2. Warm Up

2.1

$$\begin{tikzcd} V^k \arrow[loop right, "GL(V)"] \arrow[d, shift left, "a"'] \\ W^r \arrow[u, shift left, "b"'] \end{tikzcd}$$

$ab = \zeta$ First assume $\zeta \neq 0$

$\Rightarrow$ b: injective & a: surjective

$\therefore$ $k \leq r$

$(a,b) \mapsto (ga, bg^{-1})$ $g \in GL(V)$

Prop $\zeta \neq 0$ $\{(a,b) \mid ab = \zeta\} / GL(V)$

$$\xleftrightarrow[\text{bij}]{} \left\{ \xi \in End(W) \;\middle|\; \xi \underset{\text{conj}}{\sim} \left[\begin{matrix} \zeta & & & & \\ & \ddots & & 0 & \\ & & \zeta & & \\ & & & 0 & \\ & 0 & & & \ddots \\ & & & & & 0 \end{matrix}\right] \begin{matrix} \}k \\ \\ \}r-k \end{matrix} \right\}$$

(Proof) $\xi := ba \in End(W)$

$\xi^2 = baba = \zeta ba = \zeta \xi$ i.e. $\xi(\xi - \zeta) = 0$

$tr\,\xi = tr(ab) = k\zeta$

$\zeta \neq 0$ $\Rightarrow$ mult. free $\Rightarrow$ $\xi \sim \left[\begin{matrix} \zeta & & & & \\ & \ddots & & 0 & \\ & & \zeta & & \\ & & & 0 & \\ & 0 & & & \ddots \\ & & & & & 0 \end{matrix}\right] \begin{matrix} \}k \\ \\ \}r-k \end{matrix}$

Conversely if $\underset{\underset{ba}{\parallel}}{\xi} \sim \left[\begin{matrix} \zeta & & & & \\ & \ddots & & 0 & \\ & & \zeta & & \\ & & & 0 & \\ & 0 & & & \ddots \\ & & & & & 0 \end{matrix}\right]$ $\Rightarrow$ $W = (\zeta\text{-eigensp}) \oplus (0\text{-eigensp})$

$\xi b = bab = \zeta b$

$\therefore$ up to $GL(V)$ We can set b = inclusion of ξ-eigenspace

$a\xi = \zeta\xi$ a = (proj to ζ-eigenspace) $\times \zeta$ //

Next assume b: injective

Prop2 $\{(a,b) \mid ab = 0,\ b\text{: injective}\} / GL(V)$

$= \{(S, \xi) \mid S \subset W$ k-dim subsp. $\xi \in End(W)$ $\xi(S) = 0$, $Im\,\xi \subset S\}$

$\longrightarrow \quad S = \mathrm{Im}\, b \ , \ \xi := ba$

<u>Exercise</u> : Complete the proof

<u>Prop3</u> $\left\{ (a,b) \;\middle|\; \begin{array}{l} ab = 0 \\ GL(V)\text{-orbit is closed} \end{array} \right\} / GL(V)$

$\underset{\text{bij}}{\longleftrightarrow} \ \overline{N} = \{ \xi \in \mathrm{End}(W) \mid \xi^2 = 0, \ \mathrm{rk}\,\xi \leqq k \}$

(Idea of proof) $\xi = ba \Rightarrow \xi^2 = 0 \ , \mathrm{rk}\,\xi \leqq k$

Suppose $b = \begin{array}{c|c|c} & k' & k-k' \\ \hline k' & \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} & 0 \\ \hline n-k' & 0 & 0 \end{array} \Big\} k' \quad \hat{k}$ $\quad 0 = ab \Leftrightarrow a = \begin{array}{c|c|c} & k' & n-k' \\ \hline k' & 0 & A \\ \hline k-k' & B & C \end{array}$

$ba = \begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}$ Thus A is determined, but B & C are not

