F \overset{\rightsquigarrow}{\text{restriction}}$ cpx on $X^{\mathbb{C}^\times}$

$X^{\mathbb{C}^\times} \xrightarrow{i} X$

$i^!, i^* \leftarrow$ These are <u>not</u> useful for study of convolution algebras

$i^!, i^*$ do <u>not</u> <u>send</u> IC_X to a semi-simple complex

<u>Th</u> (Braden)

There is a natural isomorphism

$$\underset{\Phi}{\underset{\|}{(\pi_-)_*(g_-)^!}} \simeq (\pi_+)_!(g_+)^* : D^b_{\mathbb{C}^\times}(X) \to D^b_{\mathbb{C}^\times}(X^{\mathbb{C}^\times})$$

<u>Cor.</u>

$\underset{F}{\Phi}(IC_X)$ is a semisimple cpx

$$\cong \bigoplus_{\alpha, i} L_{\alpha, i} \otimes IC_{Y_\alpha}(\phi_\alpha)[i]$$

($Y_\alpha \subset X^{\mathbb{C}^\times}$ locally closed subvar.

ϕ_α: irr. local system on Y_α)

$(\mathrm{Thm} \Rightarrow \mathrm{Cor})$ follows from the theory of "weights"

decomp. thm $\Leftarrow \pi_! = \pi_*$

$\rightsquigarrow \mathrm{Ext}^*(F, F)$ can be analyzed as in the case of $F = \pi_*(\mathbb{C}_M)$

Toy model

X : smooth projective $\curvearrowleft \mathbb{C}^\times$

Bialynicki-Birula decomposition

$$X = \coprod X_k^{\pm}$$

$$X_k^{\pm} \xrightarrow{\text{affine space bdle}} F_k$$

$$(\pi_-)_* (g_-)^! \mathbb{C}_X = \bigoplus_k \mathbb{C}_{F_k}[-2 \operatorname{codim} X_k^-]$$

$$\| $$

$$(\pi_+)_! (g_+)^* \mathbb{C}_X = \bigoplus_k \mathbb{C}_{F_k}[-2 \dim \text{ fiber of } X_k^+ \to F_k]$$