hyperbolic restriction/localization

$X \curvearrowleft \mathbb{C}^\times \qquad X \xleftarrow{g_\pm} X^\pm \xrightarrow{\pi_\pm} X^{\mathbb{C}^\times}$

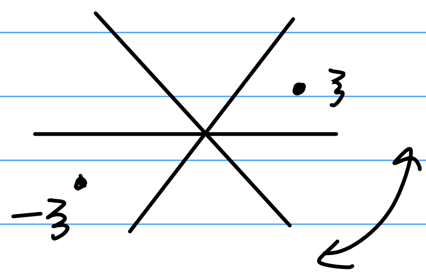
$\underline{\text{Th}}$ (Braden) $\quad (\pi_-)_*(g_-)^! \cong (\pi_+)_!(g_+)^*$ on $D^b_{\mathbb{C}^\times}(X)$

$$\underset{\Phi}{\overset{\|}{(\pi_-)_*(g_-)^!}}$$

$\underline{\text{Cor}}$. $\Phi(IC_X)$ is a semisimple cpx

generalization $\quad X \curvearrowleft T$

Choose $\xi \in \mathrm{Lie}\, T \qquad \xi = d\rho \quad \rho : \mathbb{C}^\times \longrightarrow T$



$X^\pm$ is interchanged

Φ_ξ : corresponding hyperbolic localization

$F = IC_X$ (more generally F : T-equivariant pure cpx)

$\mathrm{ev}_\xi : \underset{\mathbb{C}[\mathrm{Lie}\,T]}{\overset{}{\underset{\|}{H^*_T(pt)}}} \longrightarrow \underset{\mathbb{C}_\xi}{\mathbb{C}}$

$\mathrm{Ext}^*_T(F,F) \otimes_{H^*_T(pt)} \mathbb{C}_\xi \xrightarrow[\Phi_\xi]{} \mathrm{Ext}^*(\Phi_\xi(F), \Phi_\xi(F))$ s.s.

alg. hom.

$$\Phi_3(F) = \bigoplus_{\alpha, i} L_{\alpha,i} \otimes IC_{Y_\alpha}(\phi_\alpha)[i]$$

$\mathrm{Ext}^*(\Phi_3(F), \Phi_3(F))$ can be analyzed by the same method.

simple modules : $\{ \bigoplus_i L_{\alpha,i} \}_\alpha$

standard modules : $M_\alpha = \mathrm{Hom}_{\pi_1(Y_\alpha, y_\alpha)}(\phi_\alpha, i^!_{y_\alpha} \Phi_3(F))$

$\rightsquigarrow$ KL type multiplicity formula

Recall semismallness

$\pi : \underset{\text{smooth}}{M} \longrightarrow X = \coprod X_\alpha$ $\quad$ $2\dim \pi^{-1}(x_\alpha) + \dim X_\alpha \leq \dim M$

$\Longrightarrow \pi_*(\mathbb{C}_M[\dim M])$: perverse

Q. When Φ sends perverse sheaves to perverse sheaves?

$\longrightarrow \mathrm{End}(\Phi(F))$: semisimple

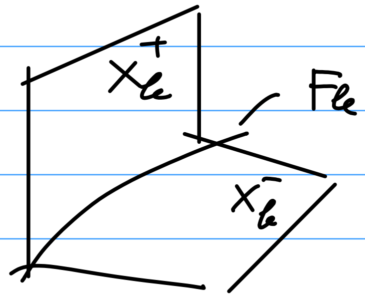
toy model $\quad$ $(\pi_-)_*(g_-)^! \mathbb{C}_X[\dim X]$

$\underset{\text{smooth proj}}{X} \curvearrowleft \mathbb{C}^\times$

$$= \bigoplus_k \mathbb{C}_{F_k}[\underbrace{\dim X - 2\operatorname{codim} X_k^-}_{\substack{\geq \;\; \| \\ \dim F_k}}]$$

$$\dim X = \dim X_\lambda^+ + \dim X_\lambda^- - \dim F_\lambda$$

$$? \iff \dim X_\lambda^- = \dim X_\lambda^+$$



$$\iff \begin{cases} \dim X_\lambda^- \leq \frac{1}{2}(\dim X + \dim F_\lambda) \\ \dim X_\lambda^+ \leq \frac{1}{2}(\dim X + \dim F_\lambda) \end{cases}$$

$$\iff \begin{cases} \underbrace{\dim X_\lambda^- - \dim F_\lambda}_{\dim \text{ fiber of } \pi_-} \leq \frac{1}{2}(\dim X - \dim F_\lambda) \\ \dim X_\lambda^+ - \dim F_\lambda \leq \frac{1}{2}(\dim X - \dim F_\lambda) \end{cases}$$

$(X, \mathbb{C}^\times F)$ is hyperbolic semismall

Def.