Now we use the closed orbit condition

$b = b \cdot \begin{bmatrix} 1 & 0 \\ 0 & t^{-1} \end{bmatrix}$ $\quad \underbrace{}_{k'} \underbrace{}_{k-k'}$ $\qquad \begin{bmatrix} 1 & 0 \\ 0 & t \end{bmatrix} a = \begin{bmatrix} 0 & A \\ tB & tC \end{bmatrix}$

$t \to 0 \quad tB, tC \to 0$

$\therefore$ (a,b) has closed orbit

$\Rightarrow B = C = 0$

(Conversely $B = C = 0 \Rightarrow$ closed orbit) //

<u>Exercise</u> Complete the proof.

<u>Remark4</u> $\xi^2 = 0 \Rightarrow \mathrm{rk}\,\xi \leqq \frac{1}{2} r$ $\quad \therefore \mathrm{rk}\,\xi \leqq k$ is automatic if $k \geqq \frac{r}{2}$

natural to assume $k \leqq \frac{r}{2} \Leftrightarrow k \leqq r - k$

✡ generalization

$$\begin{tikzcd} V_1 \arrow[r, shift left, "C_1"] \arrow[d, shift right, "b"'] & V_2 \arrow[l, shift left, "D_1"] \arrow[r, shift left, "C_2"] & \cdots \arrow[l, shift left, "D_2"] \arrow[r, shift left, "C_{n-1}"] & V_n \arrow[l, shift left, "D_{n-1}"] \\ W \arrow[u, shift right, "a"'] & & & \end{tikzcd}$$

$$\left\{\begin{array}{l} ab = D_1C_1 + \zeta_1 \\ C_1D_1 = D_2C_2 + \zeta_2 - \zeta_1 \\ \quad\vdots \\ C_{n-1}D_{n-1} = \zeta_n - \zeta_{n-1} \end{array}\right.$$

ζ_i : distinct, $\neq 0$

Prop5 $\{\text{sol's}\}/GL(V_1)\times\cdots\times GL(V_n) \longleftrightarrow \left\{\xi \in \mathrm{End}(W) \;\middle|\; \xi \sim \begin{bmatrix} \underbrace{0}_{\substack{\dim W \\ -\dim V_1}} & & & \\ & \underbrace{\zeta_1 \mathrm{id}}_{\substack{\dim V_1 \\ -\dim V_2}} & & \\ & & \ddots & \\ & & & \underbrace{\zeta_n \mathrm{id}}_{\dim V_n} \end{bmatrix}\right\}$

Proof) $n=2$ for brevity.

$$\begin{tikzcd} V_1 \arrow[r, shift left, "C_1"] \arrow[d, shift right, "b"'] & V_2 \arrow[l, shift left, "D_1"] \\ W \arrow[u, shift right, "a"'] & \end{tikzcd}$$

$$D_1C_1 \sim \left[\begin{array}{c|c} (\zeta_2-\zeta_1)\mathrm{id} & 0 \\ \hline 0 & 0 \end{array}\right]$$

$$(D_1C_1)^2 = (\zeta_2-\zeta_1)D_1C_1$$

Set $\xi := ba$ $\quad ab = D_1C_1 + \zeta_1$ $\quad \therefore\ ab \sim \left[\begin{array}{c|c} \zeta_2 \mathrm{id} & 0 \\ \hline 0 & \zeta_1 \mathrm{id} \end{array}\right]$

$$\therefore\ \xi \sim \left[\begin{array}{c|c|c} \zeta_2 \mathrm{id} & & \\ \hline & \zeta_1 \mathrm{id} & \\ \hline & & 0 \end{array}\right] \begin{array}{l} \}\ \dim V_2 \\ \}\ \dim V_1 - \dim V_2 \\ \}\ \dim W - \dim V_1 \end{array} \qquad (ab-\zeta_1)(ab-\zeta_2)=0$$

