Sheaves on ALE spaces and quiver varieties

Hiraku Nakajima (Kyoto University)

Algebraic Analysis and Around
in honor of Professor Masaki Kashiwara's 60th birthday

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1989 Summer, MSRI  Kronheimer - N

A description of Yang-Mills instantons on an ALE space
  (generalization of Atiyah-Drinfeld-Hitchin-Manin)

In particular,
their moduli space = moduli space of representations
  of a quiver

I didn't have contact with Professor Kashiwara
  at this moment.
1992~93
I defined quiver varieties as a generalization of the moduli spaces.

I've got a letter from Kashiwara:

香略
二の間は面白いかたを有難うございました。
これにつきいくつうちのついて事書きます。

But the stability conditions (parameter used to define the "quotient") are different for instanton moduli spaces and quiver varieties.

Today I want to

① study the change of quiver varieties under move of the stability conditions (stratified Mukai flop, "Jordan" flop),

② give the stability condition for torsion-free sheaves.
Today, I also want to
③ explain an application to the representation theory of affine Lie alg. (my student Nagao)

But I am sure that I will not have time.
  → see math/0703107

④ explain a relation to the representation theory of the rational DAHA, suggested by I. Gordon.

math/0703150
§1. ALE space (constructed by Kronheimer)

\[ \Gamma \subset \text{SL}_2(\mathbb{C}) \] finite subgroup

\[ \rightarrow \text{ADE classification} \]

\[ \rightarrow \text{simple Lie algebra} \]

\[ 0 \in \mathbb{C}^2/\Gamma \] simple singularity

\[ \rightarrow \text{Kronheimer constructed} \]

\text{semuniversal deformation}

\text{and its simultaneous resolution}

\text{as moduli spaces of representations of affine quivers,}

\text{together with hyperkähler metrics}

\[ \mathbb{C}^2/\Gamma = \xrightarrow{=} X_{(0,5\alpha)} \]

\[ X_{(1,0)} = \xrightarrow{=} p \otimes \mathbb{C} \]

\[ (S^1, 0) \]

\[ (S^1, 5\alpha) \]

\[ \text{the real Cartan algebra}. \]
If $S_C = 0$ and $S_R$ is "generic" (explained later)

$\Rightarrow \quad \pi : X_\mathfrak{E} = \mathbb{C}^2 / \mathcal{P} \rightarrow X_0 = \mathbb{C}^2 / \mathcal{P}$

minimal resolution of singularities

$\bar{\mathcal{E}}(0) = \bigcup C_i \quad \text{C}_i \cong \mathbb{P}^1$

Dynkin diagram
framed moduli space of instatons/torsion-free sheaves

= isomorphism classes of

- $A$: anti-self-dual connection on $E$ (C^0 vector bundle with hermitian inner product)
- $E$: torsion-free sheaf

on the compactification

- $\hat{X}_S = X_S \cup \text{pt}/\mathbb{P}$
- $\hat{X}_S = X_S \cup \text{line}/\mathbb{P}$

together with trivialization

- $E|_{pt} \cong E_0$ (framing)
- $E|_{\text{line}} \cong E_0$

- $E_0 = W/\Gamma$: orbifold bundle over pt/\mathbb{P}$
- $E_0 = (\mathcal{O}_k \otimes W)/\Gamma$: orbifold sheaf over line/\mathbb{P}$

where $W$ is a $\Gamma$-module
§2. Quiver varieties

$\mathbf{I}, \mathbf{E}$ : affine Dynkin diagram

$\mathbf{I} = \{ \text{vertexes} \}$, $\mathbf{E} = \{ \text{edges} \}$

$H = \mathbf{E} \cup \mathbf{E}^\circ = \{ \text{oriented edges} \}$

$\mathbf{U}, \mathbf{W} : \mathbf{I}$-graded vector spaces

We identify them with $(\dim V_i), (\dim W_i) \in \mathbb{Z}_{\geq 0}$.

