$[e, f] = ?$ $\quad \mathbb{P}^1_{\text{domain}} \circlearrowleft \mathbb{C}^\times \times \mathbb{C}^\times \curvearrowright \mathbb{P}^1_{\text{target}}$

$$f(z) = \frac{p(z)}{q(z)} \mapsto a\, f(tz) = \frac{t^{-d}\, a\, p(tz)}{t^{-d}\, q(tz)}$$

$$\begin{cases} p(z) = p_{d-1} z^{d-1} + \cdots + p_0 \\ q(z) = z^d + q_{d-1} z^{d-1} + \cdots + q_0 \end{cases}$$

$$\mapsto \begin{cases} a t^{-1} p_{d-1} z^{d-1} + \cdots + a t^{-d} p_0 \\ z^d + t^{-1} q_{d-1} z^{d-1} + \cdots + t^{-d} q_0 \end{cases}$$

$\therefore$ wts : $a\, t^{-1}, a\, t^{-2}, \ldots, a\, t^{-d}$

$t^{-1}, t^{-2}, \ldots, t^{-d}$

difference between d vs $d+1$ (for ef)

$a\, t^{-d-1}, t^{-d-1}$ $\quad \log a = \lambda$

$\therefore \quad -(\lambda - (d+1)t_1)(d+1)t_1$

" $\quad d$ vs $d-1$ (for fe)

$-(\lambda - d t_1) d t_1$

$\therefore ef - fe \; : \; \underbrace{-t_1 (\lambda}_{h} - \underbrace{(2d+1)}_{\rho^\vee})$

$\| $

$t_1 h$

$$\underbrace{U_\hbar(\mathfrak{g}^\vee)} \longrightarrow \mathrm{Ext}^*_{\mathbb{T}}\left(\bigoplus IC_{Z^\alpha}, \bigoplus IC_{Z^\alpha}\right) \qquad \mathbb{T} = \mathbb{C}^\times \times T$$

$$\uparrow$$

center $Z_\hbar(\mathfrak{g}^\vee)$ $\qquad$ $H^*_{\mathbb{T}}(pt)$ – algebra

$$\cong \qquad\qquad\qquad\qquad \|$$

$$\mathbb{C}[\mathrm{Lie}\,\mathbb{T}]$$

$$S_\hbar(\mathfrak{t}^\vee) = \mathrm{Sym}(\mathfrak{t}^\vee \oplus \mathbb{C}\hbar) \cong$$

$\mathfrak{t}^\vee \subset \mathfrak{g}^\vee$

Cartan subalgebra

~~standard~~ universal module : $i_0 : \underset{\underset{pt}{\|}}{Z^0} \hookrightarrow Z^\alpha$

$$H^*_{\mathbb{T}}(i_0^! IC_{Z^\alpha}) \leftarrow U_\hbar(\mathfrak{g}^\vee)\text{-module}$$

——— , ———

specialization $\quad \xi \in \mathrm{Lie}\,\mathbb{T} \quad ev_\xi : \mathbb{C}[\mathrm{Lie}\,\mathbb{T}] \longrightarrow \mathbb{C}_\xi$

Consider $U_\hbar(\mathfrak{g}^\vee) \otimes_{Z_\hbar(\mathfrak{g}^\vee)} \mathbb{C}_\xi \longrightarrow \mathrm{Ext}^*_{\mathbb{T}}(\quad) \otimes_{H^*_{\mathbb{T}}(pt)} \mathbb{C}_\xi$

$$\uparrow$$

analyze by

hyperbolic localization

Fixed pt set = ?

For simplicity, assume $\mathfrak{z} = d\rho_{\mathfrak{z}}$

$$\rho_{\mathfrak{z}} : \mathbb{C}^\times \longrightarrow \mathbb{C}^\times \times T$$
$$t \longmapsto (\underbrace{t^{m}}_{=1}, \lambda(t))$$

λ: regular dominant

$$(\mathcal{Z}^\alpha)^{\rho_{\mathfrak{z}}(\mathbb{C}^\times)} = ?$$

$\cap$ "maps $\mathbb{P}^1 \longrightarrow \mathcal{B}$ = flag variety"

$\mathcal{B}^T \cong W$: Weyl group

$b_- \longmapsto e$

$$\left(\begin{array}{ll} \mathcal{B}_w := \overline{U \cdot w} & \text{Schubert cell} \\ \mathcal{B}^w := \overline{U^- \cdot w} & \text{opposite Schubert cell} \\ \mathcal{B} \supset \mathcal{B}_e : \text{open cell} & \end{array}\right.$$

$$(\overset{\circ}{\mathcal{Z}}{}^\alpha)^{\rho_{\mathfrak{z}}(\mathbb{C}^\times)} \ni f(z) \qquad t^\lambda f(tz) = f(z) \quad {}^{\forall} t \in \mathbb{C}^\times, z \in \mathbb{P}^1$$

