Perverse Coherent Sheaves
on Blow-up

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BASED ON JOINT WORKS WITH KÔTA YOSHIOKA (KOBE)

Part I  arXiv : 0802.3120
II      : 0806.0463
III     in preparation

AND WITH KEN'ITARO NAGAO

arXiv : 0809.2992

2009.5.19 at IPMU
§ $t$-structure & torsion theory (Review)

$A$: abelian category $\Rightarrow D^b(A)$: derived category

We know many examples

$D^b(A)$ is equivalent to $D^b(A')$ ($A'$: different abelian category).

Examples

- Fourier-Mukai functor
  \[ D^b(Coh A) \cong D^b(Coh A') \]
  $A$: abelian variety $A'$: dual

- (categorical) McKay correspondence
  \[ D^b(Coh^{rig}(X)) \cong D^b(Coh M) \]
  $M$: crepant resolution of $X/\Gamma$

- tilting sheaf
  \[ D^b(Coh X) \cong D^b(mod A) \]
  $A$: (finite dimensional) noncommutative algebra

  subexample
  \[ D^b(Coh P^1) \cong D^b(mod (Kronecker quiver)) \]
Probably the most famous one (among representation theorists)

Riemann-Hilbert correspondence (Kashiwara = 柏原)

\[ D^b_{\text{reg}}(D_x) \cong D^b(\text{Cons}(X)) \quad \text{X: complex manifold} \]

regular holonomic D-module

constructible sheaves

\[ \rightarrow \text{proof of Kazhdan-Lusztig conjecture} \]

character formulas of co-diagonal rep. of \( \mathfrak{g} \)

(one of deepest results in RT)

In these examples \( D^b(\mathfrak{a}) \cong D^b(\mathfrak{a}') \), we have

an "exotic" abelian category (i.e. \( \mathfrak{a}' \)) inside \( D^b(\mathfrak{a}) \).

- \( D^b(\text{Coh} A) \supset \text{Coh} A' \)
- \( D^b(\text{Coh} X) \supset \text{mod} A \)
- \( D^b(\text{Cons} X) \supset D^\text{reg}_X \)

\( \text{Coh} A' \supset \text{Coh} A \)

\( \text{mod} A \supset \text{Coh} X \)

\( D^\text{reg}_X \supset \text{Cons}(X) \)
Beilinson-Bernstein-Deligne axiomatized this as $t$-structures.

**Def.** $D$: triangulated category.

A $t$-structure on $D$ is a pair $(D^{\leq 0}, D^{\geq 0})$ of full subcategories such that

1. $D^{\leq -1} \subset D^{\leq 0}, D^{\geq 1} \subset D^{\geq 0}$
2. $\forall X \in D^{\leq 0}, Y \in D^{\geq 1} \Rightarrow \text{Hom}_D(X,Y) = 0$
3. $\forall X \in D \exists$ distinguished triangle

$$X_0 \to X \to \cdots \text{ s.t. } X_0 \in D^{\leq 0} \text{ and } X_1 \in D^{\geq 1}$$

*trivial example*

**Example.** $A$: abelian category, $D^b(A)$: derived category

$D^{\leq 0}(A) := \{ X \in D^b(A) \mid H^i(X) = 0 \forall i < 0 \}$

$D^{\leq 0}(A) := \{ X \mid H^i(X) = 0 \forall i > 0 \}$

$D^{\leq 0}(A), D^{\leq 0}(A)$: standard $t$-structure

But through derived category equivalences, trivial examples give nontrivial examples.
Theorem (BBD) \[ C := \mathcal{D}^{\geq 0} \cap \mathcal{D}^{\leq 0} \] heart (or core) of the t-structure is an abelian category.

- For the standard t-structure, the heart \( C = \mathbf{A} \)
- \( \mathcal{D}^b(\mathbf{A}) \cong \mathcal{D}^b(\mathbf{A}') \Rightarrow (\mathcal{D}^{\geq 0}(\mathbf{A}'), \mathcal{D}^{\leq 0}(\mathbf{A}')) \) : exotic t-structure of \( \mathcal{D}^b(\mathbf{A}) \)
- In general a t-structure may not come from a derived equivalence.