$\overset{\text{def}}{\iff}$ $\exists \left(\begin{array}{l} X = \coprod X_\alpha \\ X^{\mathbb{C}^\times} = \coprod F_\beta \end{array} \right.$ stratifications

s.t. $H^*(i_\alpha^* F)$, $H^*(i_\alpha^! F)$ are local systems over X_α

$\pi_\pm : X^\pm \longrightarrow X^{\mathbb{C}^\times}$ as above

$$\dim \pi_\pm^{-1}(\underbrace{y_\beta}_{\in F_\beta}) \cap X_\alpha \leq \frac{1}{2}(\dim X_\alpha - \dim F_\beta) \quad \forall \pm, \alpha, \beta$$

are satisfied.

Th. $\Phi(F)$ is perverse (and semisimple)

Mirković-Vilonen

in the context of geometric Satake

cf. Braverman - Finkelberg - N

smooth

$M \longrightarrow X$: symplectic resolution e.g. $T^*\mathcal{B} \to \mathcal{N}$

$\circlearrowleft$ quiver variety

$\mathbb{C}^\times$ symplectic form is preserved.

$$M^{\mathbb{C}^\times} = \coprod_{k} F_k \qquad M_k^{\pm} \longrightarrow F_k \ \text{affine space bdle}$$

$$\begin{array}{c} F_k \\ \cup \\ y \end{array}$$

$$\underset{\circlearrowleft\,\mathbb{C}^\times}{T_yM} = \underbrace{\underset{\uparrow\ wt=0}{T_yF_k} \oplus \underset{\uparrow\ wt>0}{T_yM^{>0}}}_{\shortparallel\ T_yM_k^+} \oplus \underset{\uparrow\ wt<0}{T_yM^{<0}}$$

$$T_yM = \overline{T_yF_k \oplus \underbrace{T_yM^{>0} \oplus T_yM^{<0}}} \quad T_yM_k^-$$

Under sympl. form $T_yM^{>0}$ and $T_yM^{<0}$ form dual spaces each other

$\Longrightarrow$ hyperbolic semismall

§ new proof of Kazhdan-Lusztig conjecture

○ geometric side

$\mathcal{B} = G/B$ flag variety, G: simply connected

$\ni b_- = \mathrm{Lie}\, B_-$ opposite Borel

$T = B \cap B_-$: maximal torus

$\Lambda = \mathrm{Hom}(\mathbb{C}^\times, T)$ cocharacter lattice

$= $ coroot lattice

$\cup$

$\Lambda_+ =$ semigroup generated by positive roots

<u>Def.</u> $\beta \leq \alpha \overset{\text{def.}}{\iff} \alpha - \beta \in \Lambda_+$

<u>Well-known fact</u>

$$H_2(\mathcal{B}, \mathbb{Z}) \cong \Lambda$$

$$\cup \qquad\qquad \cup$$

$$\{\text{classes represented by holomorphic curves}\} = \Lambda_+$$

$\alpha \in \Lambda_+$ $\quad \overset{\circ}{Z}{}^\alpha =$ moduli space of degree α based maps $\mathbb{P}^1 \to \mathcal{B}$

$= \{ f : \mathbb{P}^1 \longrightarrow \mathcal{B} \mid \deg f = \alpha,\ f(\infty) = b_- \}$
(holo. map)

<u>Th</u> $\overset{\circ}{Z}{}^\alpha$ is a <u>smooth</u> affine algebraic variety

positivity of $\mathcal{B}$

Example $\quad G = SL_2 \qquad \mathcal{B} = \mathbb{P}^1 \qquad H_2(\mathcal{B}, \mathbb{Z}) \cong \mathbb{Z}$