$\Longrightarrow$ f is uniquely determined by $p = f(1) \in \mathcal{B}$

$\hookrightarrow f(t) = t^{-\lambda} f(1) = t^{-\lambda} p$

$f(0), f(\infty)$: determined as "limit".

$$\lim_{t \to \infty} t^{-\lambda} p = \underset{\substack{\parallel \\ e}}{b_-} \Rightarrow p \in \mathcal{B}_e$$

$$\lim_{t\to 0} t^{-\lambda} p = \underset{\substack{\uparrow \\ \mathcal{B}^T}}{f(0)} \overset{\exists}{=} \underset{\substack{\uparrow \\ \cong W}}{w} \qquad p \in \mathcal{B}^w$$

$$\therefore\ p \in \mathcal{B}_e \cap \mathcal{B}^w$$

Furthermore $\deg f = \alpha \underset{\substack{\uparrow \\ \text{calculation}}}{=} \lambda - w\lambda$

$f^*\mathcal{O}_{\mathcal{B}}(\mu)$: $\mathbb{C}^\times$-equivi. line bdle over $\mathbb{P}^1$

wts of fibers over $0, \infty$ are determined.

$\Longrightarrow$ w is determined by α

$$(Z^\alpha)^{\mathbb{P}_3(\mathbb{C}^\times)} = \coprod_{\beta \le \alpha} \Big(\mathring{Z}^\beta \times \underbrace{S^{\alpha-\beta}\mathbb{C}}_{\mathbb{P}^1 \text{ domain}}\Big)^{\mathbb{P}_3(\mathbb{C}^\times)}$$

automatically $(\alpha-\beta)[0]$ $\uparrow$ $(\mathbb{C})^{\mathbb{C}^\times}$

$$(\mathring{Z}^\beta)^{\mathbb{P}_3(\mathbb{C}^\times)} = \mathcal{B}_e \cap \mathcal{B}^y \qquad \underset{\substack{\wedge\!\| \\ \alpha}}{\beta} = \lambda - y\lambda$$

Take union over β

$$\Big(Z = \lim_{\substack{\longrightarrow \\ \alpha}} Z^\alpha\Big)^{\mathbb{P}_3(\mathbb{C}^\times)} = \mathcal{B}_e = \coprod_{y \in W} \underline{\underline{\mathcal{B}_e \cap \mathcal{B}^y}}$$

Hyperbolic localization gives IC sheaves of

$\Rightarrow$ KL multiplicity formula

Need to show $H^*_{\mathbb{T}}(i_\lambda^! \bigoplus_\alpha IC_{Z^\alpha}) \otimes_{H^*_{\mathbb{T}}(pt)} \mathbb{C}_\xi \cong$ Verma module

as $U_{\hbar}(\check{\mathfrak{g}})$-module

- comparison of characters

---

§ geometric Satake correspondence

Ref. Zhiwen Zhu : 1603.05593

Lusztig, Ginzburg, Mirkovic–Vilonen, Beilinson–Drinfeld, ……

G : a complex reductive group

$\mathcal{O} = \mathbb{C}[[z]] \subset K = \mathbb{C}((z))$

$\operatorname{Spec} \mathcal{O} = D$ formal disk $\qquad \operatorname{Spec} K = D^\times$ formal punctured disk

$G(\mathcal{O}) \subset G(K)$

Gr_G : affine grassmannian $= G(K)/G(\mathcal{O})$

"∞-dim'l" partial flag variety $\mathbb{G}/\mathbb{P}$ ($\mathbb{G}$ cpx red, $\mathbb{P}$ parabolic)

$\mathbb{G} \overset{\text{diagonal}}{\curvearrowright} \mathbb{G}/\mathbb{P} \times \mathbb{G}/\mathbb{P}$

$\overset{1:1}{\longleftrightarrow}$ (orbits) $\mathbb{P} \curvearrowright \mathbb{G}/\mathbb{P}$

$G(\mathcal{O})$-orbits in $G(\mathcal{K})/G(\mathcal{O})$

$\underset{1:1}{\longleftrightarrow}$ $G(\mathcal{O})\backslash G(\mathcal{K})/G(\mathcal{O})$ $\qquad T \subset G$ max torus

$\underset{1:1}{\longleftrightarrow}$ $\{ z^\lambda \mid \lambda \in \underset{\text{coweight}}{\mathrm{Hom}(\mathbb{C}^\times, T)}$ dominant $\}$