$M(\mathbf{U}, \mathbf{W}) = \bigoplus_{\mathbf{a} \in H} \text{Hom}(U_0, U_{a(a)}) \otimes B_{\mathbf{a}}$

$\bigoplus_{\mathbf{a} \in H} \bigoplus_{i \in \mathbf{I}} \text{Hom}(U_i, U_{a_i}) \otimes \text{Hom}(U_{a_i}, W_i)$

\[ \text{e.g. } A_{2,1} \]

\[ \begin{array}{c}
\mathbf{U}_0 \\
\mathbf{U}_1 \\
\mathbf{W}_1 \\
\mathbf{U}_2 \\
\mathbf{W}_2
\end{array} \]

\[ \begin{array}{c}
B_{1,0} \\
B_{2,1} \\
B_{2,0}
\end{array} \]
group action 
\[ \text{IM}(\mathcal{G}, \mathcal{W}) \rtimes G = \prod_{i \in I} GL(\mathcal{W}_i) \]

moment map 
\[ \mu : \text{IM}(\mathcal{G}, \mathcal{W}) \to \text{Lie} G \]

\[ (B_\alpha, a_i, b_i) \mapsto \sum_{i \in \alpha} \varepsilon(\alpha) \delta_\alpha b_i \]

where \( \varepsilon(\alpha) = \pm 1 \)
\[ H = E \cup i \]
• complex parameter
  \[ S_{\mathbb{C}} = (s_{\mathbb{C}}^{(i)})_{i \in \mathbb{Z}_+} \in \mathbb{C}^I \]
  \[ \mu_{\mathbb{C}}^{-1}(S_{\mathbb{C}}^{(i)}, \text{id}_{V_i}) : \text{level set} \subset G_V \]

• real parameter (stability condition)
  \[ S_{\mathbb{R}} = (s_{\mathbb{R}}^{(i)}) \in \mathbb{R}^I \]

**Def:** \( x = (b, a, b) \) is \( S_{\mathbb{R}} \)-semi stable

\[ \iff \forall S = \oplus S_i \quad \text{I-graded subspace of } V \]

\[ \forall S_i \quad b_i(S_i(a)) \subset S_i(c(a)), S_i \subset \text{ker } b_i \]

\[ \Rightarrow \sum_i s_{\mathbb{R}}^{(i)} \dim S_i \leq 0 \]

\[ < \text{ unless } S = 0 \]

and \( \forall T = \oplus T_i \)

\[ \forall T_i \quad b_i(T_i(a)) \subset T_i(c), \text{ Im } a_i \subset T_i \]

\[ \Rightarrow \sum_i s_{\mathbb{R}}^{(i)} \dim T_i \leq \sum_i s_{\mathbb{R}}^{(i)} \dim V_i \]

\[ < \text{ unless } T = V \]
Write
\[ S = (S_R, S_C) = (R^1 \oplus C^1) = (\text{Im } H)^I \]

We define quiver varieties as quotients:
\[ M^s_S(V, W) \overset{\text{det.}}{=} \mu^{-1}(S_C)^{S_R\text{-semistable}} \sslash G_V \]
\[ \cup \text{ open} \]
\[ M^s_S(V, W) \overset{\text{det.}}{=} \mu^{-1}(S_C)^{S_R\text{-stable}} \sslash G_V \]

\[ M^s_S(V, W) : \text{nonsingular of dim. } = \sum_i 2 \text{dim } V_i \text{dim } W_i - \sum_{i,j} C_{ij} \text{dim } V_i \text{dim } W_j \]
\[ M^s_S(V, W) \setminus M^s_S(V, W) : \text{singularities} \]

Thus it is natural to ask when \( M^s_S \setminus M^s_S = \emptyset \), i.e.,
when \( \exists S_R\text{-semistable, non } S_R\text{-stable point?} \)

**Prop.** If \( S \in (R \oplus C) \otimes D_\theta \)
for any root hyperplane \( D_\theta \subset \theta^{S_R} \)
with \( \theta = \sum \theta_i \text{vol } i \quad \theta_i \leq \text{dim } V_i \)

then \( M^s_S(V, W) = M^s_S(V, W) \).
§3. Wall-crossing

Fix $\Sigma_c$, and move $\Sigma_R$.
\[ \mathcal{R}(\Sigma_c, \nu) \text{ (possibly empty)} \]
\[ \overset{\text{def}}{=} \{ \theta = \sum \theta_i \alpha_i : \text{positive root} \mid \theta_i \leq \dim V_i, \langle \theta, \Sigma_c \rangle = 0 \} \]

Then $\mathbb{R}^I$ has the chamber structure as
\[ \mathbb{R}^I \setminus \bigcup_{\theta \in \mathcal{R}(\Sigma_c, \nu)} D_\theta \]

If $\Sigma_R$ stays in the connected component, then $M_\Sigma$ is unchanged.
Example $A^G_1$, $S_C = 0$

real roots = $\{ n\alpha_0 + (n+1)\alpha_1, (n+1)\alpha_0 + n\alpha_1 \mid n \in \mathbb{Z} \}$

imaginary roots = $\{in\delta \mid n \in \mathbb{Z} \geq 0 \}$

$D_0 = \{S^{(0)}_R + S^{(1)}_R = 0 \}$

(Level 0 type plane)

Fix $V$ so we choose finitely many positive roots $R(S_C, V)$
Suppose $5r$ cross the wall $D_0$.

