☆ Supplement

<u>Remark 4</u>

$$\begin{tikzcd} V^k \arrow[d, shift right, "b"'] \\ W^r \arrow[u, shift right, "a"'] \end{tikzcd}$$

$$M = \{(a,b) \mid ab = 0,\ \text{closed orbit}\} \overset{}{\underset{\text{bij}}{\longleftrightarrow}} \overline{N} = \{\xi \mid \xi^2 = 0,\ \mathrm{rk}\,\xi \leq k\}$$

∘ $\mathrm{rk}\,\xi = k \Rightarrow GL(V) \curvearrowright (a,b)$ free

may assume $b = \left[\begin{array}{c} 1 \ \ddots \ 1 \\ \hline 0 \end{array}\right]$ $\quad bg = b \Rightarrow g = \mathrm{id}$

Conversely if $\mathrm{rk}\,\xi < k$, $GL(V) \curvearrowright (a,b)$: nonfree

(∵ Recall $b = \begin{array}{c} \left[\begin{array}{c|c} 1 \ \ddots \ 1 & 0 \\ \hline 0 & 0 \end{array}\right] \begin{array}{l} \}k' \\ \}r-k' \end{array} \\ \underbrace{\quad}_{k'} \ \underbrace{\quad}_{k-k'} \end{array}$ $(\mathrm{rk}\, b = k')$, $\quad a = \begin{array}{cc} & \begin{array}{cc} k' & r-k' \end{array} \\ & \left[\begin{array}{cc} 0 & \xi \\ 0 & 0 \end{array}\right] \begin{array}{l} k' \\ k-k' \end{array} \end{array}$

$$\begin{array}{r} k'\{ \\ k-k'\{ \end{array} \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & g \end{array}\right] \text{ fixes } a, b$$
$$\qquad\ \underbrace{\quad}_{k'} \ \underbrace{\quad}_{k-k'}$$

∘ If $\exists$ free orbit $\Longrightarrow k \leq r/2$ $(\Leftrightarrow k \leq r-k)$

max rank

2.2 Triangle

$$\begin{tikzcd}
V_1 \arrow[loop above, "B_1"] \arrow[rr, "A"] \arrow[dr, "b"'] & & V_2 \arrow[loop above, "B_2"'] \\
& \mathbb{C} \arrow[ur, "a"'] &
\end{tikzcd}$$

$$B_2 A - A B_1 + ab = 0$$

(S1) $B_1(S_1) \subset S_1$, $A(S_1) = 0 = b(S_1)$
$\Longrightarrow S_1 = 0$

(S2) $B_2(T_2) \subset T_2$, $T_2 \supset \mathrm{Im} A + \mathrm{Im}\, a$
$\Longrightarrow T_2 = V_2$

$$\widetilde{M} \equiv \widetilde{M}_{tri} = \{ (A, B_1, B_2, a, b) \mid \text{above} \}$$

$\circlearrowleft$ $G(V) = GL(V_1) \times GL(V_2)$ $\quad$ ⚠ not taking quotient

$v_i = \dim V_i$

Example $V_1 = 0$ $\quad$ (S2) $\Longleftrightarrow$ $\mathbb{C}a + \mathbb{C}B_2 a + \cdots = V_2$
$\Longleftrightarrow$ $\langle a, B_2 a, \cdots, B_2^{v_2 - 1} a \rangle$ basis of V_2

$\because$ If $\alpha_1 a + \alpha_2 B_2 a + \cdots + \alpha_k B_2^{k-1} a = B_2^k a$
$\qquad k \leq v_2 - 1$

$\Rightarrow$ $B_2^{k+1} a = \alpha_1 B_2 a + \cdots + \alpha_{k-1} B_2^{k-1} a + \underbrace{\alpha_k B_2^k a}_{\cap \ \langle a, B_2 a, \ldots, B_2^{k-1} a \rangle}$

$\vdots$

contradicts with $\mathbb{C}a + \cdots = V_2$

B_2 is represented as

$$\left[\begin{array}{ccc|c}
0 & \cdots & 0 & * \\
\hline
1 & & 0 & * \\
 & \ddots & & \vdots \\
0 & & 1 & *
\end{array}\right]$$