In \( \mathcal{D}^b(\text{Cons} X) \cong \mathcal{D}^b_{\text{r}}(\text{D}X) \),

the t-structure \((\mathcal{D}^{\geq 0}_{\text{r}}(\text{D}X), \mathcal{D}^{\leq 0}_{\text{r}}(\text{D}X))\) can be defined entirely in the language of \( \mathcal{D}^b(\text{Cons} X) \).

This definition makes sense e.g. when \( X/\mathbb{F}_p \) : finite field \( \implies \) theory of (mixed) perverse sheaves

many applications to RT.
not only the sol. of KL conjecture.
Remark. I do **NOT** review Bridgeland’s stability condition. But one of several equivariant definitions is based on $t$-structures.

Stability condition

$t$-structure + $\mathbb{Z}$: central charge on $C$: Heat

s.t. Harder-Narasimhan filtration exists
Another example of $t$-structures (studied in representation theory of noncommutative algebras)

**Def.**: abelian category

A torsion pair $(\mathcal{T}, \mathcal{F})$ is a pair of full subcategories such that

1. $\text{Hom}(T, F) = 0$ \quad $\forall T \in \mathcal{T}, \forall F \in \mathcal{F}$
2. $\forall X \in \mathcal{A}$ \quad exists short exact sequence
   
   $0 \to T \to X \to F \to 0$

$T = \text{torsion part of } X$

$F = \text{free part of } X$

**Example**

$\mathcal{A} = \text{Coh} \mathcal{X}$

$\mathcal{T} = \text{torsion sheaves} = \{ E \in \text{Coh} \mathcal{X} \mid \text{stalk at generic pt } = 0 \}$

$\mathcal{F} = \text{torsion free sheaves}$

$E \in \text{Coh} \mathcal{X} \Rightarrow T(E) = \{ s \in E \mid f s = 0 \text{ for } f \in \mathcal{O}_X \}$

$\text{torsion part}$
**Theorem (Happel-Reiten-Small)**

\[ D = D^b(\mathcal{A}) \]

\[ D^{\leq 0} := \{ X \in D \mid H^i(X) = 0 \text{ for } i > 0, \quad H^0(X) \in \mathcal{F} \} \]

\[ D^{> 0} := \{ X \in D \mid H^i(X) = 0 \text{ for } i < -1, \quad H^{-1}(X) \in \mathcal{F} \} \]

\[ \Rightarrow (D^{\leq 0}, D^{> 0}) : t \text{-structure} \]

\[ \mathcal{B} := D^{= 0} \cap D^{> 0} : \text{abelian category} \]

\( (\mathcal{F} \oplus [1], \mathcal{F}) : \text{torsion pair in } \mathcal{B} \)

\[ D^b(\mathcal{A}) \]

\[ \mathcal{A}(\mathcal{F}) \]

\[ \mathcal{A} \mathcal{C}(\mathcal{F}) \]

\[ \mathcal{B} \]

\[ \mathcal{B}[1] \]

**Exercise** What is \( \mathcal{B} \) for \( \text{Coh } \mathbb{P}^1 \to (\text{torsion, torsion free}) \)?

**Remark.** This transition \( \mathcal{A} \to \mathcal{B} \) occurs when we cross a certain wall in the space of stability conditions.
§ Review of Bridgeland's perverse coherent sheaves

( another source & Bridgeland stability condition )

Side Remark. Several other people use "perverse coherent sheaves" in other context.

Bridgeland's motivation

Want to show $\mathcal{D}^b(\text{Coh} Y) \cong \mathcal{D}^b(\text{Coh} Y^+) \text{ for } Y \times \text{flip of 3-folds}$

via Fourier-Mukai transform

If it is possible, $Y^+ = \{ \exists \gamma(y) | y \in Y^+ \}$

a skyscraper sheaf

\[ Y^+ : \text{moduli space of objects in } \mathcal{D}^b(\text{Coh} Y). \]

To define moduli space, we need to use the geometric invariant theory, abelian category + slope (+ quas schemes etc.)