<u>Ex.</u> $G = GL_n$ (elementary divisors)

$$G(\mathcal{O})\backslash G(\mathcal{K})/G(\mathcal{O}) \ni \begin{bmatrix} z^{\lambda_1} & & 0 \\ & z^{\lambda_2} & \\ 0 & & \ddots \quad z^{\lambda_n} \end{bmatrix}$$

$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$

G_c = max. cpt subgrp

ΩG_c : based (polynomial) loop space $= \{\phi : S^1 \to G_c \mid \phi(1) = e\}$ polynomial

$\mathrm{Pol}(S^1, G_c)/G_c$

<u>Fact</u>. $\Omega G_c \xrightarrow[\simeq]{} G(\mathcal{K})/G(\mathcal{O}) = Gr_G$

<u>Fact</u>. $G(\mathcal{O})z^\lambda$: $G(\mathcal{O})$-orbit $\subset Gr_G$

$\downarrow$

G/P_λ : partial flag variety

vector bdle (finite rank)

$\dim G(\mathcal{O})z^\lambda$
$= 2\langle \rho, \lambda \rangle$

$Gr_G^\lambda := G(\mathcal{O})z^\lambda$

$$\overline{Gr_G^\lambda} := \bigcup_{\mu \leq \lambda} Gr_G^\mu$$

$\overline{Gr_G^\lambda}$ has a structure of a projective variety (in the usual sense)

$\lambda \leq \mu$ $\quad \overline{Gr_G^\lambda} \subset \overline{Gr_G^\mu}$

closed embedding

singular in general

$$Gr_G = \varinjlim_{\lambda} \overline{Gr_G^\lambda}$$ direct limit of proj. varieties

We consider $G(\mathcal{O})$-equivariant perverse sheaves on $\overline{Gr_G^\lambda}$ where λ : arbitrary

11:28 ~

$$\mathrm{Perv}_{G(\mathcal{O})} Gr_G$$

<u>Fact.</u> $\mathrm{Perv}_{G(\mathcal{O})} Gr_G$ is a semisimple abelian category

simple objects : $IC_{\overline{Gr_G^\lambda}}$ (trivial local system)

$\mathrm{Stab}_{G(\mathcal{O})}(\cdot)$ is connected

semisimple : $\mathrm{Ext}^1(IC_{\overline{Gr_G^\lambda}}, IC_{\overline{Gr_G^\mu}}) = 0$

<u>convolution diagram</u>

$$\begin{array}{ccccccc} Gr_G \times Gr_G & \xleftarrow{p} & G(K) \times Gr_G & \xrightarrow{q} & G(K) \times_{G(\mathcal{O})} Gr_G & \xrightarrow{m} & Gr_G \\ \| & & \rotatebox{90}{$\in$} & & & & \rotatebox{90}{$\in$} \\ G(K)/G(\mathcal{O}) & & ([g_1(z)], [g_2(z)]) & & [g_1(z), [g_2(z)]] & \mapsto & [g_1(z) g_2(z)] \end{array}$$

proper

$m: [g_1(z) g_2(z)]$: well-defined

$A, B \in \mathrm{Perv}_{G(\mathcal{O})} \mathrm{Gr}_G$

$A \boxtimes B \in \mathrm{Perv}_{G(\mathcal{O}) \times G(\mathcal{O})} \mathrm{Gr}_G \times \mathrm{Gr}_G$

$p^*(A \boxtimes B) \cong q^*(\underset{\sim}{A \tilde{\boxtimes} B})$ $\in D^b_{G(\mathcal{O})}(G(K) \times^{G(\mathcal{O})} \mathrm{Gr}_G)$

$m_*(A \tilde{\boxtimes} B) =: A * B$

<u>Fact</u>. (m : (stratified) semi-small morphism)

$A * B \in \mathrm{Perv}_{G(\mathcal{O})} \mathrm{Gr}_G$

Moreover $A * B \cong B * A$

$\exists$ natural isom.

- <u>cf</u> 0414 Exercise Hecke alg $(GL_n(K), GL_n(\mathcal{O}))$

commutative.

- Conceptual proof Beilinson-Drinfeld

idea: deformation of Gr_G

$0 \in D \rightsquigarrow 0 \in \mathbb{C}$, z

$\mathrm{Gr}_G \times \mathrm{Gr}_G$ $\in \mathbb{C}$ z_1, z_2 $z_1 \neq z_2$

Gr_G $\longleftarrow$ $z_1 = z_2$

$\therefore$ $\mathrm{Perv}_{G(\mathcal{O})}\mathrm{Gr}_G$: commutative
monoidal
semisimple abelian
category

Th. $\mathrm{Perv}_{G(\mathcal{O})}\mathrm{Gr}_G \cong \mathrm{Rep}\, G^\vee$ $\quad G^\vee$ = Langlands dual grp

$* \longleftarrow \otimes$ $\quad$ Rep : category of finite dim'l rep.