$\mathbb{C}^{v_2}$

Prop 1. Suppose $V_1 = 0 \Rightarrow$ $\widetilde{M}_{tri} \cong GL(V_2) \times \{ \text{matrix of the above form} \}$

$GL(V_2) \circlearrowleft$ $\quad \xrightarrow{\text{right mult.}}$ $u \qquad \eta$

$$(B_2, a) \longmapsto \begin{cases} u = [a \; B_2 a \cdots B_2^{v_2 - 1} a]^{-1} \\ \eta = u B_2 u^{-1} \end{cases}$$

$$(u^{-1} \eta u, \; u^{-1} \left[\begin{smallmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{smallmatrix}\right]) \longleftarrow (u, \eta)$$

Note $\det\left(t - \begin{bmatrix} 0 & & & \eta_1 \\ 1 & \ddots & 0 & \eta_2 \\ & \ddots & & \vdots \\ 0 & & 1 & \eta_n \end{bmatrix}\right) = t^n - \eta_n t^{n-1} - \cdots - \eta_1$

$$\left(\begin{array}{l} \det \begin{bmatrix} t & -\eta_1 \\ -1 & t-\eta_2 \end{bmatrix} = t^2 - \eta_2 t - \eta_1 \\ \det \begin{bmatrix} t & & -\eta_1 \\ -1 & t & -\eta_2 \\ & -1 & t-\eta_3 \end{bmatrix} = t^3 - \eta_3 t^2 - \eta_2 t - \eta_1 \end{array} \right.$$

$\therefore$ eigenvalues with multiplicities determine η

$GL(V_2) \curvearrowright End(V_2)$ adjoint action $\quad \{\text{orbits}\} \longleftrightarrow \{\text{Jordan normal forms}\}$

$End(V_2)/GL(V_2)$ is not Hausdorff as $GL(V_2)$-orbit may not be closed

e.g. $\overline{GL(V_2)\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}} \ni 0$

More natural to consider $End(V_2) /\!/ GL(V_2) := End(V_2)/\sim$

$$A \sim B \iff \overline{GL(V_2)A} \cap \overline{GL(V_2)B} \neq \phi$$

In each equivalence class

- $\exists 1$ (smallest) closed orbit $\cdots\cdots$ semisimple
- $\exists 1$ (largest) open orbit $\cdots\cdots$ regular

$$g\begin{bmatrix} 0 & & & \eta_1 \\ 1 & \ddots & & \vdots \\ 0 & & 1 & \eta_n \end{bmatrix} = \begin{bmatrix} g_{12} & & g_{1n} & * \\ \vdots & & \vdots & \\ g_{n2} & & g_{nn} & \end{bmatrix}$$

$$\Updownarrow$$

$$\begin{bmatrix} 0 & & & \eta_1 \\ 1 & \ddots & 0 & \vdots \\ 0 & & 1 & \eta_n \end{bmatrix} g = \begin{bmatrix} \eta_1 g_{n1} & \\ \eta_2 g_{n1} + g_{11} & \cdots \\ \eta_n g_{n1} + g_{n-1,1} & \end{bmatrix}$$

g is determined by $\begin{bmatrix} g_{11} & \\ \vdots & * \\ g_{n1} & \end{bmatrix}$ $\therefore \dim Stab(g) = n$

$\therefore$ A matrix of the above form is regular.

$$\begin{tikzcd}
 & & \mathrm{End}(V_2)/\!\!/GL(V_2) \arrow[d, "\text{eigenvalues}"'] \arrow[d, phantom, "\cong" right, shift left=3ex] \\
\{\eta \in \mathrm{End}(V_2) \mid \text{regular}\} \arrow[urr, bend left, "\text{surjection}"] & & \mathbb{C}^{V_2} \arrow[ll, color=red]
\end{tikzcd}$$

