Plan of lectures

1. Geometric Satake correspondence for Kac–Moody
2. Warm Up
   2.1. quiver varieties of type A and nilpotent orbits
   2.2. triangles and slices
3. Quiver description of Cherkis bow varieties
   弓箭
4. Hanany–Witten transition and its application
5. Analysis of fixed pts
6. Construction of repr. of affine Lie alg. $\widehat{\mathfrak{sl}}(n,\mathbb{C})$
   via Cherkis bow varieties

3, 4 :	N–Takayama	1606.02002
5, 6 :	N	1810.04293
	BFN	1604.03625

Quiver varieties : introduced 25 years ago
studied by many people

bow varieties : introduced 10 years ago
(by Cherkis)
but restarted by NT
$\longrightarrow$ only two papers to read
(+ basics on geometric rep. theory)
studied only by N and students

1. Geometric Satake correspondence for Kac-Moody

quiver $Q = (Q_0, Q_1)$

Q_0: vertex, Q_1: oriented edge

$$\begin{tikzcd} o(h) \arrow[r, "h"] & i(h) \end{tikzcd}$$

V, W : Q_0-graded finite dim'l cpx vector spaces

$$\mathbf{N} \equiv \mathbf{N}(V,W) = \bigoplus_{h \in Q_1} \mathrm{Hom}(V_{o(h)}, V_{i(h)}) \oplus \bigoplus_{i \in Q_0} \mathrm{Hom}(W_i, V_i)$$

e.g.

$$\begin{tikzcd} V_1 \arrow[r, "B_{21}"] & V_2 \\ W_1 \arrow[u, "a_1"] & W_2 \arrow[u, "a_2"'] \end{tikzcd}$$

$$G \equiv G(V) := \prod_{i \in Q_0} GL(V_i) \curvearrowright \mathbf{N}$$

quiver variety $\mathfrak{M} \equiv \mathfrak{M}(V,W)$: hamiltonian reduction of $\mathbf{N} \oplus \mathbf{N}^*$ by G [N]90s

Coulomb branch [N, BFN] 2015

$\rightsquigarrow \mathcal{M} \equiv \mathcal{M}(V,W)$: affine variety $\curvearrowleft \pi_1(G)^\wedge = (\mathbb{Z}^{Q_0})^\wedge \cong T^{Q_0}$

holomorphic symplectic on smooth locus

☆ geometric Satake correspondence for Kac-Moody conjectured by [BFN]

<u>Conj 1</u>. $\mathcal{M}^{T^{Q_0}} = \phi$ or $\{pt\}$

Take a generic 1PS $\chi : \mathbb{C}^\times \longrightarrow T^{Q_0}$ and consider the attracting set

$$\mathcal{A}_\chi \equiv \mathcal{A}_\chi(V,W) := \{ x \in \mathcal{M}_C \mid \lim_{t \to 0} \chi(t) \cdot x \text{ exists} \}$$

If $\mathcal{M}_C$ would be smooth, $\mathcal{A}_X$ is lagrangian
$\leftarrow T^{Q_0}$ preserves the symp. form

<u>Conj 2</u> $\mathcal{M}_C \overset{\exists}{=} \bigsqcup_\alpha \mathcal{M}_C^\alpha$ stratification
symplectic

and $\mathcal{A}_X \cap \mathcal{M}_C^\alpha$: lagrangian

Consider $H_{top}^{BM}(\mathcal{A}_X, \mathbb{C}) = \bigoplus \mathbb{C}[Y]$
Y: irreducible components

<u>Main Conjecture</u>
Assume Q has no loop
$\rightsquigarrow$ $\mathfrak{g}_{KM}$: corresponding Kac-Moody Lie alg. Cartan matrix $= 2I - (\mathrm{adj})$

$$\overset{??}{\Longrightarrow} \bigoplus_V H_{top}^{BM}(\mathcal{A}_X(V, W), \mathbb{C})$$

has a structure of $\mathfrak{g}_{KM}$-representation
- integrable highest weight (hence irreducible)
  h.w. $= \sum \dim W_i \cdot \Lambda_i$
- each summand is a weight space

<u>Remark</u> $H_{top}^{BM}(\mathcal{A}_X(V, W), \mathbb{C}) = \bigoplus \mathbb{C}[Y]$
Y: irreducible component of $\mathcal{A}_X$

$\Longrightarrow$ basis of integrable highest wt representation

Conjecturally the set of irreducible components has a Kashiwara crystal structure

Remark Superficially this is very similar to quiver variety side :

$$\mathfrak{M}(V, W) \xrightarrow{\pi} \mathfrak{M}_0(V, W)$$

quasi proj. affine

$$\mathfrak{L}(V, W) = \pi^{-1}(0)$$

lagrangian

$\bigoplus_V H^{BM}_{top}(\mathfrak{L}(V, W), \mathbb{C})$ has $\mathfrak{g}_{KM}$-module str. integrable h.w $\sum \dim W_i \Lambda_i$

each summand is a wt space

$\coprod_V \mathrm{Irr}\, \mathfrak{L}(V, W)$ has a Kashiwara crystal structure.

