§ (quantized) Coulomb branch

G : cpx reductive group

$\mathbb{N}$: representation

$\rightsquigarrow \mathcal{M}_C \equiv \mathcal{M}_C(G, \mathbb{N})$: affine algebraic variety

$\mathrm{Gr}_G = G(\mathcal{K})/G(\mathcal{O})$ $\qquad \mathcal{O} = \mathbb{C}[[z]] \subset \mathcal{K} = \mathbb{C}((z))$.

$$\begin{array}{ll} \mathcal{T} := G(\mathcal{K}) \times_{G(\mathcal{O})} \mathbb{N}(\mathcal{O}) & \text{vector bdle with fiber } \mathbb{N}(\mathcal{O}) \\ \quad \downarrow & \\ \quad \mathbb{N}(\mathcal{K}) \qquad [g, s] \mapsto g(z)s(z) & \end{array}$$

cf. $T^*(G/B) = G \times_B (\mathfrak{g}/\mathfrak{b})^* \longrightarrow \mathfrak{g}^*$

Springer resolution

$\mathcal{Z} := \mathcal{T} \times_{\mathbb{N}(\mathcal{K})} \mathcal{T}$ $\qquad$ fiber product

$G(\mathcal{K}) \rtimes \mathbb{C}^\times \curvearrowright \mathcal{Z}$

$\llcorner$ "loop rotation" $z \mapsto \lambda z$

note : it acts on $G(\mathcal{K})$

We want to consider the convolution algebra

$$H_*^{G(\mathcal{K}) \rtimes \mathbb{C}^\times}(\mathcal{Z})$$

But technical reasons, the construction does <u>not</u> work.

Observe $\mathcal{Z} \cong G(K) \times_{G(\mathcal{O})} \mathcal{R}$

where $\mathcal{R} = \{ [g(z), s(z)] \in \mathcal{T} \mid g(z)s(z) \in N(\mathcal{O}) \}$

$$([g_1(z), g(z)s(z)], [g_1(z)g(z), s(z)]) \longleftarrow\!\shortmid [g_1(z), [g(z), s(z)]]$$

$$([g_1(z), s_1(z)], [g_2(z), s_2(z)]) \longmapsto [[g_1(z), [g_1(z)^{-1}g_2(z), s_2(z)]]$$

s.t. $g_1(z)s_1(z) = g_2(z)s_2(z)$

?. formal application of the induction thm "$H_*^{G(K) \rtimes \mathbb{C}^\times}(\mathcal{Z})$" $\cong H_*^{G(\mathcal{O}) \rtimes \mathbb{C}^\times}(\mathcal{R})$

$$\begin{array}{ccc} \mathcal{R} & \longrightarrow & Gr_G \\ \| & & \| \\ \bigcup \mathcal{R}_{\leq \lambda} & & \bigcup Gr_G^{\lambda} \end{array}$$

$\rightsquigarrow \mathcal{R}_{\leq\lambda}$ / truncated formal power series z^d

finite dim'l

$$\mathcal{R} = \varinjlim_{\lambda} \varprojlim_{d} \mathcal{R}^d_{\leq \lambda}$$

well-defined

<u>convolution diagram</u>

$$\begin{tikzcd}
\mathcal{R} \times \mathcal{R} \arrow[d, phantom, "\cap\, i_1"] & p^{-1}(\mathcal{R}\times\mathcal{R}) \arrow[l] \arrow[r] \arrow[d, phantom, "\cap\, i_2"] & q(p^{-1}(\mathcal{R}\times\mathcal{R})) \arrow[r] \arrow[d, phantom, "\cap\, i_3"] & \mathcal{R} \arrow[d, phantom, "\cap\, i_4"] \\
\mathcal{T} \times \mathcal{R} & G(K) \times \mathcal{R} \arrow[l, "p"'] \arrow[r, "q"] & G(K) \times_{G(\mathcal{O})} \mathcal{R} \arrow[r] & \mathcal{T}
\end{tikzcd}$$

$$([g_1, g_2 s], [g_2, s]) \longleftarrow\!\shortmid (g_1, [g_2, s]) \longmapsto [g_1 g_2, s]$$