An above form matrix gives a representative of the equiv. rel. $\sim$

$J = \begin{bmatrix} 0 & & 0 \\ 1 & \ddots & \\ 0 & 1 & 0 \end{bmatrix}$

Also consider $T_N(\mathcal{O}(J))$ (orbit through N) $= \left\{ \left[\xi, \begin{bmatrix} 0 & & 0 \\ 1 & \ddots & \\ 0 & 1 & 0 \end{bmatrix}\right] \;\middle|\; \xi \in \mathfrak{gl}(V_2) \right\}$

$$\| $$

$$\begin{bmatrix} \xi_{12} & \cdots & \xi_{1n} & 0 \\ \vdots & & \vdots & \vdots \\ \xi_{n2} & \cdots & \xi_{nn} & 0 \end{bmatrix} - \begin{bmatrix} 0 & \cdots & 0 \\ \xi_{11} & \cdots & \xi_{1n} \\ \vdots & & \vdots \\ \xi_{n-1,1} & \cdots & \xi_{n-1,n} \end{bmatrix}$$

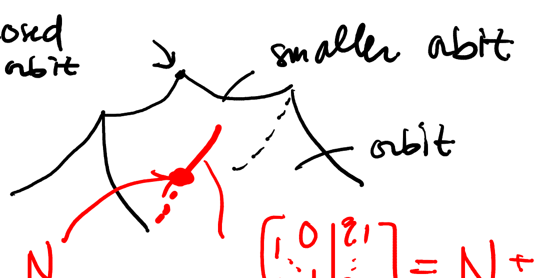
$$\cap \left\{ \left[\begin{array}{c|c} 0 & \begin{matrix} \eta_1 \\ \vdots \\ \eta_n \end{matrix} \end{array}\right] \right\} \Rightarrow \begin{array}{l} \xi_{12} = \cdots = \xi_{1n} = 0, \quad \eta_1 = 0 \\ \xi_{22} = \xi_{11},\ \xi_{23} = \cdots = \xi_{2n} = 0, \quad \eta_2 = \xi_{1n} = 0 \end{array}$$

$$\Rightarrow \eta = 0$$

- codim orbit in $\mathrm{End}(V_2) = \dim \mathrm{Stab} \begin{bmatrix} 0 & & 0 \\ 1 & \ddots & \\ 0 & 1 & 0 \end{bmatrix} = n$

$\therefore T_N \mathrm{End}(V_2)$

$= T_N(\mathcal{O}(J)) \oplus \left[\begin{array}{c|c} 0 & \begin{matrix} * \\ \vdots \\ * \end{matrix} \end{array}\right]$



$$\left[\begin{array}{c|c} \begin{matrix} 0 & & 0 \\ 1 & \ddots & \\ 0 & 1 & 0 \end{matrix} & \begin{matrix} \eta_1 \\ \vdots \\ \eta_n \end{matrix} \end{array}\right] = N + \left[\begin{array}{c|c} 0 & \begin{matrix} \eta_1 \\ \vdots \\ \eta_n \end{matrix} \end{array}\right]$$

This construction is available only in type A.
For general type Kostant, Slodowy considered

Kostant–Slodowy slice

$$\begin{tikzcd}
 & \mathfrak{g}/\!\!/G \arrow[d] \\
\mathfrak{g}_{\mathrm{reg}} \arrow[ur] & \mathfrak{h}/W \arrow[d, phantom, "\cong"] \\
 & \mathbb{C}^{\ell} \arrow[ul, color=red]
\end{tikzcd}$$

We say slice. $\mathcal{M}_{\mathrm{tri}} = GL(V_2) \times \text{slice}$

General case

Rem2

$$\begin{tikzcd} V_1 \arrow[r, "A"] \arrow[loop above, "B_1"] \arrow[dr, "b"'] & V_2 \arrow[loop above, "B_2"] \\ & \mathbb{C} \arrow[u, "a"'] \end{tikzcd} \quad \overset{\text{dual}}{\rightsquigarrow} \quad \begin{tikzcd} V_1^* \arrow[loop above, "{}^tB_1"] & V_2^* \arrow[l, "{}^tA"'] \arrow[loop above, "{}^tB_2"] \arrow[d, "{}^ta"] \\ & \mathbb{C} \arrow[ul, "{}^tb"] \end{tikzcd}$$