He constructed an abelian category $\text{Perv}(Y/X) \subset \mathcal{D}^b(\text{Coh} Y)$ perverse coherent sheaves
\( p : Y \to X \)

- birational, \( \dim p^!(x) \leq 1 \ \forall x \) (cf. Bryan's talk)
- \( Rp_* \mathcal{O}_Y = \mathcal{O}_X \)

\[
\begin{align*}
C & := \{ K \in \text{Coh} Y \mid Rp_* K = 0 \} \\
\mathcal{J} & := \{ E \in \text{Coh} Y \mid R^i p_* E = 0, \text{Hom}(E, K) = 0 \ \forall K \in C \} \\
\mathcal{F} & := \{ E \in \text{Coh} Y \mid p_* E = 0 \}
\end{align*}
\]

\( \to (\mathcal{J}, \mathcal{F}) : \text{torsion pair of Coh} Y \)

\[
\text{Pen}_n(Y/X) := \{ E \in D^b(\text{Coh} Y) \mid H^i(E) = 0 \ \forall i \neq 0, -1 \}
\]

\( H^0(E) \in \mathcal{J}, H^{-1}(E) \in \mathcal{F} \)

abelian category

This definition looks artificial, but it turns out to be correct, after some studies......

Starting observation: \( E \in \text{Pen}_n(Y/X) \Rightarrow Rp_* (E) : \text{coherent sheaf on} X \)

In fact, afterwards Van den Bergh: \( \text{Pen}_n(Y/X) \subset \text{Coh}(X) \)

coherent \( \mathfrak{A} \)-modules \( \mathfrak{A} \): noncommutative \( \mathcal{O}_X \)-algebra
Next we construct moduli space of stable perverse coherent sheaves.

This part is highly technical. A modification of the GIT quotient construction of moduli spaces (of Simpson). Also it will become unnecessary if powerful machinery will be developed in more general situation.

**Step 1.** Define a stability condition on $\text{Perv}(Y/X)$ by using Hilbert polynomials.

Under a mild assumption, show that

$$E: \text{semistable} \Rightarrow H^i(E) = 0$$

i.e., $E$: coherent sheaf

Very roughly, $H^i(E)[1] \subset E$ violates the semistability inequality.

**Step 2.** Follow Simpson's argument.

Anyway we have moduli space of stable perverse coherent sheaves $\text{MP}(X/Y)$ and a morphism $\text{MP}(X/Y) \to \text{M}_H(X)$
Remark.

Perverse coherent sheaves are "semi-global" objects.
They are local w.r.t. X, but not w.r.t. Y.

For example, on the conifold $Y = \text{Tot}(\mathcal{O}_\mathbb{P}(1) \oplus \mathcal{O}_\mathbb{P}(-1)) \rightarrow \{xy = 2w^2 = 0\}$
ordinary ideal sheaves can be studied by the topological vertex

But this is not possible in $\text{Perv}(Y/X)$.
§ Coherent sheaves on blow-up

\( X \): nonsingular projective surface over \( \mathbb{C} \)

\( H \): ample line bundle (\( \cdots \text{Kähler metric} \) )

\( O \in X \): a point (fixed)

\( p: \hat{X} \to X \) blow-up of \( X \) at \( O \)

\( \mathbb{P} \to O \)

locally \( \hat{X} \) : total space of \( \mathcal{O}_{\mathbb{P}}(-1) \) over \( \mathbb{P} \)

\( \text{Tot} \mathcal{O}_{\mathbb{P}}(H) \subseteq \text{Tot} (\mathcal{O}_{\mathbb{P}}^\oplus) = \mathbb{P} \times \mathbb{C}^2 \to \mathbb{C}^2 \)

Zero section: \( \mathbb{P} \to O \)

\( p^*H \): not ample since \( \langle p^*H,[C]\rangle = 0 \)

but \( H_\varepsilon = p^*H - \varepsilon \text{P.D.}[C] \) : ample for \( \forall \varepsilon > 0 \)