In fact,

$$\begin{aligned} \xi(\xi-\zeta_1)(\xi-\zeta_2) &= b\underline{a\ (ba-\zeta_1)}(ba-\zeta_2) \\ &= b\underline{(ab-\zeta_1)\ a}(ba-\zeta_2) \\ &= b(ab-\zeta_1)(ab-\zeta_2)a = 0 \end{aligned}$$

//

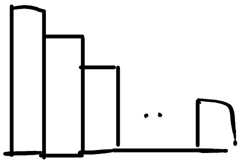
Prop6 $\{\text{sol's s.t. } b, D_1, D_2, \dots, D_{n-1} : \text{injective}\} / \prod GL(V_i)$

$\underset{\text{bijective}}{\longleftrightarrow}$ T^*(partial flag variety $0 \subset S_n \subset S_{n-1} \subset \cdots \subset S_1 \subset W$ } of type $\dim S_i = \dim V_i$

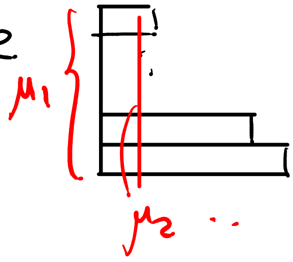
$= \{ (S_\bullet \text{ as above}, \xi \in \mathrm{End}(W) \mid \xi(S_i) = S_{i+1} \}$ where $S_0 = W$, $S_{n+1} = 0$

Next consider generalization of Prop3.
With regard of Rem4,
Assume $\dim W - \dim V_1 \geq \dim V_1 - \dim V_2 \geq \dim V_2 - \dim V_3 \geq \cdots$

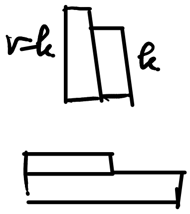
Consider this as a partition



and its transpose



eg.



$\Rightarrow \mu = (\underbrace{2\,2 \cdots 2}_{k}\ \underbrace{1 \cdots 1}_{r-2k})$

Prop7 Let $J_\mu = \begin{bmatrix} \begin{matrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ & & & 0 \end{matrix} \Big\} \mu_1 & & \\ & \begin{matrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ & & & 0 \end{matrix} \Big\} \mu_2 & \\ & & \ddots \end{bmatrix}$

$\Rightarrow \{ \text{sol's s.t. closed } GL(V_1) \times \cdots \times GL(V_n)\text{-orbits} \} / GL(V_1) \times \cdots \times GL(V_n)$

$\longleftrightarrow \overline{J_\mu} = \bigcup_{\nu \leq \mu} J_\nu$ "$\leq$" dominance order
closure in End(W)

2.2 Triangle

$$\begin{tikzcd}
V_1 \arrow[loop above, "B_1"] \arrow[r, "A"] \arrow[dr, "b"'] & V_2 \arrow[loop above, "B_2"] \\
& \mathbb{C} \arrow[u, "a"']
\end{tikzcd}$$

$B_2A - AB_1 + ab = 0$

(S1) $B_1(S_1) \subset S_1$, $A(S_1) = 0 = b(S_1)$

$\Rightarrow S_1 = 0$

(S2) $B_2(T_2) \subset T_2$, $T_2 \supset \mathrm{Im}\, A + \mathrm{Im}\, a$

$\Rightarrow T_2 = V_2$

$\tilde{M} \equiv \tilde{M}_{tri} = \{ (A, B_1, B_2, a, b) \mid \text{above} \}$

$\circlearrowleft G(V) = GL(V_1) \times GL(V_2)$

$v_i = \dim V_i$

Example $V_1 = 0$ (S2) $\Leftrightarrow \mathbb{C}a + \mathbb{C}B_2 a + \cdots = V_2$

$\Leftrightarrow \langle a, B_2 a, \cdots, B_2^{v_2-1} a \rangle$ basis of V_2

$\because$ If $\alpha_1 a + \alpha_2 B_2 a + \cdots + \alpha_k B_2^{k-1} a = B_2^k a$