$d \in \mathbb{Z}_{\geq 0} \qquad \overset{\circ}{Z}{}^d = \{ f : \mathbb{P}^1 \to \mathbb{P}^1 \mid f(\infty) = 0,\ \deg f = d \}$

$$f(z) = \frac{p(z)}{q(z)} \qquad \begin{aligned} p(z) &= \underline{p_{d-1}} z^{d-1} + \underline{p_{d-2}} z^{d-2} + \cdots + \underline{p_0} \\ q(z) &= z^d + \underline{q_{d-1}} z^{d-1} + \cdots + \underline{q_0} \end{aligned}$$

$$\overset{\circ}{Z}{}^d = \{ (\vec{p}, \vec{q}) \in \mathbb{C}^{2d} \mid \text{no common divisor} \}$$

$\Leftrightarrow$ Resultant$(p, q) \neq 0$

$$\text{Resultant}(p, q) = \prod_{i,j} (a_i - b_j) = \text{polynomial in } \vec{p}, \vec{q}$$

roots of $p(z)$: $a_1, \cdots, a_{d-1}$

" $\quad q(z)$: $b_1, \cdots, b_d$

$\overset{\circ}{Z}{}^d \subset \mathbb{C}^{2d}$

open dense subset

$\overset{\circ}{Z}{}^\alpha \subset Z^\alpha \quad$ (partial compactification)

Zastava space

(affine algebraic variety) Finkelberg-Mirković

As a set, $\displaystyle Z^\alpha = \coprod_{\beta \leq \alpha} \overset{\circ}{Z}{}^\beta \times S^{\alpha - \beta} \mathbb{C}$ ← "defect"

colored symmetric power

$$\alpha - \beta = \sum_i \underset{\substack{\uparrow \\ \mathbb{Z}_{\geq 0}}}{d_i} \underbrace{\alpha_i^\vee}_{\text{simple coroot}}$$

$$S^{\alpha-\beta}\mathbb{C} = \underbrace{\prod S^{d_i}\mathbb{C}}_{} = \mathbb{C}^{d_i} / \mathfrak{S}_{d_i}$$

Example $G = SL_2$ $\quad Z^d = \{(\vec{p}, \vec{q}) \in \mathbb{C}^{2d} \mid \text{no condition}\}$

$$(p(z), q(z)) = (\bar{p}(z) \prod_i (z - a_i), \bar{q}(z) \prod_i (z - a_i))$$

$a_i \in \mathbb{C}$

$$\underbrace{\bar{p}(z), \bar{q}(z)}_{\in \overset{\circ}{Z}{}^{d'}} \text{ have no common div.} \qquad d' \leq d$$

$\{a_i\}$ up to order $\in S^{d-d'}\mathbb{C}$

In general, $\quad Z^\beta \longrightarrow Z^\alpha \quad \beta \leq \alpha$

add "defect $= \underset{\in \mathbb{C}}{0}$ with mult $(\beta - \alpha)$"

$$Z := \varinjlim Z^\alpha$$

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$\mathbb{T} = \mathbb{C}^\times \times T \curvearrowright Z^\alpha, Z$

action on $\mathbb{P}^1$ fixing ∞ $\qquad$ action on $\mathcal{B}$ fixing b_-

IC_{Z^α} : $\mathbb{T}$-equivariant perverse sheaf on Z^α, regarded as a perverse sheaf on Z

Consider $\mathrm{Ext}^*_{\mathbb{T}}\left(\bigoplus_{\alpha} IC_{Z^\alpha}, \bigoplus_{\alpha'} IC_{Z^{\alpha'}}\right)$

ext-algebra

graded

$H^*_{\mathbb{T}}(pt)$-algebra

$\| \; \mathbb{C}[\mathrm{Lie}\,\mathbb{T}]$

$$\begin{tikzcd} Z^\alpha \arrow[r] & Z^\beta \\ Z^{\alpha'} \arrow[ur] & \end{tikzcd} \qquad \beta \geq \alpha, \alpha'$$

$H^*_{\mathbb{C}^\times}(pt) = \mathbb{C}[\hbar]$

☆ representation theoretic side

$\mathfrak{g}^\vee$ = Langlands dual Lie algebra

$\hbar$: variable

$U_\hbar(\mathfrak{g}^\vee)$ = asymptotic universal enveloping alg.

graded $\mathbb{C}[\hbar]$-algebra generated by $\mathfrak{g}^\vee$

relation $XY - YX = \hbar[X, Y]$

$(X, Y \in \mathfrak{g}^\vee)$

$\deg X = 2 \ (X \in \mathfrak{g}^\vee)$, $\deg \hbar = 2$

Set $\hbar = 1 \longrightarrow$ universal enveloping algebra $U(\mathfrak{g}^\vee)$

$\hbar = 0 \longrightarrow S(\mathfrak{g}^\vee)$

filtered algebra

associated graded

($U_\hbar(\mathfrak{g}^\vee)$ is an example of Rees alg of a filtered alg.

