Cluster algebras and canonical base  H.Nakajima

Project
Relate a cluster algebra $A$ to Lusztig’s canonical base / perverse sheaves on the spaces of quiver representations, or related spaces.

Goal
$A$ has a (dual) canonical base $B$ containing all cluster monomials. (In fact, in our example today, $B$ = cluster monomials ?)

- positivity of Laurent expansions with respect to any seed.
- factorization of dual canonical base elements.

Why?
- Original motivation (Fomin-Zelevinsky)
- Canonical base elements should reflect various properties of quiver representations.

So, want to relate canonical base / tilting theory to cluster category

- Also gives a monoidal categorification (Hernandez-Leclerc), as the canonical base is the set of simple objects in an abelian category.
Today, I restrict myself to the very special case:

- $A$ = cluster algebra for \[
\begin{array}{cccc}
\text{frozen vertex} & 0 & 1 & 2 & \cdots & l-1 & l \\
0 & \rightarrow & \rightarrow & \rightarrow & \cdots & \rightarrow & 0
\end{array}
\] type $A_{l+1}$

- $y_0, y_1, \ldots, y_l$ : initial variables
- $y[\alpha]$ : cluster variable corresponding to a positive root $\alpha$ of $A_l$

In the article (0905.0002), I studied a cluster algebra associated with a bipartite quiver. As I want to avoid non-essential complications, I consider the above example, where the same technique applies.

Recently Kimura and Qin announce that they generalize my result to an acyclic cluster algebra.

Cluster category does not fit well with Lusztigs theory. So I enlarge the algebra, instead of the category.

In this example, we use rather ad-hoc construction:

Representations of the quiver

\[
\begin{array}{ccccccc}
1 & 2 & 3 & \cdots & l & l+1 \\
\rightarrow & \rightarrow & \rightarrow & \cdots & \rightarrow & \rightarrow
\end{array}
\]

Initial variable $y_i$ corresponds to indecomposable module

\[
\begin{array}{ccccccc}
0 & \cdots & 0 & C & C & \cdots & C \\
\leftarrow & \leftrightarrow & \leftrightarrow & \cdots & \leftrightarrow & \leftrightarrow
\end{array}
\]
Other cluster variables (and monomials) are given by the **Caldero-Chapoton formula**.

Let \( W = W(1) \oplus W(2) \oplus \cdots \oplus W(k) \) be a graded vector space over \( \mathbb{C} \).

Let \( w_i := \dim W(i). \)

Then

\[
y_i[W] = \frac{1}{y_1^{w_1} \cdots y_k^{w_k}} \sum \operatorname{Euler}(\operatorname{Gr}_W(x)) \prod_i y_i^{w_i - 1} \prod_{i < j} y_i^{w_i - w_j} \]

where \( \operatorname{Gr}_W(x) \) = quiver Grassmann for a general representation \( x \) such that the underlying vector space = \( W \)

and \( v \in \mathbb{Z}_{\geq 0} \) dimension of submodule.

\( x \) indecomposable \( \iff y[W] = y[x] \) is a cluster variable.

I use this CC formula to show that cluster monomials correspond to perverse sheaves on the space of quiver rep's.
Graded quiver varieties ($A_1$ type)

Consider the opposite quiver $\alpha_1 \alpha_2 \ldots \alpha_l \alpha_{l+1}$

$W = W(1) \oplus W(2) \oplus \ldots \oplus W(l+1)$: graded vector space over $\mathbb{C}$

$E_W := \bigoplus_{i} \text{Hom}(W(i+1), W(i)) \ni x = \bigoplus_{i=1}^{l} x_i \quad W(1) \subset W(2) \subset W(3) \subset \ldots \subset W(l+1)$

$G_W = \text{PGL}(W(i))$ the space of quiver representations with bases

We introduce a closed subvariety: (this is an affine graded quiver variety of type $A_1$)

$M^*_0(W) := \{ x \in E_W \mid x^2 = 0 \} \subset E_W$

We will study $G_W$-invariant (constructible) $\mathbb{Z}$-valued functions on $M^*_0(W) \subset E_W$

Let $K(Q_W)$ = the set of all such functions.

**NB.** In the original article, I used constructible sheaves, instead of functions. This is necessary even here for the proof of our main result. But in this exposition, I suppose the audience is not familiar with sheaves, and use functions instead. This has a drawback. I cannot explain what are perverse sheaves. I will only say they are nice constructible functions...
K(Qw) has a basis \( \{ 1_{O(x)} \} \) consisting of characteristic functions of orbits \( O(x) \) through \( x \).

Later we will move \( W \). So we change the notation \( 1_{O(x)} \) to \( M_w(T) \), where \( T \) is the graded vector space, defined by \( T = \text{Im } x \).