$k \leq v_2 - 1$

$\Rightarrow B_2^{k+1} a = \alpha_1 B_2 a + \cdots + \alpha_{k-1} B_2^{k-1} a + \underbrace{\alpha_k B_2^k a}_{\in \langle a, B_2 a, \cdots, B_2^{k-1} a \rangle}$

$\vdots$

contradicts with $\mathbb{C}a + \cdots = V_2$

B_2 is represented as
$$\left[\begin{array}{ccc|c} 0 & \cdots & 0 & * \\ \hline 1 & \ddots & 0 & * \\ & \ddots & & \vdots \\ 0 & & 1 & * \end{array}\right]$$

Prop Suppose $V_1 = 0 \Rightarrow \tilde{M}_{tri} \cong GL(V_2) \times \overset{\mathbb{C}^{v_2}}{\overset{\wr\!\wr}{\{\text{matrix of the above form}\}}}$

$GL(V_2) \circlearrowright \xrightarrow{\text{right mult.}}$ u η

$$\begin{cases} u = [a\ B_2 a \cdots B_2^{v_2-1} a]^{-1} \\ \eta = u B_2 u^{-1} \end{cases}$$

Note $\det\left(t - \left[\begin{array}{ccc|c} 0 & & & \eta_1 \\ 1 & \ddots & 0 & \eta_2 \\ & & & \vdots \\ 0 & & 1 & \eta_n \end{array}\right]\right) = t^n - \eta_n t^{n-1} - \cdots - \eta_1$

$$\left(\begin{array}{l} \det \begin{bmatrix} t & -\eta_1 \\ -1 & t-\eta_2 \end{bmatrix} = t^2 - \eta_2 t - \eta_1 \\ \det \begin{bmatrix} t & & -\eta_1 \\ -1 & t & -\eta_2 \\ & -1 & t-\eta_3 \end{bmatrix} = t^3 - \eta_3 t^2 - \eta_2 t - \eta_1 \end{array}\right.$$

∴ eigenvalues with multiplicities determine η

$\underset{GL(V_2)}{\curvearrowright} End(V_2)$ adjoint action $\{$orbits$\} \longleftrightarrow \{$Jordan normal forms$\}$

$End(V_2)/GL(V_2)$ is not Hausdorff as $GL(V_2)$-orbit may not be closed

e.g. $\overline{GL(V_2)\begin{bmatrix} 1 & \ddots & 0 \\ 0 & \ddots & 1 \end{bmatrix}} \ni 0$

More natural to consider $End(V_2) /\!/ GL(V_2) := End(V_2)/\sim$

$A \sim B \iff \overline{GL(V_2)A} \cap \overline{GL(V_2)B} \neq \phi$

In each equivalence class

- $\exists 1$ (smallest) closed orbit …. semisimple
- $\exists 1$ (largest) open orbit …. regular

$$g \left[\begin{array}{ccc|c} 0 & & & \eta_1 \\ 1 & \ddots & & \vdots \\ 0 & & 1 & \eta_n \end{array}\right] = \left[\begin{array}{ccc|c} g_{12} & & g_{1n} & \\ \vdots & & \vdots & * \\ g_{n2} & & g_{nn} & \end{array}\right] \qquad \Big\Updownarrow$$

$$\left[\begin{array}{ccc|c} 0 & & & \eta_1 \\ 1 & \ddots & 0 & \vdots \\ 0 & & 1 & \eta_n \end{array}\right] g = \left[\begin{array}{c|c} \eta_1 g_{n1} & \\ \eta_2 g_{n1} + g_{11} & \cdots \\ \eta_n g_{n1} + g_{n-1,1} & \end{array}\right]$$

g is determined by $\left[\begin{array}{c|c} g_{11} & \\ \vdots & * \\ \underbrace{g_{n1}} & \end{array}\right]$ $\therefore \dim Stab(g) = n$

∴ A matrix of the above form is regular.