\( H_\varepsilon \) : approximation of \( p^*H \)

**Problem**

Study relations between

- moduli space \( \hat{M} \) & \( H \)-stable sheaves on \( \hat{X} \)
- \( \hat{M} \) & \( H^\varepsilon \)-stable sheaves on \( \hat{X} \)

for sufficiently small \( \varepsilon > 0 \)
Rem. moduli of sheaves ... algebro-geometric model of moduli space of instantons

Thus these are related to Donaldson invariants for C^4-mfd.
(brother of DT invariants)

Witten twisted N=2 SUSY Yang-Mills
Why do we bother blow-up?

- Most basic operation in birational geometry of complex surfaces (or smooth 4-manifold)

  - More specifically
    - Fintushel-Stern's blowup formula of Donaldson invariants is given by theta functions $\mapsto$ Seiberg-Witten curves for $N=2$ SUSY YM

  - Proof of Nekrasov conjecture on instanton counting (N-Yoshioka 2003)
  - Proof of the wall-crossing formula of Donaldson invariants with cpx surfaces with $p_g=0$
    - via modular forms (Göttsche - N-Yoshioka + T. Mochizuki 2006)

- Hope to have relations to Donaldson-Thomas type invariants for (Calabi-Yau) 3-fold.
  - Possibly relevant to refined version

Remark: blow-up of a Calabi-Yau surface is not CY.
Warm-up (rank 1 case: Hilbert scheme of points)

\[ \mathcal{X}^{[n]} : \text{Hilbert scheme of } n \text{ points in } X \]
\[ = \{ \mathfrak{I} \subset \mathcal{O}_X : \text{ideal sheaf, colength } = n \} \]

crepant resolution of \( S^nX = X^n/\mathcal{S}_n \)

e.g. \( \mathcal{X}^{[2]} \): either 2 distinct points \( \{x, y\} \)
on projectified tangent vector \( C\mathfrak{u} \subset T_xX \)

We have a morphism \( S^n\mathcal{X} \to S^nX \).

But we do not have \( \mathcal{X}^{[n]} \to \mathcal{X}^{[n]} \).

e.g. \( n=2 \)

\[ \begin{array}{c}
\mathbb{C}^2 \\
\downarrow \phi \\
\mathbb{C}^2
\end{array} \quad \begin{array}{c}
\mathbb{C}^2 \\
\downarrow \phi \\
\mathbb{C}^2
\end{array} \]

We cannot map \( (C\mathfrak{u} \subset T_x\mathcal{X}) \in \mathcal{X}^{[2]} \) to \( \mathcal{X}^{[2]} \)
as \( dp(\mathfrak{u}) = 0 \).
Therefore \( X[n] \rightarrow X \) is only defined partly, (biration

\( p^*(g) \) does not behave well in families,
\( \text{as } R^* p^*(g) \) may not vanish.

This makes the relation nontrivial and interesting.

For example, we know interesting relations among invariants:

\[ \text{FAMOUS Göttsche formula} \]
\[ \sum_{n=0}^{\infty} e(X^{[n]}) \, c^n = \prod_{d=1}^{\infty} (1 - g^d)^{-e(n)} \]

We have \( e(X) = e(X) + 1 \).

\[ \frac{\sum_{n=0}^{\infty} e(X^{[n]}) \, g^n}{\sum_{n=0}^{\infty} e(X^{[n]}) \, c^n} = \prod_{d=1}^{\infty} \frac{1}{1 - g^d} \quad (\text{essentially Dedekind } \gamma \text{-function}) \]
Perverse coherent sheaves on blow-up

As I told you, there is no morphism $\hat{X}^{[n]} \to X^{[n]}$.

Idea: Use perverse coherent sheaves on blow-up and wall-crossing to analyze relations.