<u>Observation</u> $p : G(\mathcal{K}) \times \mathcal{R} \longrightarrow G(\mathcal{K}) \times \underbrace{\mathbb{N}(\mathcal{O}) \times \mathcal{R}}_{\text{smooth}} \longrightarrow \mathcal{J} \times \mathcal{R}$

smooth $\quad$ $G(\mathcal{O})$-bdle

$$\therefore \quad p^* \omega_{\mathcal{J}\times\mathcal{R}} = \omega_{G(\mathcal{K})\times\mathcal{R}}$$

dualizing cpx

$\rightsquigarrow$ pull-back with support

$$H^*_{G(\mathcal{O})\times G(\mathcal{O})}(\omega_{\mathcal{R}\times\mathcal{R}}) = H^*(i_1^! \omega_{\mathcal{J}\times\mathcal{R}})$$

$$\xrightarrow{\text{adj.}} H^*(\underbrace{i_1^! p_*}_{\cong\ p_* i_2^!} \underbrace{p^* \omega_{\mathcal{J}\times\mathcal{R}}}_{\omega_{G(\mathcal{K})\times\mathcal{R}}})$$

$$p_* \underbrace{i_2^! \omega_{G(\mathcal{K})\times\mathcal{R}}}_{"\ \omega_{p^{-1}(\mathcal{R}\times\mathcal{R})}}$$

$$\cong H^*_{G(\mathcal{O})\times G(\mathcal{O})}(p^{-1}(\mathcal{R}\times\mathcal{R}))$$

$\rightsquigarrow$ convolution product is defined.

<u>Th.</u> $H_*^{G(\mathcal{O})}(\mathcal{R})$ is commutative. $\quad a*b$

cf. $A*B \cong B*A$ in geometric Satake

<u>Rem</u> (1) $H_*^{G(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathcal{R})$ is noncommutative

(2) $H^*_{G(\mathcal{O})\rtimes\mathbb{C}^\times}(pt) \cong H^*_{G\times\mathbb{C}^\times}(pt)$ is <u>not</u> in the center of $H_*^{G(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathcal{R})$

mult. is $H^*_{G\times\mathbb{C}^\times}(pt)$-linear in the 1st var, but <u>not</u> in the 2nd var.

But it is true $H^*_{G(\mathcal{O})\rtimes\mathbb{C}^\times}(pt) \longrightarrow H_*^{G(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathcal{R})$

commutative subalgebra

Def. $\mathcal{M}_C = \operatorname{Spec} H_*^{G(\mathcal{O})}(\mathcal{R})$

$T \subset G$ max torus

$\operatorname{Spec} H_G^*(pt) = \mathfrak{t}/W$ $\quad$ $\mathfrak{t} = \operatorname{Lie} T$

$W = $ Weyl group

$\mathcal{M}_C \xrightarrow{\varpi} \mathfrak{t}/W \cong \mathbb{C}^\ell$ $\quad$ $\ell = \operatorname{rank} G$

$$\underbrace{H^*_{G\times\mathbb{C}^\times}(pt)}_{\text{commutative.}} \longrightarrow H_*^{G(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathcal{R})$$

$H^*_{\mathbb{C}^\times}(pt) = \mathbb{C}[\hbar]$

Poisson bracket $\{f, g\} := \dfrac{\tilde{f}\tilde{g} - \tilde{g}\tilde{f}}{\hbar} \mod \hbar$

$H_*^{G(\mathcal{O})}(\mathcal{R}) \supset \varpi^*(H_G^*(pt))$: Poisson commutative subalg.