- $(B,a,b) : \pm 5\text{-stable} \Rightarrow 5\text{-semistable.}$
  \[ \Rightarrow \exists \text{ Jordan-Hölder type filtration by } \]
  \[ \Downarrow \text{filtration invariant under } (B,a,b) \text{ of } \text{gr}_{\chi_{\text{int}}}(B,a,b) : 5\text{-stable} \]
  \[ \Rightarrow \text{Taking direct sum of } \text{gr}_{\chi_{\text{int}}} \]
  \[ \text{we have morphisms } M_{5s}, M_{-5} \rightarrow M_{5s} . \]

- Let $^+x^g : ^+5r\text{-stable, but not } ^-5r\text{-stable ,}$
  $^-x^g : ^-5r\text{-stable, but not } ^+5r\text{-stable .}$
  \[ \Rightarrow \text{the images of } ^+x^g, ^-x^g \text{ are in } M_{5s} \setminus M_{5g} \]
Next, we need to understand $O_5^{sr}$-stable representations. The picture is different for a real root and an imaginary root.

**Case 1.** real root $\Theta = \sum \theta_i \alpha_i$:

- An $O_5^{sr}$-stable representation is either

  a) $M_o^S(V, W)$  \( \forall \Theta 

  b) the unique point $B_\Theta$ in $M_o^S(\Theta, 0)$ (i.e. $\dim V = \Theta$)

  \[
  \begin{pmatrix}
  \text{e.g. } A^{(1)}_{1n} \\
  \theta = n\alpha_0 + (n+1)\alpha_1 \\
  \begin{bmatrix}
  \alpha_0 \\
  \alpha_1
  \end{bmatrix}
  \end{pmatrix}
  \]

  \[
  \begin{pmatrix}
  C^n \rightarrow C^{n+1} \\
  \begin{bmatrix}
  \alpha_0 \\
  \alpha_1
  \end{bmatrix}
  \end{pmatrix}
  \]

\[
M_o^S(V, W) \setminus M_o^S(V, W) = \bigcup_{m > 0} M_o^S(V', W) \times \{ [B^\Theta_\Theta] \}
\]

\[
V' + m\Theta = V
\]

(stratification)
Moreover the lattice has no self-extension.

\[ \Rightarrow \]

- \((B, a, b) \in \delta \) is an extension of the form:
  
  \[ 0 \to (B', a', b') \to (B, a, b) \to B_{\delta}^{\oplus_m} \to 0 \]

\[ M_{\ominus_{5R}}^{5} (v', w) \]

- \((B, a, b) \in \delta \) is an extension of the form:
  
  \[ 0 \to B_{\delta}^{\oplus_m} \to (B, a, b) \to (B', a', b') \to 0 \]

\[ \Rightarrow \text{Over each stratum (i.e., fixed } m \in \mathbb{Z}_{\geq 0}), \]

we have a Grassmann bundle.

\[ \Rightarrow \delta^+ \equiv \bigcup_{m > 0} \bigcup_{v = v' \in M_5} \mathbb{S}(m, \text{Ext}^1(B_{\delta}, (B', a', b'))(B, a, b)) \]

\[ \delta^- \equiv \bigcup_{m} \bigcup_{(B', a', b') \in M_{\ominus_{5R}}^{5}} \mathbb{S}(m, \text{Ext}^1((B', a', b'), B_{\delta})) \]

Thus \( M_{\delta}^{5} \) and \( M_{-5} \) are related by a stratified Mukai flop.
Case 2°: \( \Theta = 8 \) \( \rightarrow \) discussed later
§4. Torsion-free sheaves on an ALE space

The usual stability parameter, which I have been used to construct the Kac-Moody affine Lie algebra action on homology groups, is

$$s_i > 0 \quad \forall i$$

The ALE space $X_5$ is an example of quiver varieties with

$$\begin{cases} V = S \text{ (imaginary root)} \\ W = 0 \\ s \text{ : in the level 0 hyperplane} \end{cases}$$

If $s \notin (\mathbb{R} \oplus \mathbb{C}) \oplus \mathbb{D}$ for the real root $\Theta$, $X_5$ is nonsingular.

Remark. When $W=0$, $C^*(\subset G_v = \text{Lie} \text{GL}(V_i))$ acts trivially on $\text{IM}(V,W)$. So we should consider $G_v/C^*$ instead of $G_v$.