$Z_\hbar(\mathfrak{g}^\vee)$ = center of $U_\hbar(\mathfrak{g}^\vee)$

HC $\cup$

$S_\hbar(\mathfrak{t}^\vee)^W \underset{\text{Harish-Chandra}}{\cong} \mathbb{C}[\hbar, \mathfrak{t}]^W$ $\left(\begin{array}{l}\mathfrak{t} = \mathrm{Lie}\,\mathbb{T} = (\mathrm{Lie}\,T^\vee)^* \\ W = \text{Weyl grp}\end{array}\right.$

$\mathbb{C}[\hbar, \mathfrak{t}] \,\|\, H^*_{\mathbb{T}}(pt)$

Thm. [Braverman–Finkelberg–N, 2007.09799]

$$U_\hbar(\mathfrak{g}^\vee) \underset{Z_\hbar(\mathfrak{g}^\vee)}{\otimes} \mathbb{C}[\hbar, t] \xrightarrow[\underbrace{\quad}_{H^*_T(pt)\text{-}}]{\exists\,\text{alg. hom}} \mathrm{Ext}^*_T\left(\bigoplus_\alpha IC_{Z^\alpha}, \bigoplus_\alpha IC_{Z^\alpha}\right)$$

NB We do not know is surj / inj.

But we know Verma $\leftrightarrow$ std
simple $\leftrightarrow$ simple
LHS RHS

$\rightsquigarrow$ multi. can be computed from RHS.

(sketch of proof for $G = SL_2$)

$i : Z^d \longrightarrow Z^{d+1}$ adding defect at 0

Take f in $\prod_d \mathrm{Ext}^*_T(i_! \underbrace{IC_{Z^d}}_{\mathbb{C}_{Z^d}[2d]}, IC_{Z^{d+1}})$.

$i_! i^! \to \mathrm{id}$ adj

$$i^! \underbrace{IC_{Z^{d+1}}}_{\mathbb{C}_{Z^{d+1}}[2d+2]} = (\mathbb{C}_{Z^d}[2d])[-2]$$

$f \in \prod \mathrm{Ext}^2$ given by $(i_! i^! \to \mathrm{id}) \times (-1)$

e in $\prod_d \mathrm{Ext}^*_{\mathbb{T}}(IC_{Z^{d+1}}, i_* \underbrace{IC_{Z^d}}_{\substack{\| \\ i^* IC_{Z^{d+1}}[-2] \\ ?}})$

$\uparrow$

$id \xrightarrow[\text{adj.}]{} i_* i^*$

$h_\xi = ef - fe$

$$(ef-fe)\underbrace{1_d}_{\substack{\uparrow \\ \mathbb{C}_{Z^d}}} = -\big(e(N_{Z^d/Z^{d+1}}) - e(N_{Z^{d-1}/Z^d})\big)1_d$$

$\uparrow$ normal b'dle of $Z^d \subset Z^{d+1}$

$\mathbb{T} \curvearrowright Z^d \cong \underbrace{\mathbb{C}^{2d}}_{\substack{\| \\ \{(\vec{p}, \vec{q})\}}}$ $\quad$ wt = ?

$$f(z) = \frac{p(z)}{q(z)} \mapsto a^{-1} f(tz) = \frac{t^{-d} a^{-1} p(tz)}{t^{-d} q(tz)}$$

$a \in \mathbb{C}^\times = T$

$t \in \mathbb{C}^\times$

$\rightsquigarrow$ $h_\xi = -(\lambda + (2d+1)h)$ on IC_{Z^d}

$d \in \mathbb{Z}_{\geq 0}$

highest wt $d=0$: $-\lambda \underbrace{-h}_{\substack{\| \\ -\underbrace{\rho}_{\substack{\| \\ \frac{1}{2}\sum_{\alpha \in \Delta_+} \alpha}} h}}$

<u>Rem</u>. This shift $-\rho h$ is natural and compatible with HC isomorphism.

$W \curvearrowright \{x_4$ $\quad$ $\mu = w\lambda$ $\quad$ Verma $\quad$ $\bullet$

$\Uparrow$ $\quad$ HC $\quad$ dot action

$$\Downarrow$$

$$w \cdot (\lambda - \rho) = \mu - \rho$$