$\{f, g\} = 0, \ \forall f, g \in \varpi^*(H_G^*(pt))$

Fact. $\mathcal{M}_C^{reg}$: symplectic manifold

$\dim \mathcal{M}_C = 2\ell$

$$H_*^{G(\mathcal{O})}(\mathcal{R}) = H_*^{T(\mathcal{O})}(\mathcal{R})^W$$

$\cdot \ \otimes_{H^*_{T(\mathcal{O})}(pt)} \operatorname{Frac} H_T^*(pt)$

$H^*_{T(\mathcal{O})}(pt) \cong H_T^*(pt)$

$$\underset{\text{localization}}{\cong} H_*^{T(\mathcal{O})}(\mathcal{R}^T)^W \otimes \operatorname{Frac}$$

$$\mathrm{Gr}_G^T \cong (\Omega G_c)^T \cong \mathrm{Hom}(\mathbb{C}^\times, T)$$

$$\mathrm{Gr}_G^T = T(K)/T(\mathcal{O})$$

$$z^\mu = \begin{bmatrix} z^{\mu_1} & & 0 \\ & \ddots & \\ 0 & & z^{\mu_n} \end{bmatrix}$$

discrete

$$\mathbb{N}(\mathcal{O})^T = \mathbb{N}^T(\mathcal{O})$$

$$\mathcal{T}^T = T(K) \times_{T(\mathcal{O})} \mathbb{N}^T(\mathcal{O}) = \mathrm{Gr}_G^T \times \mathbb{N}^T(\mathcal{O})$$

$T(\mathcal{O})$ trivial

$$\mathcal{R}^T = \mathcal{T}^T$$

may forget.

$$H_*^{T(\mathcal{O})}(\mathcal{R}^T) = \underbrace{H_T^*(pt)}_{= \mathbb{C}[\mathfrak{t}]} \otimes \mathbb{C}[\mathrm{Hom}(\mathbb{C}^\times, T)]$$

group ring

☆ localization is compatible with convolution product.

$$\begin{tikzcd} H_*^{G(\mathcal{O})}(\mathcal{R}) \arrow[rr, dashed, leftrightarrow, "\text{isom.}", "\text{over } \mathrm{Frac}\, H_T^*(pt)"'] & & H_*^{T(\mathcal{O})}(\mathcal{R}^T)^W \\ & H_G^*(pt) = H_T^*(pt)^W \arrow[ul] \arrow[ur] & \end{tikzcd}$$

$\overset{\mathrm{Spec}}{\Longrightarrow}$

$$\begin{tikzcd} \mathcal{M}_C \arrow[rr, dashed, leftrightarrow, "\text{birational}"] \arrow[dr, "\varpi"'] & & (\mathfrak{t} \times T^\vee)/W \arrow[dl, "1^{st}\text{ proj.}"] \\ & \mathfrak{t}/W & \end{tikzcd}$$

$T^\vee$ = dual torus

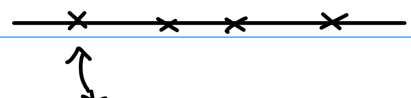
$\mathfrak{t} \times T^\vee$ = cotangent bdle of $T^\vee$

Cor.
- $\dim \mathcal{M}_C = 2\ell$
- generic fiber of $\varpi \cong T^\vee$

Examples

(1) $G = \mathbb{C}^\times$, $N = 0$

$\mathcal{R} = \mathrm{Gr}_G \cong \mathbb{Z}$



$H^*_G(pt) = \mathbb{C}[w]$

$$H_*^{G(\mathcal{O})}(\mathcal{R}) = \bigoplus_{\lambda \in \mathbb{Z}} H^*_G(pt)\, r^\lambda$$

$$r^\lambda \cdot r^\mu = r^{\lambda+\mu}$$

$$H_*^{G(\mathcal{O})}(\mathcal{R}) = \mathbb{C}[w, r^{\pm 1}]$$

$$\mathcal{M}_C = \mathrm{Spec} \quad = \mathbb{C} \times \mathbb{C}^\times$$

quantization: $H_*^{G(\mathcal{O}) \rtimes \mathbb{C}^\times}(\mathcal{R})$

$$r^\lambda \cdot r^\mu = r^{\lambda+\mu}$$

Prop. $r^\lambda w - w \cdot r^\lambda = \underline{\underline{\lambda \hbar}}\, r^\lambda$