(S1) $\Leftrightarrow$ (S2 for dual)

(S2) $\Leftrightarrow$ (S1 for dual)

(S1) $S_1 \subset V_1$ $B_1(S_1) \subset S_1$, $b(S_1) = 0 = A(S_1)$

$\Rightarrow S_1 = 0$

$\Updownarrow$

$S_1^\perp \subset V_1^*$ ${}^tB_1(S_1^\perp) \subset S_1^\perp$, $\mathrm{Im}\,{}^tb \subset S_1^\perp$

$\mathrm{Im}\,{}^tA \subset S_1^\perp$

This is (S2) for dual

$\left(\langle {}^tB_1(\underset{S_1^\perp}{t}), \underset{S_1}{s} \rangle = \langle t, B_1(s) \rangle \right)$

$\Rightarrow S_1^\perp = V_1^*$

Prop3 $\varphi(A, B_1, B_2, a, b) = B_2A - AB_1 + ab$

$d\varphi$: surjective on $\widetilde{\mathcal{M}}$

(:) Suppose $\xi \in \mathrm{Hom}(V_2, V_1) \perp \mathrm{Im}\, d\varphi$.

$\Rightarrow \mathrm{tr}(\xi(\dot{B}_2A + B_2\dot{A} - \dot{A}B_1 - A\dot{B}_1 + \dot{a}b + a\dot{b}))$

$\Rightarrow$ $A\xi = 0$, $\xi A = 0$

$\xi B_2 = B_1 \xi$, $b\xi = 0$, $\xi a = 0$

$\mathrm{Im}\,\xi \subset \mathrm{Ker}\, A$, $\mathrm{Ker}\, b$ B_1-inv.

$\Rightarrow \mathrm{Im}\,\xi = 0$ //

Cor4. $\widetilde{\mathcal{M}}$ smooth and $\dim \widetilde{\mathcal{M}} = v_1^2 + v_2^2 + v_1 + v_2$

Lemma 4 [Takayama] A is full rank

$\because$ Let $\begin{cases} x \in \operatorname{Ker} A \\ y \in \operatorname{Ker} {}^tA \end{cases}$ $\quad 0 = \langle (B_2A - AB_1 + ab)x, y\rangle$

$\qquad = bx \times {}^tay$

If $\exists y\ {}^tay \neq 0 \Rightarrow \forall x \in \operatorname{Ker} A\ \ bx = 0$
If $\exists x\ {}^tbx \neq 0 \Rightarrow \forall y \in \operatorname{Ker}{}^tA\ \ {}^tay = 0$

$\therefore \operatorname{Ker} A \subset \operatorname{Ker} b$ or $\operatorname{Ker}{}^tA \subset \operatorname{Ker}{}^ta$

$\Downarrow \qquad\qquad \Downarrow$

$B_1(\operatorname{Ker} A) \subset \operatorname{Ker} A \qquad {}^tB_2(\operatorname{Ker}{}^tA) \subset \operatorname{Ker}{}^tA$

$\uparrow$
$B_2A - AB_1 + ab = 0$

$\therefore \operatorname{Ker} A = \{0\}$ by (S1) or $\operatorname{Ker}{}^tA = \{0\}$ by (S2)

└ A : injective $\qquad$ └ A : surjective //

Prop 5 [Takayama]

(1) If $V_1 \neq V_2$

$$\widetilde{\mathcal{M}}_{tri} \cong GL(n) \times \left\{ \eta \in \mathfrak{gl}(n) \,\middle|\, \begin{array}{cc} & \begin{matrix} m & \quad n-m \end{matrix} \\ \begin{matrix} m \\ \\ n-m \end{matrix} & \left[\begin{array}{c|c} * & \begin{matrix} 0 & \begin{smallmatrix} * \\ \vdots \\ * \end{smallmatrix} \end{matrix} \\ \hline \begin{matrix} * \cdots * \\ 0 \end{matrix} & \begin{matrix} 1 & & 0 & * \\ & \ddots & & \vdots \\ 0 & & 1 & * \end{matrix} \end{array}\right] \end{array} \right\}$$