Example

$Q : A_1$ i.e. $\circ$

$\dim W = 1$

$$\dim V = 0 \Rightarrow \mathcal{M}_C = pt, \ \mathcal{A}_X = pt$$

$$= 1 \Rightarrow \mathcal{M}_C = \mathbb{C}^2, \ \mathcal{A}_X = \mathbb{C}$$

$$(x, y) \mapsto (tx, t^{-1}y)$$

$$> 1 \Rightarrow \mathcal{M}_C^{\mathbb{C}^\times} = \phi \qquad (\text{NB. } \mathcal{M}_C \neq \phi)$$

$$\therefore \bigoplus_V H_{top}(\mathcal{A}_X(V, W)) = \mathbb{C}[pt] \oplus \mathbb{C}[\mathcal{A}_X \text{ for } \dim V = 1]$$

f e

std 2-dim'l repr. of $\mathfrak{sl}(2)$

Conjectural definition of e_i, f_i ($i \in Q_0$) [N]

$$\chi_i : \mathbb{C}^\times \longrightarrow T^{Q_0 \setminus \{i\}}$$

Conj 3. $\mathcal{M}^{\chi_i(\mathbb{C}^\times)}$ = Coulomb branch for A_1-quiver gauge theory

$$\mathcal{A}_{\chi_i} := \{ x \in \mathcal{M} \mid \lim_{t \to 0} \chi_i(t) x \text{ exists} \}$$

$$\searrow \mathcal{M}^{\chi_i(\mathbb{C}^\times)} = \mathcal{M}_{A_1} \supset \mathcal{A}_{A_1}$$

$$\mathcal{A}_X = \mathcal{A}_{\chi_i} \times_{\mathcal{M}_{A_1}} \mathcal{A}_{A_1} = \{ x \in \mathcal{M} \mid \lim_{t \to 0} \chi_i(x) \text{ exists and it is in } \mathcal{A}_{A_1} \}$$

$\rightsquigarrow$

One can define e_i, f_i from the study of $\mathcal{A}_{A_1}$.

Tensor product

Suppose $W = W^1 \oplus W^2$ decomposition

$\rightsquigarrow \widetilde{\mathcal{M}} \longrightarrow \mathcal{M}$ partial resolution

$\mathbb{C}^* \circlearrowright \quad T_{Q_0} \circlearrowright$ birational projective morphism

equivariant

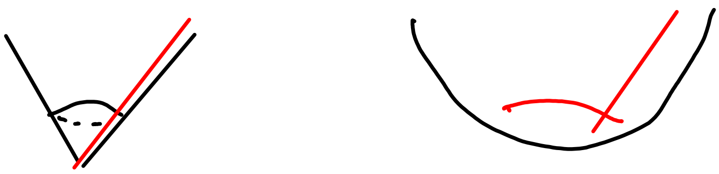
$\widetilde{\mathcal{A}}_x$: attracting set in $\widetilde{\mathcal{M}}$

Main Conj II

$$\bigoplus_{V} H_{top}(\widetilde{\mathcal{A}}_x(V, W = W^1 \oplus W^2), \mathbb{C}) \cong \underset{W^1}{V(\lambda^1)} \otimes \underset{W^2}{V(\lambda^2)}$$

Ex. A_1 $\dim W = 2 = 1+1$
$\dim V = 1$

$$\mathcal{M} = \{xy + z^2 = 0\} \longleftarrow \widetilde{\mathcal{M}} = T^*\mathbb{P}^1$$



$H_{top}(\mathcal{A}_x)$: 1-dim $\qquad$ $H_{top}(\widetilde{\mathcal{A}}_x)$: 2dim

As $\mathfrak{sl}(2)$ rep. $\qquad \mathbb{C}^2 \otimes \mathbb{C}^2 = \underset{\text{3 dim}}{S^2(\mathbb{C}^2)} \oplus \underset{\text{trivial}}{\mathbb{C}}$

The extra component corresponds to the trivial rep.

More generally
$\dim W = m$ $\qquad \underbrace{\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{m} = S^m(\mathbb{C}^2) \oplus \text{smaller}$

can be understand in the same way

$\rightarrow A_1$ case is OK

usual geometric Satake

G : complex reductive group e.g. $SL(n, \mathbb{C})$

$\cup$ $\qquad\qquad\qquad\qquad\qquad\qquad$ $\cup$

T : maximal torus $\qquad\qquad$ $\begin{bmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{bmatrix}$

$$G_{\mathcal{K}} = G(\mathbb{C}((z))) \supset G_{\mathcal{O}} = G(\mathbb{C}[[z]])$$

$$Gr_G = G_{\mathcal{K}} / G_{\mathcal{O}} \quad \text{affine Grassmannian}$$

$\circlearrowleft$

T

$$Gr_G^T = \mathrm{Hom}(\mathbb{C}^\times, T) \ni z^\mu$$

$z^\lambda \in Gr_G^T$ $\qquad$ $\overline{Gr}_G^\lambda := \overline{G_{\mathcal{O}} z^\lambda} \subset Gr_G$

analog of Schubert variety

depending only on $W\lambda$

Weyl group

So assume λ : dominant

Take $\chi : \mathbb{C}^\times \to T$ generic

Consider the attracting set

$$\mathcal{A}(\lambda, \mu) := \{ x \in \overline{Gr}_G^\lambda \mid \lim_{t \to 0} \chi(t) \cdot x = z^\mu \}$$

Th [Mirković-Vilonen]

$\bigoplus_\mu H_{top}(\mathcal{A}(\lambda, \mu))$ has a structure of

a finite dim. Irredecible $G^\vee$-module

Langlands dual group

highest weight $= \lambda$

coweight of G = weight of $G^\vee$

Q : type ADE $\qquad \lambda = \sum \dim W_i \Lambda_i$

$\lambda - \mu = \sum \dim V_i \alpha_i$

Th [Krylov: 1709.0039 based on BFN: 1604.03625]

$\mathcal{A}(\lambda, \mu) = \mathcal{A}_\chi$ for Q, V, W as above

<u>Remark</u> $\otimes$ and reduction to $\mathfrak{sl}(2)$ appeared already in the usual geom. Satake

So the new point is an introduction of $\mathcal{A}_\chi \subset \mathcal{M}$ by Coulomb branch.