$[g_1(z), [g_2(z)]] \mapsto (g_1 g_2(z))$

☺ $\mathrm{Gr}_G \times \mathrm{Gr}_G \leftarrow G_K \times \mathrm{Gr}_G \longrightarrow G_K \times_{G_\mathcal{O}} \mathrm{Gr}_G \longrightarrow \mathrm{Gr}_G$

rep. is twisted by this process //

$$r^\lambda \underset{\sim}{f(w)} = f(w+\lambda\hbar)\, r^\lambda$$

$\ni$ $H^*_G(pt)$ $\therefore$ r^λ is a difference operator //

(2) $G = \mathbb{C}^\times$, $N = \mathbb{C}$ wt 1

$$\begin{array}{ccccc} \mathcal{R} & = & \bigsqcup_{\lambda\in\mathbb{Z}} & z^\lambda \mathbb{C}[[z]] \cap \mathbb{C}[[z]] \\ \cap & & & \cap \\ \mathcal{T} & = & \bigsqcup_{\lambda\in\mathbb{Z}} & z^\lambda \mathbb{C}[[z]] \end{array}$$

- $\lambda \geq 0$ above = below
- $\lambda < 0$ codim $|\lambda|$ subsp.

${}'r^\lambda :=$ fund class of fiber of $\mathcal{T}$ at λ ($\rightsquigarrow [z^\lambda]$ in Gr_G)

$r^\lambda :=$ 〃 $\mathcal{R}$ at λ

$$\therefore\ r^\lambda = \begin{cases} {}'r^\lambda & \lambda \geq 0 \\ w^{-\lambda}\, {}'r^\lambda & \lambda < 0 \end{cases}$$

☆ ${}'r^\lambda \cdot {}'r^\mu = {}'r^{\lambda+\mu}$ $\Longleftarrow$ same computation as in (1).

$$\therefore\ r^1 \cdot r^{-1} = w \cdot \underbrace{{}'r^1\, {}'r^{-1}}_{=1} = w$$

$$\therefore\ H_*^{G(\mathcal{O})}(\mathcal{R}) = \mathbb{C}[w, \underset{r^1}{\underset{\parallel}{x}}, \underset{r^{-1}}{\underset{\parallel}{y}}] / xy = w$$

$$= \mathbb{C}[x,y] \underset{\mathrm{Spec}}{\rightsquigarrow} \mathbb{C}^2$$

$$
\begin{tikzcd}
(x,y)\in\mathbb{C}^2 \arrow[dr, "\varpi"'] & \approx & \mathbb{C}\times\mathbb{C}^\times \arrow[dl, "\Omega"] \\
 & \mathbb{C} &
\end{tikzcd}
$$

$(x,y) \mapsto w = xy$, $\quad (w,r) \mapsto w$

$r = x$
$r^{-1} = w^{-1}y$

$\varpi^{-1}(0) = \{xy = 0\}$

$\varpi^{-1}(t) = \mathbb{C}^\times$
$t \neq 0$

<u>quantization</u> $\quad [r^{1}, r^{-1}] = \hbar$

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✡ If $G = T \implies$ explicit computation is possible.

$\mathcal{M}_C =$ toric hyperkähler manifold.