$n = \max(V_1, V_2)$
$m = \min(V_1, V_2)$

$$(A, B_1, B_2, a, b) = \left(-u^{-1}\begin{bmatrix} id \\ 0 \end{bmatrix}, -\underbrace{h}_{m\times m \text{ block}}, -u^{-1}\eta u, u^{-1}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, [* \cdots *]\right)$$

(2) If $V_1 = V_2$ $\quad \widetilde{\mathcal{M}}_{tri} \cong GL(n) \times \mathfrak{gl}(n) \times \mathbb{C}^n \times (\mathbb{C}^n)^*$

$$(A, B_1, B_2, a, b) = (u, u^{-1}\eta u, \eta - IJ, I, Ju)$$

(proof) (2) A : isom. $\Rightarrow$ clear

(1) Assume $v_1 < v_2$

A : injective

S2

$$\begin{cases} u^{-1} = [A\ a\ B_2 a \cdots B_2^{v_2-v_1-1} a] \\ \eta = u B_2 u^{-1} \end{cases}$$

$$B_2 A - A B_2 + ab = 0 \quad \Longrightarrow \quad B_2(\mathrm{Im} A) \subset \mathrm{Im} A + \mathrm{Im}\, a$$

$$\therefore \quad \begin{array}{c|cc} & m & n-m \\ \hline m & * & 0\ \ \begin{matrix} * \\ \vdots \end{matrix} \\ n-m & \begin{matrix} * \cdots * \\ 0 \end{matrix} & \begin{matrix} 1 & 0 & \\ & \ddots & \\ 0 & & 1 \end{matrix}\ \ \begin{matrix} \vdots \\ * \end{matrix} \end{array}$$

//

$$J := \begin{bmatrix} 0 & 0 \\ 0 & \underbrace{\begin{matrix} 1 & 0 \\ & \ddots \ 1 \end{matrix}}_{J'} \end{bmatrix} \qquad O(N) : \text{adjoint orbit through } N$$

$$T_J O(J) = \{ [\xi, J] \mid \underset{\begin{bmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{bmatrix}}{\xi} \in gl(n) \} = \left\{ \begin{bmatrix} 0 & \xi_{12} J' \\ -J'\xi_{21} & [\xi_{22}, J'] \end{bmatrix} \right\}$$

<u>Claim</u> $T_J \mathrm{End}(V_2) = T_J O(J) \oplus \begin{bmatrix} * & 0\ \vdots^{*} \\ *\cdots* & 0\ \vdots_{*} \end{bmatrix}$

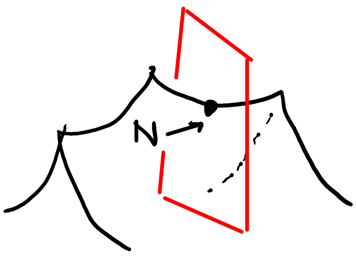
∘ $\cap \begin{bmatrix} * & 0 & \begin{matrix} * \\ \vdots \end{matrix} \\ *\cdots* & & \\ 0 & 0 & \begin{matrix} \vdots \\ * \end{matrix} \end{bmatrix} = 0$

- (m, m)-block $= 0$
- The last column of $\xi_{12} J' = 0$
- The 1st row of $J'\xi_{21} = 0$
- $\{[\xi_{22}, J']\} \cap \left[0 \ \middle| \begin{matrix} * \\ \vdots \\ * \end{matrix} \right] = \{0\}$

∘ $\mathrm{codim}\, O(J) = \dim \mathrm{Stab}(J)$

$$\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & J' \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & J' \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \iff \begin{matrix} g_{12} J' = 0, \ J' g_{21} = 0 \\ g_{22} J' = J' g_{22} \end{matrix}$$