Recall $C := \{ K \in \mathrm{Coh}(Y) \mid R^p\pi_* K = 0 \}$
$\mathcal{C} := \{ E \in \mathrm{Coh}(Y) \mid R^i\pi_* E = 0, \text{Hom}(E, K) = 0 \forall K \in C \}$
$\mathcal{C} := \{ E \in \mathrm{Coh}(Y) \mid p^*_E = 0 \}$
\[
\operatorname{Perv}(Y/X) = \{ E \in D^b(Y) \mid H^i(E) = 0 \text{ for } i \neq 0, -1 \}
\]
\[
H^{-1}(E) \notin \mathcal{C}, \quad H^0(E) \in \mathcal{C}
\]

When $Y = \hat{X}$

Lemma. $C = \{ \mathcal{O}_C(-1)^{\oplus s} \mid s \in \mathbb{Z}_{\geq 0} \}$

Cor. $\operatorname{Perv}(\hat{X}/X) = \{ E \in D^b(\hat{X}) \mid H^i(E) = 0 \text{ for } i \neq 0, -1 \}$
$\pi^*(H^{-1}(E)) = 0, \quad \pi^*(H^0(E)) = 0$
$\pi^*(\mathcal{O}_C(-1)) = 0$
$\text{Hom}(H^0(E), \mathcal{O}_C(-1)) = 0$

In fact, this is automatic.
Let \( c \in H^*(\hat{X}) \).

\[ \hat{M}^0(c) = \text{moduli space of stable perverse coherent sheaves } E \text{ on } \hat{X} \]

with \( \text{ch}(E) = c \)

More generally \( \hat{M}^m(c) := \hat{M}^0(c.e^{-m[c]}) \) i.e. \( E(-mC) \) : stable perverse.

\[ \text{[Rem. category of perverse coherent sheaves is not invariant under } \otimes \Theta(-c)] \]

We have \( \hat{M}^0(c) \rightarrow M^X_H(p\ast c) \).
Main Observation \( (\text{Assume } \text{GCD}(c, (c_1, d))) = 1) \)

1)  
\[ \begin{array}{c} \text{diagram} \quad \hat{\mathcal{M}}^m(c) \leftarrow \hat{\mathcal{M}}^{m+1, \alpha}(c) \rightarrow \hat{\mathcal{M}}^{m+2}(c) \end{array} \]

\[ \text{induced by the wall-crossing} \]

2) \( m \gg 0 \) (compared with \( c \)) \( \text{w.r.t. } H_2 \)
\[ \hat{\mathcal{M}}^m(c) = \text{moduli space of stable torsion-free sheaves on } X \]

3) \( \text{If } (c_1, [c]) = 0, \)
\[ \hat{\mathcal{M}}^0(c) = \hat{\mathcal{M}}^X_{H_2} (k,c) \]
\[ \text{moduli space of stable torsion-free sheaves on } X \]
Suppose \( E \in \hat{M}^m(c) \setminus \hat{M}^{m+1}(c) \).

Then there exists a short exact sequence
\[
0 \to \mathcal{O}_c(-m-1)^{\oplus n} \to E \to E' \to 0
\]
with \( E \subseteq \hat{M}^m(c) \setminus \hat{M}^{m+1}(c) \).

For \( E' \in \hat{M}^{m+1}(c) \setminus \hat{M}^m(c) \),

there exists a short exact sequence
\[
0 \to E' \to E^+ \to \mathcal{O}_c(-m-1)^{\oplus n} \to 0
\]
with \( E' \subseteq \hat{M}^m(c) \setminus \hat{M}^{m+1}(c) \).

\( \hat{M}^{m,m+1}(c) \) parametrizes \( \{ E' \oplus \mathcal{O}_c(-m-1)^{\oplus n} \} \) (\( n \) may vary).