K(Qw) has a **nicer** basis \( \{ IC_w(T) \} \), given by the simple perverse sheaf associated with \( O(x) \).

I don't explain what perverse sheaves are. We have
\[
IC_w(T) = M_w(T) + \sum a_{uv} M_w(T') \quad \text{for some } a_{uv} \in \mathbb{Z}
\]
with \( M_w(T') \) corresponding to an orbit in the closure of \( O(x) \).

As notation suggests, \( K(Qw) \) is the Grothendieck ring of an additive category \( Qw \). In fact, \( \text{Hom}(K(Qw), \mathbb{Z}) \) is the module category of a quantified algebra \( Aw \).
The dual of \( \{ IC_w(T) \} \) is a base given by simple modules.

The definition of \( Aw \) is geometric, and in this particular case it is probably possible to give a presentation. But I don't know how to do it in general.
\( \bigoplus_{W} \mathbb{K}(Q_{W}) \) has a structure of cocommutative coalgebra:

Fix \( W \to W' \) and set \( W^2 := \ker. \) Consider the diagram

\[ \begin{array}{c}
\mathbb{K} \\
\downarrow \phi(W,W^2) := \{ x \in M_{0}^{\bullet}(W) \mid x(W^2) \subseteq W^2 \} \hookrightarrow M_{0}^{\bullet}(W) \\
M_{0}^{\bullet}(W') \times M_{0}^{\bullet}(W^2)
\end{array} \]

Define \( \mathbb{K}(Q_{W}) \hookrightarrow \mathbb{K}(Q_{W'}) \otimes \mathbb{K}(Q_{W^2}) \) by

\[ \Delta \psi := \mathbb{K}! \psi \oplus \psi \]

where \( (\mathbb{K}! \psi)(x) = \sum_{m \in \mathbb{Z}} m \cdot \text{Euler}(k^{-1}(x) \cdot \psi(m)) \)

NB. \( \bigoplus_{W} \mathbb{K}(Q_{W}) \) is a \textit{comodule algebra} of Lusztig's construction of \( \mathbb{U}(\mathfrak{g}) \).

We introduce an equivalence relation \( \sim \) on the set of \( \bigcup_{W} \text{IC}_{W}(\mathfrak{g}) \) generated by \( \text{IC}_{W}(\mathfrak{g}) \sim \text{IC}_{W^\perp}(\mathfrak{g}) \) where \( W^\perp = \ker x/\text{Im} x \).

Then we define

\[ R = \{ (f_{W}) \in \prod_{W} \text{Hom}(\mathbb{K}(Q_{W}), \mathbb{Z}) \mid \langle f_{W}, \text{IC}_{W}(\mathfrak{g}) \rangle = \langle f_{W'}, \text{IC}_{W}(\mathfrak{g}) \rangle \} \]

if \( \text{IC}_{W}(\mathfrak{g}) \sim \text{IC}_{W'}(\mathfrak{g}) \).
One can show $R$ is compatible with the comultiplication $\Delta$. Therefore $R$ is an algebra.

It has a base dual to $\{[ICW(0)] : W \text{ graded vector space }\}$. Denote it by $\{L(W)\}$.

I mentioned that $K(\mathcal{O})^* = K(\text{mod } \mathcal{A})$. The idea for $\mathcal{O}$ comes from the fact that there exists a Hopf algebra $\mathcal{U}$ ($U_q(\hat{sl}_2)$: quantum affine $sl_2$ in our example) and a family of homomorphisms

\[
\mathcal{U} \to \mathcal{A}
\]

compatible with $\Delta$:

\[
\begin{array}{ccc}
\mathcal{U} & \longrightarrow & \mathcal{A} \\
\downarrow & & \downarrow \\
\mathcal{A} & \longrightarrow & \mathcal{A}
\end{array}
\]

\[
\begin{array}{ccc}
\Delta & & \Delta \\
\downarrow & \circlearrowleft & \downarrow \\
\mathcal{U} \otimes \mathcal{U} & \longrightarrow & \mathcal{A} \otimes \mathcal{A}
\end{array}
\]

We have $R \cong K(\text{mod } \mathcal{U})$, and $L(W)$ is the class of a simple module.

Thus we have a monoidal categorification of $R$.

Goal $R \cong \mathcal{A}$ so that $L(W) \leftrightarrow$ a cluster monomial corresponding to a generic representation of $\mathcal{T}_W$. 