$\therefore\ 2v_1 + v_1^2 + (v_2 - v_1)$ dim



NB slice $\cap$ bigger orbits are no longer a single point

Example6 $\dim V_1 = 1$, $\dim V_2 = n$ $(m=1)$

$$\eta = \begin{bmatrix} a & & & \eta_1 \\ \zeta & 1 & & \vdots \\ & & \ddots 1 & \eta_n \end{bmatrix} \qquad \det(t-\eta) = (t-a)(t^{n-1} - \eta_n t^{n-2} - \cdots - \eta_2) + \zeta\eta_1$$

$$\det \begin{bmatrix} t-a & -\eta_1 \\ -\zeta & t-\eta_2 \end{bmatrix} = (t-a)(t-\eta_2) + \zeta\eta_1$$

$$= t^2 - (a+\eta_2)t + a\eta_2 + \zeta\eta_1$$

$$\det \begin{bmatrix} t-a & 0 & -\eta_1 \\ -\zeta & t & -\eta_2 \\ 0 & -1 & t-\eta_3 \end{bmatrix} = t(t-a)(t-\eta_3) + \zeta\eta_1 - \eta_2(t-a)$$
$$= (t-a)(t^2 - \eta_3 t - \eta_2) + \zeta\eta_1$$

$$\det \begin{bmatrix} t-a & & & -\eta_1 \\ -\zeta & t & & \vdots \\ & -1 & \ddots & t & -\eta_{n-1} \\ & & & -1 & t-\eta_n \end{bmatrix} = (t-a)(t^{n-1} - \eta_n t^{n-2} - \cdots - \eta_2)$$
$$+ \zeta \det \begin{bmatrix} 0 & \cdots & 0 & -\eta_1 \\ -1 & t & 0 & t-\eta_3 \\ & \ddots & \ddots & \vdots \\ 0 & & -1 & t-\eta_n \end{bmatrix}$$
$$= (t-a)(t^{n-1} - \eta_n t^{n-2} - \cdots - \eta_2) + \zeta\eta_1$$

$$= t^n - (a+\eta_n)t^{n-1} - (a\eta_n + \eta_{n-1})t^{n-2} + \cdots + a\eta_2 + \zeta\eta_1$$

$\therefore$ nilpotent $\cap$ slice

$$a + \eta_n = 0$$
$$a\eta_n + \eta_{n-1} = 0$$
$$\vdots$$
$$a\underbrace{\eta_2}_{\substack{\| \\ (-a)^{n-1}}} + \zeta\eta_1 = 0 \qquad \zeta\eta_1 = (-a)^n$$

Type A_n hypersurface in $\mathbb{C}^3$

in general (eigenvalues are fixed) $\cap S$: deformation

<u>symplectic form</u>

$\mathcal{M}_{\sigma\lambda}$ has a symplectic form defined by

$$\begin{cases} \mathrm{tr}(d\eta\wedge du\,u^{-1}+\eta\, du\,u^{-1}\wedge du\,u^{-1}) & V_1\neq V_2 \\ \mathrm{tr}(d\eta\wedge du\,u^{-1}+h\, du\,u^{-1}\wedge du\,u^{-1}+dI\wedge dJ) & V_1=V_2 \end{cases}$$

$GL(V_1)\times GL(V_2)$-action is hamiltonian

moment map is given by $(-B_1, B_2)$

Generalization

$$\begin{tikzcd}
V_0 \arrow[loop above] \arrow[r] \arrow[d] & V_1 \arrow[loop above] \arrow[r] \arrow[d] & \cdots \arrow[r] & V_{l-1} \arrow[loop above] \arrow[d] & 0 \\
\mathbb{C} \arrow[ur] & \mathbb{C} & \mathbb{C} \arrow[ur] & \mathbb{C} &
\end{tikzcd}$$

$\tilde{\mathcal{M}} =$ each triangle : same as before
$/GL(V_1) \times \cdots \times GL(V_{l-1})$ not quotient by $GL(V_0)$

This can be viewed as a hamiltonian reduction :

$$\begin{tikzcd}
V_0 \arrow[loop above, "B_1"] \arrow[r] \arrow[d] & V_1 \arrow[loop above, "B_1"] \\
\mathbb{C} \arrow[ur] &
\end{tikzcd}
=
\begin{tikzcd}
V_1 \arrow[loop above, "B_1'"] \arrow[r] \arrow[d] & V_2 \\
\mathbb{C} \arrow[ur] &
\end{tikzcd}$$