$IC_{\overline{\mathrm{Gr}_G^\lambda}} \longleftrightarrow V(\lambda)$ — irr. highest weight rep.

$\lambda \in \mathrm{Hom}(\mathbb{C}^\times, T)$ dominant coweight $\quad$ $\lambda \in \mathrm{Hom}(T^\vee, \mathbb{C})$ weight

$T^\vee$ = dual torus of T

$V(\lambda)$ can be realized by hyperbolic localization
$= \bigoplus_\mu V_\mu(\lambda)$ weight space decomposition

$T \subset G \qquad T \curvearrowright \mathrm{Gr}_G = \Omega G_c$

$\mathrm{Gr}_G^T = \Omega G_c^{T_c} \qquad t\phi(z)t^{-1} = \phi(z)$
$\phi : S^1 \to G \qquad \forall t, z$, $t \in T_c$, $z \in S^1$

$\phi(z) \in T_c \overset{\text{pole}}{\Longrightarrow} \phi(z) = z^\lambda$

$\therefore\ Gr_G^T \cong \mathrm{Hom}_{grp}(\mathbb{C}^\times, T)$

coweight lattice

$$Gr_G \xleftarrow{g_\pm} X^\pm \xrightarrow{\pi_\pm} Gr_G^T = \mathrm{Hom}_{grp}(\mathbb{C}^\times, T)$$

<u>Fact</u> [MV]

This diagram is hyperbolic semismall w.r.t. $Gr_G = \coprod Gr_G^\lambda$

$$\Longrightarrow \underset{\text{hyperbolic localization}}{\Phi} : \mathrm{Perv}_{G(\mathcal{O})} Gr_G \longrightarrow \mathrm{Perv}\, Gr_G^T = \bigoplus_{\mu \in \mathrm{Hom}_{grp}(\mathbb{C}^\times, T)} \mathrm{Vect}$$

$$\Phi(IC_{\overline{Gr_G^\lambda}}) \,(= V(\lambda)) \cong \bigoplus_\mu V(\lambda)_\mu$$

---

§ quantized Coulomb branch (Braverman-Finkelberg-N)

G : cpx reductive group

$\mathbb{N}$: (finite dim'l) representation of G

$\mathbb{N}$ could be 0.

$\rightsquigarrow\ \mathcal{M}_C = \mathcal{M}_C(G, \mathbb{N})$: affine algebraic variety

$\rightsquigarrow \mathcal{A}_\hbar \equiv \mathcal{A}_\hbar(G, \mathbb{N})$: noncommutative alg. $/\mathbb{C}[\hbar]$

(s.t. $\mathcal{A}_\hbar\big|_{\hbar=0} = \mathbb{C}[\mathcal{M}_C]$ coordinate ring

quantized Coulomb branch

$\mathrm{Gr}_G = G(K)/G(\mathcal{O})$ as before

$\mathcal{T} = G(K) \times_{G(\mathcal{O})} \mathbb{N}(\mathcal{O})$ associated vector b'dle

$G(\mathcal{O}) \curvearrowright \mathbb{N}(\mathcal{O}) = \mathbb{N} \otimes_{\mathbb{C}} \mathbb{C}[[z]]$

∞-rank ($=\dim \mathbb{N}(\mathcal{O})$)

vector b'dle over Gr_G

$$\begin{array}{ccc} \mathcal{T} & \longrightarrow & \mathbb{N}(K) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ [g(z), s(z)] & \longmapsto & g(z)s(z) \end{array}$$

well-defined

Springer resol.

cf. $T^*(G/B) = G \times_B (\mathfrak{g}/\mathfrak{b})^* \longrightarrow \mathfrak{g}^*$

$[g.\xi] \longmapsto g\xi$

$G(K) \curvearrowright \mathcal{Z} = \mathcal{T} \times_{\mathbb{N}(K)} \mathcal{T}$ fiber product

$H_*^{G(K)}(\mathcal{Z})$ + convolution product : define algebra

$G(\mathcal{O}) \curvearrowright \mathcal{R}$

$$\begin{aligned}\mathcal{R} &\subset \mathcal{T} \\ &\| \\ \{[g(z), s(z)] &\mid g(z)s(z) \in N(\mathcal{O})\}\end{aligned}$$

$H_*^{G(\mathcal{O})}(\mathcal{R})$ + convolution product $\hookrightarrow \mathbb{C}[\mathcal{M}_C)$

$\mathcal{A}_h$

By technical resons