Review of geometric Satake correspondence

\[ G : \text{reductive grp} \ \ackslash C \]

\[ \mathcal{K} = C(L(s)) \Rightarrow \mathcal{O} = C(L(s)^\perp) \]

\[ \text{Gr}_G = G(\mathcal{K}) / G(\mathcal{O}) : \text{affine Grassmannian} \]

- \( \mathfrak{g}_0 \)-dimensional variety
- \( \text{Gr}_G \cong \Omega G_{\text{cpt}} \) based loops

\( G(\mathcal{O}) \)-orbits on \( \text{Gr}_G \)
\[ \leftrightarrow \lambda \in \Lambda^+ : \text{dominant coweights} \]

\[ \Lambda^+ \subset \Lambda = \text{coweight lattice of } G = \text{Hom}(G_\text{sc}, T) \subset G(\mathfrak{k}) \]

\[ \cong \text{weight lattice of } \mathfrak{g} \]

\[ \lambda \in \Lambda^+ \leftrightarrow \text{dominant weight of } ^L G \leftrightarrow \text{f.d. irr. rep } \mathfrak{V}(\lambda) \otimes ^L G \]

\[ \text{Gr}_G = \bigsqcup_{\lambda \in \Lambda^+} \text{Gr}_G^\lambda : \text{stratification} \]

(Analog of Schubert cells)

Closure \( \overline{\text{Gr}_G^\lambda} : \) finite dimensional projective variety

Usually singular
$IC(\overline{Gr}_{G^\lambda})$: intersection cohomology complex of $\overline{Gr}_G$ (Goresky-MacPherson) 
(not sheaf, cpx of constructible sheaves)

(extend $C_{Gr_G}$ rather nontrivial way to $G_{G^\lambda}$ so that Poincare duality holds)

$\mathcal{P} = \text{Perv}_{G(\emptyset)}G_{G^\lambda}$: abelian category of $G(\emptyset)$-equiv perv. sheaves on $Gr$

It has a tensor structure via "convolution diagram"

$$G(\emptyset) \times G_G = G_G \times G_G \to G_G$$

$G_G$-b.i.c. over $\overline{Gr}_G$ 

$A \ast B := \text{diag}(A \otimes B)$

Th. (Lusztig, Ginzburg, Beilinson-Drinfeld, Mirkovic-Vilonen)

$(\mathcal{P}, \ast) \cong (\text{Rep}(G^\vee), \otimes)$ as $\otimes$-categories

s.t. $H^\ast(IC(\overline{Gr}_{G^\lambda})) \cong \nabla(\lambda)$

Highest weight representation
How about weight space?

$\mathcal{V}(\lambda)_\mu$: weight space = stalk of $IC(\overline{Gr}_G)$ at $s^\mu \in Gr_G^\mu$

This is the starting point of the geometric Langlands.

More suitable for double affine generalization

$\overline{M}_\mu^\lambda$: transversal slice to $Gr_G^\mu$ in $\overline{Gr}_G^\lambda$

$\overline{M}_\mu^\lambda \cap Gr_G^\mu =: M_\mu^\mu \subset \overline{M}_\mu^\lambda$ open

$\mathcal{V}(\lambda)_\mu \cong IC(\overline{Gr}_G^\lambda)$ at $s^\mu \cong IC(\overline{M}_\mu^\lambda)$ at $s^\mu$

**Question**

What is the affine analog of the affine Grassmann

= double affine Grassmann?

$\mathcal{V}(\lambda)$: $\infty$-dimensional

$\mathcal{V}(\lambda) \otimes \mathcal{V}(\mu)$: $\infty$-direct sum of $\mathcal{V}(\nu)$'s

Consider only integrable highest weight rep.

$\Rightarrow$ controllable $\infty$ sum!
But geometric side: \( \text{Gr}_G \cong \frac{G_{\mathfrak{aff}}(K)}{G_{\mathfrak{aff}}(O)} \)

and orbits are highly co-dimensional!

difficult to define IC sheaves .......

Proposal (Braverman - Finkelberg 07/11/2083)

analog of \( \overline{M}_\mu^\lambda \) = Uhlenbeck partial compactification of \( G \)-instantons on \( IR^+ \times \mathbb{Z}_l \)

\( l = \text{level of the rep. of aff. KM group} \)

\( H^*(IC(\text{analog of } \overline{M}_\mu^\lambda)) \cong \bigoplus \lambda \mu \)

\( \text{rep. of } (G_{\mathfrak{aff}}) \)

• certain diagram \( \longleftrightarrow \otimes \)

explained later
$G$: simple & simply-connected

$\text{Bun}_G^\mathbb{C}(\mathbb{C}^2) = \text{framed moduli space of } G_{\mathbb{C}}\text{-instantons on } S^4 \text{ with } C_2 = \mathbb{R}$

trivialization at $\infty$

$= \text{framed moduli space of algebraic } G\text{-bundles on } \mathbb{C}P^2$

trivialization at $l_\infty \subset \mathbb{C}P^2$

smooth & dim = $2k^4$

$\text{Bun}_G^\mathbb{C}(\mathbb{C}^2) \subset \mathcal{U}_G^\mathbb{C}(\mathbb{C}^2) := \bigsqcup_{0 \leq k' \leq k} \text{Bun}_G^{k'}(\mathbb{C}^2) \times S^{k-k'} \mathbb{C}^2$