Impose $B_1 - B_1' = 0$ & $/GL(V_1)$

Assume $\underset{\mu_1}{\dim V_0 - \dim V_1} \geq \underset{\mu_2}{\dim V_1 - \dim V_2} \geq \cdots \geq \underset{\mu_{l-1}}{\dim V_{l-2} - \dim V_{l-1}} \geq \underset{\mu_l}{\dim V_l} > 0$

Prop 7(1) $\tilde{\mathcal{M}} = GL(V_0) \times \{ \eta \in \mathrm{End}(V_0) \mid$

$$\eta = \begin{array}{c} \mu_l \\ \mu_{l-1} \\ \vdots \\ \mu_1 \end{array}
\overset{\begin{array}{cccc} \mu_l & \mu_{l-1} & \quad & \mu \end{array}}{\left[\begin{array}{c|c|c|c}
\begin{matrix} 0 & & * \\ 1 & \ddots & \vdots \\ & 1 & * \end{matrix} & \begin{matrix} & & * \\ & 0 & \vdots \\ & & * \end{matrix} & & \begin{matrix} * \\ \vdots \\ \vdots \end{matrix} \\ \hline
\begin{matrix} * & \cdots & * \\ & 0 & \end{matrix} & \begin{matrix} 0 & & * \\ 1 & \ddots & \vdots \\ & 1 & * \end{matrix} & & \\ \hline
\begin{matrix} * & \cdots & * \end{matrix} & \begin{matrix} * & \cdots & * \end{matrix} & \ddots & \begin{matrix} * \\ \vdots \\ * \end{matrix} \\ \hline
 & & & \begin{matrix} 0 & & * \\ 1 & \ddots & \vdots \\ & 1 & * \end{matrix}
\end{array}\right]}$$

$\}$

This is true under a weaker assumption $\dim V_0 > \dim V_1 > \cdots > \dim V_l$

(2) 2^{nd} Term : slice to $\mathcal{O}(J_\mu)$

Lemma 8.

$$J_\mu = \begin{array}{c} \mu_l \\ \mu_{l-1} \\ \\ \mu_1 \end{array}\begin{bmatrix} \begin{smallmatrix}1 & & 0\\ & \ddots & \\ & & 1\end{smallmatrix} & & \\ & \begin{smallmatrix}1 & & 0\\ & \ddots & \\ & & 1\end{smallmatrix} & \\ & & \ddots \end{bmatrix} \qquad \mu_1 \geq \mu_2 \geq \cdots \geq \mu_l > 0$$

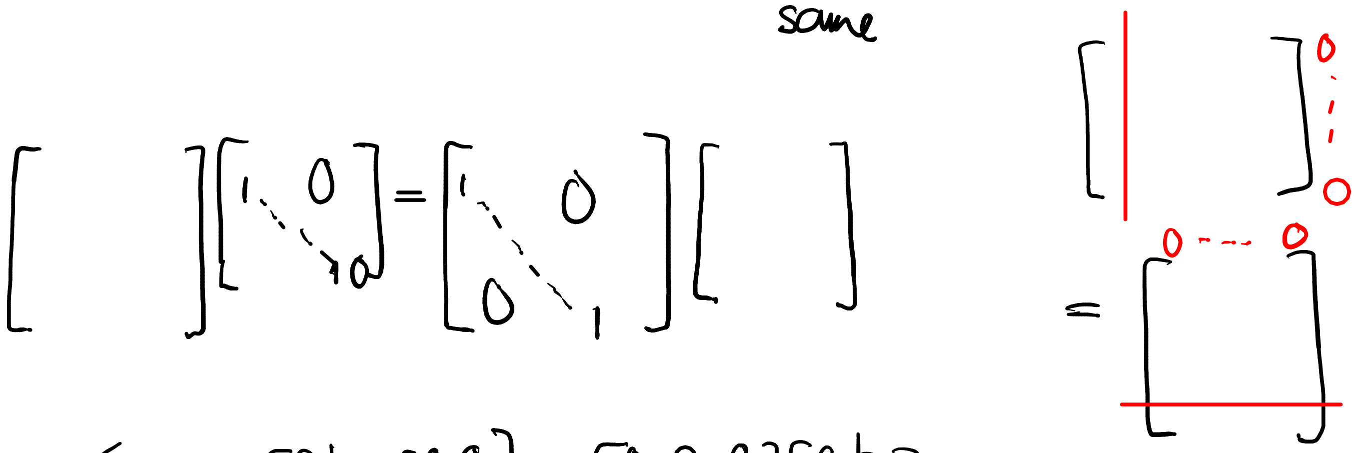
$$\dim \mathrm{Stab}(J_\mu) = (2l-1)\mu_l + (2l-3)\mu_{l-1} + \cdots + \mu_1$$