This is the easiest case of the wall-crossing formula since \( \mathcal{O}_c(-m-1) \) does not have the self-extension.
Wall-crossing formula

- Betti numbers (virtual Hodge polynomials)

For simplicity, rank 1 case, i.e., \( C = 1 - N \) \( \mathbb{P}^d \) pt \( (\mathcal{N} \in \mathbb{Z}_{\geq 0}) \)

\[
\sum_{N=0}^{\infty} \frac{P_C(\hat{\mathcal{N}}^{m}(1-N[\text{pt}])) g^N}{\sum_{N=0}^{\infty} P_C(\hat{\mathcal{N}}^{m}(1-N[\text{pt}])) g^N} = \frac{1}{1 - t^2 g^{2m}}
\]

\( \text{Rem.} \quad m \to \infty \) compatible with the famous Göttsche's formula

\[
\sum P_C(X^{m}) g^m = \frac{1}{\prod_{m=1}^{\infty} \frac{1}{1 - t^{2m} g^{m}}}
\]

\( \text{Rem.} \quad I am not sure whether this follows from KS formula. But anyway very easy to prove. Enough to show } X = \text{toric surface use torus action}

\( \text{Rem.} \quad \text{Göttsche's formula can be understood as a representation of Heisenberg algebra (oscillator). How about well-crossing ?} \)
Blow-up formula for Donaldson type invariants

comparing $\sum_{\mathcal{M}^m(c)} \Xi(\varepsilon)$ and $\sum_{\mathcal{M}^{m+1}(c)} \Xi(\varepsilon)$.

$\Xi$: universal family over $\mathcal{M}^m(c)$
$\Xi(\varepsilon)$: a cohomology class constructed from $\Xi$, e.g., $\mathcal{A}(\varepsilon)/\Sigma$, $\Sigma \in H^*(\mathbb{R})$

The integral are not motivic at all.
The approach is based on Modulizaki's master spaces.
(We need to use the ADHM type description by a technical reason.)

\[
\sum_{\mathcal{M}^m(c)} \Xi(\varepsilon) - \sum_{\mathcal{M}^{m+1}(c)} \Xi(\varepsilon)
\]

\[
= i \sum_{j=1}^{\infty} \frac{1}{j!} \int_{\mathcal{M}^{m}(c-j \mathcal{H}(\Theta_{c}(-m-1)))} \operatorname{Res}_{\xi_{j}=0} \cdots \operatorname{Res}_{\xi_{1}=0} \left[ \Xi(\mathcal{E}_{b} \oplus \Theta_{c}(-m-1) \otimes e^{-\xi_{i}}) \right]
\]

\[
\prod_{i=1}^{j} \left( -\xi_{i} + 2 \pi i \right)
\]

\[
\prod_{i=1}^{j} e(\mathcal{E}_{b}, \Theta_{c}(-m-1) \otimes e^{-\xi_{i}}) e(\mathcal{E}_{b} \otimes \Theta_{c}(-m-1) \otimes e^{-\xi_{i}}, \mathcal{E}_{b})
\]

$\mathcal{M}^{m}(c-j \mathcal{H}(\Theta_{c}(-m-1))) \times \{ \oplus_{\xi_{k}} \Theta_{c}(-m-1) \otimes e^{-\xi_{k}} \}$: "piece" of $\mathcal{M}^{m+1}(c)$

$\mathcal{E}_{b}$: universal family for $\mathcal{M}^{m}(c-j \mathcal{H}(\Theta_{c}(-m-1)))$ "holomorphic bundle"
This gives us a recursive formula expressing 
\[ \sum_{M_{H_0}(c)} \mathbb{F}(\mathfrak{c}) \] in terms of 
\[ \sum_{\mathcal{M}_{H_1}(c')} \mathbb{F}(\mathfrak{c}) \] for various \( \mathfrak{c}, \mathfrak{c}'. \)

But they are very complicated.

We can recover FS blow-up formula or its variations, but we need to combine above with something completely different.
Remark: \( X = \text{resolved conifold} \)  

We have a similar wall-crossing formula for DT type invariants \([\text{Nagao - N}]\):

\[
\sum = M(-g)^2 \prod_{m=1}^{\infty} (1 - (-g)^m t^m)
\]

\[
\sum = \prod_{m=1}^{\infty} (1 - (-g)^m t^m)
\]

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\sum = M(-g)^2 \prod_{m=1}^{\infty} (1 - (-g)^m t^m)
\]

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\]