$\pi_0(\mathrm{Gr}_G) = \pi_0(\Omega G_c) = \pi_1(G)$

$\mathrm{Gr}_G \curvearrowleft G(\mathcal{O})$ each connected component has the smallest orbit

closed, smooth $\quad \overline{\mathrm{Gr}_G^\lambda} = \mathrm{Gr}_G^\lambda$

vector bundle $\downarrow$ G/P_λ

homology class supported on $\mathcal{R}_\lambda = \pi^{-1}(\mathrm{Gr}_G^\lambda)$ can be computed explicitly by localization.

$\leftarrow$ explicit difference

(operator acting on $\mathbb{C}[\mathfrak{t}]^W$

written in terms of

$$(\mathrm{Gr}_G^\lambda)^T \qquad e(T_\mu \mathrm{Gr}_G^\lambda)$$

$\cong$

$W\cdot\lambda \ni \mu$

Weyl group

e.g. Macdonald difference operator.

Recall the setting of quiver Hecke algebra:

$Q = (Q_0, Q_1)$: quiver

V : Q_0-graded vector space $\quad V_{o(h)} \longrightarrow V_{i(h)}$ $\quad \circlearrowleft GL(V_{o(h)})$

$$G = GL_Q(V) = \prod_i GL(V_i)$$

$$N = N_Q(V) = \bigoplus_{h \in Q_1} \mathrm{Hom}(V_{o(h)}, V_{i(h)})$$

quiver Hecke algebra: $H_*^G(\coprod \tilde{\mathcal{F}}_{\vec{\nu}} \times_{N_Q} \tilde{\mathcal{F}}_{\vec{\nu}})$

$$\tilde{\mathcal{F}}_{\vec{\nu}} = G \times_{B_{\vec{\nu}}} \underset{\sim}{N^{\geq 0}}$$

"Borel" $\quad$ upper triangular for the std flag.

cf $\quad H_*^{G(K)}(\mathcal{T} \times_{N(K)} \mathcal{T}) \quad \mathcal{T} = G(K) \times_{G(\mathcal{O})} N(\mathcal{O})$

$\quad \quad \quad \quad \quad \quad \parallel H_*^{G(\mathcal{O})}(R)$

$N^{\geq 0} \subset N$

$\circlearrowleft \quad \quad \circlearrowleft$

$B_{\vec{\nu}}$-invar. $\quad G$-rep.

cf. $\quad N(\mathcal{O}) \subset N(K)$

$\circlearrowleft \quad \quad \circlearrowleft$

$G(\mathcal{O})$-inv $\quad G(K)$-rep.

Thus Coulomb branch for this case is very similar to quiver Hecke algebras

<u>Fact</u>. [BFN] Suppose Q: finite ADE type

$\Rightarrow \mathcal{M}_C \cong \overset{\circ}{Z}{}^\alpha$ $\quad \alpha = \overrightarrow{\dim} V$

$\overset{\circ}{Z}{}^\alpha$: degree α based maps $\mathbb{P}^1 \to \mathcal{B}$

$\mathcal{B}$: flag variety of type Q

We use a certain recipe to show a candidate space is $\mathcal{M}_C$

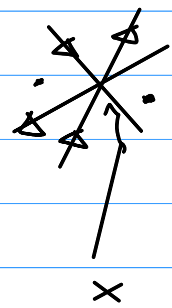
1°. Equip $\overset{\circ}{Z}{}^\alpha \xrightarrow{\varpi} \mathfrak{t}/W$

2°. Show that $\overset{\circ}{Z}{}^\alpha$ is normal and all fibers have the same dimension.
(The corresponding result for $\mathcal{M}_C$ is known)

3°. Define a candiate isomorphism

$$\begin{tikzcd} \Xi : \mathcal{M}_C \arrow[rr, dotted] \arrow[dr] & & \overset{\circ}{Z}{}^\alpha \arrow[dl] \\ & \mathfrak{t}/W & \end{tikzcd}$$

over complements of hyperplanes in $\mathfrak{t}(/W)$
(compatible with $\mathfrak{t} \times \mathbb{P}^\vee/W$)



4°. Show that Ξ extends across

$\bigcup$ hyperplanes $\setminus$ intersections of several hyperplanes

i.e. Ξ is defined over an open subset whose complement is codim ≥ 2 by 2°.