$[\xi, J_\mu] = 0$

$$\begin{bmatrix} \xi_{ll} & \cdots & \xi_{l1} \\ \vdots & & \vdots \\ \xi_{1l} & \cdots & \xi_{11} \end{bmatrix}$$

$$\xi_{ij} J_{\mu_j} = J_{\mu_i} \xi_{ij}$$

<u>Claim</u> $\xi_{ij} = \begin{bmatrix} & 0 \\ \diagdown\diagdown\diagdown & \end{bmatrix}$ same $\quad \therefore \min(\mu_i, \mu_j) - \dim$



Check
$$\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$$

$$\begin{bmatrix} b & 0 \\ d & 0 \\ f & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ a & b \\ c & d \end{bmatrix} \quad \therefore \begin{bmatrix} 0 & 0 \\ f & 0 \\ e & f \end{bmatrix}$$

Now
$$\begin{bmatrix} \xi_{ll} & & \xi_{l1} \\ & & \\ \xi_{1l} & & \xi_{11} \end{bmatrix} \begin{array}{l} \mu_l \quad \dim \\ \mu_{l-1} \end{array}$$

$\therefore$ dim match

enough to check $T_{J_\mu} O(J_\mu) \wedge$ above form $= 0$

$$\begin{bmatrix} a_{11} & a_{1q} \\ a_{p1} & a_{pq} \end{bmatrix} \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ & & 1 \;\; 0 \end{bmatrix} - \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

$$= \begin{bmatrix} a_{12} & a_{1q} & 0 \\ & & \vdots \\ a_{p2} & a_{pq} & 0 \end{bmatrix} - \begin{bmatrix} 0 & \cdots & 0 \\ a_{11} & & a_{1q} \\ \vdots & & \vdots \\ a_{p-1,1} & & a_{p-1,q} \end{bmatrix} \overset{\text{Suppose}}{=} \begin{bmatrix} * & \cdots & \cdot & * \\ & 0 & & \end{bmatrix}$$

③ $=0$ ① $=0$

② $=0$

$= 0$ by induction //

<u>Check</u>

$$\begin{tikzcd} \mathbb{C}^3 \arrow[loop above] \arrow[r] \arrow[d] & \mathbb{C}^2 \arrow[loop above] \arrow[dr] & \\ \mathbb{C} \arrow[ur] & & \mathbb{C} \end{tikzcd} \qquad \left[\begin{array}{cc|c} 0 & * & * \\ 1 & * & * \\ \hline * & * & 0 \end{array}\right] \qquad 6 \text{ dim}$$

$$\begin{tikzcd} \mathbb{C}^3 \arrow[loop above] \arrow[r] \arrow[d] & \mathbb{C} \arrow[loop above] \arrow[dr] & \\ \mathbb{C} \arrow[ur] & & \mathbb{C} \end{tikzcd} \qquad \left[\begin{array}{c|cc} * & 0 & * \\ \hline * & 0 & * \\ 0 & 1 & * \end{array}\right] \qquad 5 \text{ dim}$$

$\mu = (21)$ $\qquad \begin{matrix} 2 & 1 \\ 1 & 1 \end{matrix}$ $\quad \therefore$ 5 dim