Umkehr partial optification

Fix a hom $\mu: \mathbb{Z}_k \to G$

$\cap_{\text{SL}(2) \times \text{GL}(2)}$

$\mathbb{Z}_k \curvearrowright \text{Bun}_G^\mathbb{C}(\mathbb{C}^2) \subset \mathcal{U}_G^\mathbb{C}(\mathbb{C}^2)$

through the action of diagonal emb. to $(\text{ind} \times \mu): \mathbb{Z}_k \to \text{GL}(2) \times G$

fixed pts $=: \text{Bun}_G^\mathbb{C}(\mathbb{C}^2/\mathbb{Z}_k)$
another inv. \( \lambda : \mathbb{Z}_2 \longrightarrow G \) hom. 
action corr. to
the fiber at \( 0 \in \mathbb{C}^2 \)

\[ U_{G, \mu} : \text{fixed pt set} \]

**Technical conjecture**

\( U_{G, \mu} : \text{irreducible} \)

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**Lemma (BF)**

\( \lambda, \mu \in \text{Hom}(\mathbb{Z}_2, G) \) \text{ bijec} \text{t} \text{m} \text{\( \Downarrow \)} \text{ level 2 wts of} \text{\( \bar{G} \)}

\( \bar{G} \) does not contain the degree operator \( d \)
lifts to \( (\bar{G}_\text{aff})^\vee \) unique up to \( \mathbb{C} \mathfrak{g} \)

\( \bar{\lambda}, \bar{\mu} \) lifts s.t. \( \langle \bar{\lambda} - \bar{\mu}, d \rangle = \mathbb{R} e \mathbb{C}_2 \)

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**Main Conjecture**

\[ H^*(IC(U_{G, \mu})) = \mathcal{U}(\bar{\lambda})_{\bar{\mu}} \]
\[ \mathcal{U}(X + c_5^\mu + c_5) \cong \mathcal{U}(X)^\mu \]

- \exists \text{ graded version}
  - LHS: cohomological grading
  - RHS: principal nilpotent

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Tensor product
\[ l = l_1 + l_2 \]
\[ \mathbb{C}^2 / \mathbb{Z}_l \leftarrow X_{l_1, l_2} \leftarrow \]
\[ (l-1) \mathbb{P}^1 \text{'s} = (l_1-1) + (l_2-1) + 1 \]

Consider Okounkov space on \( X_{l_1, l_2} \)
\[ \{ \lambda_1, \lambda_2, \mu \} \subset G, \mu \]
\[ \lambda_1, \lambda_2 : \mathbb{Z}/l_1, \mathbb{Z}/l_2 \rightarrow G \]
level \( l_1, l_2 \) weights
Technical conjecture

\[ \pi : U_{G, \mu}^{\lambda_1, \lambda_2, d}(X_{\lambda_1, \lambda_2}) \to U_{G, \mu}^{\lambda_{1+\lambda_2}}(\mathbb{P}^2) \]

Main Conjecture 2

\[ \pi_* \text{IC}(U_{G, \mu}^{\lambda_1, \lambda_2, d}(X_{\lambda_1, \lambda_2})) = \bigoplus \text{IC}(U_{G, \mu}^{\lambda_1, \lambda_2}) \]

\[ \oplus \text{other} \]

with

\[ (\bigoplus \bigotimes \bigotimes)_{\mu} = \bigoplus \bigotimes \bigotimes \]

\[ \bigotimes \bigotimes \bigotimes \]

The conjectures (except graded version) are true for \( G = \text{SL}(r) \) of MC1

\( G = \text{SL}(r) \) ---- \( U_{G, \mu}^{\lambda, d} \) is an (affine) quiver variety

its IC sheaf was computed

---- related to rep. theory of

\[ \hat{sl}_r \] at level = \( r \)
I. Frenkel level-rk duality

\[ \hat{\mathfrak{sl}}(r)_e \leftrightarrow \hat{\mathfrak{sl}}(2)_r \]

\[ \otimes \leftrightarrow \text{branching to } \hat{\mathfrak{sl}}(1)_e \times \hat{\mathfrak{sl}}(2)_r \]

I develop the theory for the branching in the quiver variety

Remark technical advantage for \( G = \mathfrak{SL}(r) \)

= nice resolution of \( \mathcal{U}_G^\mu \)

(Gieseker quotification)

② quiver variety generalization to other \( \Gamma \subset \mathfrak{SL}(2) \)

\[ \leftrightarrow \text{affine ADE} \]

But the gauge group is \( \mathfrak{SL}(r) \times \mathfrak{GL}(r) \)

Question. What kind of algebraic structure controls e.g. \( G_{E_8} \) – instantons on \( \mathbb{R}^4/\Gamma_{E_8} \)?

I. Frenkel’s joke: Monster?