The normality implies Ξ extends.

Remarks.

For 3^0: We use birational coordinates

$$\begin{tikzcd} \mathcal{M}_C \arrow[r, dashed] & \mathfrak{t} \times T^\vee / W \\ \mathring{Z}^\alpha \arrow[ur, dashed] & \end{tikzcd}$$

need to construct.

Δ

For 4^0. locally around a hyperplane \ intersection

$\mathcal{M}_C$ looks like $\underbrace{2^{\dim} \text{ Coulomb branch}}$ $\times\ \mathfrak{t}' \times T'^\vee$

(up to cover)

Show $\mathring{Z}^\alpha$ have the same description.

Quantization

Thm (BFKKNWW)

$$\begin{tikzcd} \mathcal{A}_\hbar \arrow[d] & \text{"shifted Yangian"} \arrow[l] \arrow[dl] \\ \left(\begin{array}{l}\text{ring of}\\ \text{rational}\\ \text{difference operators}\end{array}\right) & \end{tikzcd}$$

quantization of $\mathcal{M}_C$

Fact. surjective (Weekes)

representations of $\mathcal{A}_\hbar$ can be studied by

$\mathcal{R}^{\rho_\xi}$: fixed pt set w.r.t. $\rho_\xi : \mathbb{C}^\times \to T \times \mathbb{C}^\times_{loop}$

$\xi = d\rho_\xi$ $\quad \tau \mapsto (\lambda(\tau), \tau)$

$\lambda : \mathbb{C}^\times \to T$ for brevity

$g(z)s(z) = B(z)$

$\mathbb{N}(K)^{(\lambda(\tau),\tau)}$ fixed

$$\begin{array}{ccc} V_i & \xrightarrow{B_a(z)} & V_j \\ \circlearrowleft & & \circlearrowleft \\ \lambda_i(\tau) & & \lambda_j(\tau) \end{array}$$

$\Updownarrow$

$$\lambda_j(\tau) B_a(z\tau) \lambda_i(\tau)^{-1} = B_a(z) \quad \forall z, \tau$$

$$\Longrightarrow \quad B_a(\tau) = \lambda_j(\tau)^{-1} B_a(z=1) \lambda_i(\tau)$$

$\therefore$ $B_a(z)$ is determined by $B_a(z=1)$

i.e. $\mathbb{N}(K)^{(\lambda(\tau),\tau)} \cong \mathbb{N}$

$Gr_G \cong \Omega G_c$

$\qquad\qquad \Psi$

$\qquad g : S^1 \longrightarrow G_c$

is fixed by $\rho_z \iff \lambda(\tau) g(z\tau) g(\tau)^{-1} \lambda(\tau)^{-1} = g(z)$

$\forall z, \tau \in S^1$

$\therefore\ z \mapsto \lambda(z) g(z)$ is a group homomorphism

$\underset{\text{conj.}}{\sim}\ z^\mu \qquad \mu \in \mathrm{Hom}(\mathbb{C}^\times, T)$

connected component of $Gr_G^{(\lambda(\tau),\tau)}$ containing z^μ

$\cong G \cdot z^\mu \subset Gr_G \cong G/P_\mu$

$\quad \ni g \cdot z^\mu$

parabolic corresponding to μ.

We further have

$(g z^\mu)^{-1} s(z) \in N(\mathcal{O})$ — limit exists when $z \to 0$

$\lambda_j(z)^{-1} B_a(1) \lambda_i(z)$

$\longrightarrow$ preserves the corresponding flag.

$\Longrightarrow$ Get essentially $\widetilde{\mathcal{F}}_{\nu}$ except partial flag.

space used quiver Hecke alg.

perverse canonical $\leftarrow$ Lusztig base