In order to relate $\mathbb{R}$ with the cluster algebra, we introduce several other spaces:

$$\mathbb{E}_W^*: \text{dual space to } \mathbb{E}_W = \bigoplus_i \text{Hom}(W(i), W(i+1)) \quad W(1) \rightarrow W(2) \rightarrow \ldots \rightarrow W(b+1)$$

Choose another graded vector space $V = V(1) \oplus \ldots \oplus V(n)$.

$$\text{Gr}_V(W) := \{ (x^*, S \subseteq W) \mid S \supseteq V, x^*(s) < 0 \}$$

$$\pi^+: \mathbb{E}_W^* \rightarrow \mathbb{E}_W$$

**fiber of** $\pi^+ = \text{quiver Grassmann}$

$$\text{Gr}_V(W)$$ is a vector bundle over the product $\prod_i \text{Gr}(V_i, W_i)$ & (usual) Grassmann manifolds. ($V_i = \dim V_i$)

It is a subbundle of a trivial bundle $\mathbb{E}_W^* \times \prod_i \text{Gr}(V_i, W_i)$.

We consider annihilator:

$$M^*(V, W) := \{ (x, S \subseteq W) \subseteq \mathbb{E}_W^* \times \prod_i \text{Gr}(V_i, W_i) \mid \langle x, x^* \rangle = 0, \forall x^*_s, s \in S \}$$

$$= \{ (x, S \subseteq W) \mid \text{Im} x < S \subseteq \text{ker} x \}$$

$$x = \begin{bmatrix} * & * \\ 0 & 0 \end{bmatrix}$$
Let \( \pi : M^0(V, W) \rightarrow E_W \) : natural projection

Note \( x^2 = 0 \) if \( x \in \text{image of } \pi \).

Thus \( \pi : M^0(V, W) \rightarrow M_0^0(W) \).

\[ \text{These are graded quiver varieties of type } A_1. \]

non-singular / affine

Let \( \mathcal{P}_W(V) := \pi_!(\mathbb{R} M^0(V, W)) \in K(CW) \).

Lemma \( \text{ch} : \mathbb{R}^\hat{\mathbb{R}} \rightarrow \mathbb{R}^\hat{\mathbb{R}} \)

\( \hat{\mathbb{R}} = \text{all graded vector spaces } V = \mathbb{Z}_{\geq 0}^{\mathbb{R}} \)

\( \text{L}(W) \rightarrow \{ < \mathcal{P}_W(V), \text{L}(W) > \} \) is injective.
Therefore it is enough to calculate

\[ \langle \tau_w(v), L(w) \rangle = \text{coeff. of } IC_w(b) \text{ in } \tau! (1 m^v(v,w)) \]

\[ = m_0 \text{ where } \tau! (1 m^v(v,w)) = \sum_{v'} m_{v'} IC_w(v') \]

**Key Observation**

\[ \tau_w(v) \text{ is related to CC formula via Fourier transform } \Phi : \text{Func}(E_w) \rightarrow \text{Func}(E^*_w) \]

\[ \Phi(\tau_w(v)) = \tau^+_1 \left( 1 g_{w,v}(w) \right) \]

Recall \( \tau^+_1 \left( 1 g_{w,v}(w) \right)(x^*) = E_1 \mu \left( G_{w,v}(x^*) \right) \quad \text{equivalent Grassmann} \)

If \( x^* \) is general, RHS appears in CC formula.

The Fourier transform is defined by \( \Phi(\varphi) = p_2! \left( \varphi^* \varphi \cdot 1_{p} \right) \)

where \( p_2 \left( E \times E^* \right) > p = \{ (x, x^*) \mid \langle x, x^* \rangle \leq 0 \} \)

\[ E \quad E^* \]

\[ \downarrow \quad \downarrow \]

\[ \text{due } \]

\[ \text{annihilator} \]

\[ G_{w,v}(w) \xrightarrow{\text{due}} M_{w,v}(w) \]

\[ E_1 \]
It is known that $\Phi$ maps a simple perverse sheaf to a simple perverse sheaf.

\[ \langle \pi_{W}(T), L(W) \rangle = \text{coeff. of } \pi_{1}^{+}(1_{Gr_{W}(w)}) \text{ in } \Phi(IG_{W}(o)) \]

Now $IG_{W}(o) = 1_{o}$ (analog of $\delta$-function)

\[ \Rightarrow \text{ Fourier transform } \Phi(IG_{W}(o)) = 1_{F_{W}^{\ast}} \text{ (constant function).} \]

All other $\Phi(IG_{W}(T)) = \text{char. func. of an orbit } + \Sigma \text{ smaller}$

that have smaller support $\neq F_{W}^{\ast}$

\[ \Rightarrow \langle \pi_{W}(T), L(W) \rangle = \pi_{1}^{+}(1_{Gr_{W}(w)}) \uparrow \text{ general element} \]

\[ \Rightarrow \text{ CC formula } L(W) \text{ is a cluster monomial.} \]