In particular, $\dim = 2 - \sum C_{ij} \dim V_i \dim V_j$

$$= 2 \quad \text{if} \quad V = S$$
$E^R_{\infty} (Kronheimer + N.)$

$s$: as above

$$M^S_s(V,W) = \text{framed moduli space of Yang-Mills instantons on the ALE space } X_s.$$  

Later I found that if $s_C = 0$ and $s^{(i)}_c < 0 \forall i$ 

$$\Rightarrow M^S_s(V,W) \cong \text{framed moduli space of } \Gamma\text{-equivariant torsion-free sheaves on } \mathbb{C}^2.$$  

**Rem. categorical McKay correspondence**

Gonzalez-Sprinberg + Verdier, Kapranov-Vasserot

$$D^b(\Gamma - \text{Gr}(\mathbb{C})) \cong D^b(\text{Coh } \mathbb{C}^{\frac{2}{\Gamma}}).$$
Main Theorem

$s$: parameter for ALE space as above, i.e. level 0 hyperplane in $(\mathbb{R} \oplus \mathbb{C}) \otimes D_0$ for real root $\Theta$

$s$: from the adjacent chamber

\[ M_s(v, w) = \text{framed moduli space of torsion-free sheaves on } X_s \]

NB. $s = 0, w = \mathbb{C}$ at vertex $0$, $V = \mathbb{C}^n$

\[ M_s(v, w) = \text{Hilbert scheme of } n \text{ points on } \mathbb{P}^2 \]

This special case was proved by Kuznetsov.
About proof:

straightforward combination of two techniques

a) Kronheimer + N

b) special case $\Gamma \cong \mathbb{C}^*$ torsion-free sheaf on $\mathbb{C}^2$

(Barth, reproduced in my lecture notes.)

More words on the proof:

We need

1) a resolution of the diagonal $\Delta \subset X_5 \times X_5$.

2) vanishing thin $H^i(\tilde{X}_5, \mathbb{R} \otimes \mathbb{R})$.

Both were proved in Kronheimer + N.
§5. Wall–crossing revisited

Yoshioka, in private communication, tells me that another $\mathcal{M}_{5}(v,w)$s are also framed moduli spaces of sheaves on $X$'s, not necessarily torsion-free, if $\sigma_{v} \cdot s < 0$.

Then the exact sequence

$$0 \rightarrow (B', a', b') \rightarrow (B, a, b) \rightarrow B_{s}^{\oplus m} \rightarrow 0$$

can be identified with the exact sequence in $\text{Coh}(X)$.

$B_{s} \leftrightarrow \text{a torsion sheaf supported on curves}$

After the wall–crossing,

$$0 \rightarrow B_{s}^{\oplus m} \rightarrow (B, a, b) \rightarrow (B', a', b') \rightarrow 0$$

Thus the sheaf corresponding to $(B, a, b)$ contains torsion.
Example $B_0 \leftrightarrow O_{\mathbb{P}^1}(-n)$

for $nS_R^{(a)} + (n+1)S_R^{(a)} = 0$

Case 2°. $\Theta = \delta$ (level 0 Hyperplane)

A $S_{\mathbb{R}}$-stable representation is either

1. $M_5^S(V, W)$ with
2. a point $B$ in $M_5^S(\delta, 0) \cong X_0$
stratification of $M_{05}(v,w)$

$$M_{05}(v,w) \cong \bigsqcup_{n=1}^{\infty} M_{05}^s(v',w) \times \bigcup_{\lambda \vdash \{x_1, x_2, \ldots\}} S_{\lambda}^n X_{05}$$

where $\lambda = (\lambda_1, \lambda_2, \ldots)$, $S_{\lambda}^n X_{05} = \{ \sum \lambda_i x_i \mid x_i : \text{distinct points in } X_{05} \}$

$M_{05} \setminus M_{05}^s$; union of stratum with $|\lambda| \neq 0$.

- different points have no extensions.
- But $\text{Ext}^1(x_i, x_i) \cong T_{x_i} X_{05}$; 2-dimensional
  $\implies$ difference from real root case.
Over each stratum, fibers of the projection $M_{\tilde{5}} \to M_{\tilde{5}}$ are

\[ \{ 0 \text{rank} \to Q \mid \text{Supp} Q = \sum \lambda_i x_i \} \]

Opposite fibers are hard to describe …… ("dual quot scheme")
But isomorphic to the quot scheme as in
\[ \text{Gr}(m, N) \cong \text{Gr}(N-m, N) \] non-canonically.

"Jordan" flop.

If we consider more general quiver varieties, and cross the wall defined by a root $\theta$ with
\[ (\theta, \theta) = 2 - 2g \quad (g \geq 0) \]
we have "(g+1)-Jordan" flop.