## §3. Cherkis bow varieties

### 3.1 Definition

Example 1  combine examples studied in §2 Warm up.

$$\mathcal{M} = \begin{tikzcd} V_1 \arrow[r, shift left] & V_2 \arrow[l, shift left] \arrow[r, shift left] & \cdots \arrow[l, shift left] \arrow[r, shift left, "C"] & V_n \arrow[l, shift left, "D"] \arrow[loop above, "B"] \arrow[r] \arrow[dr] & V_{n+1} \arrow[loop above] \arrow[r] \arrow[dr] & \cdots \arrow[r] & V_m \arrow[loop above] \arrow[dr] & 0 \\ & & & & \mathbb{C} \arrow[u] & \mathbb{C} \arrow[u] & \mathbb{C} \arrow[u] & \mathbb{C} \end{tikzcd} \Big/ \prod GL(V_i)$$

$CD+B=0$

$$\begin{tikzcd} V_1 \arrow[r, shift left] & \cdots \arrow[l, shift left] \arrow[r, shift left] & V_{n-1} \arrow[l, shift left] \arrow[d, shift left] \\ & & V_n \arrow[u, shift left] \end{tikzcd} \quad \times \quad \begin{tikzcd} V_n \arrow[loop above] \arrow[r] \arrow[dr] & V_{n+1} \arrow[loop above] \arrow[r] \arrow[dr] & \cdots & V_m \arrow[loop above] \arrow[dr] & 0 \\ & \mathbb{C} \arrow[u] & \mathbb{C} & \mathbb{C} \arrow[u] & \mathbb{C} \end{tikzcd}$$

quotient by $GL(V_1)\times\cdots\times GL(V_{n-1})$ $\quad$ quotient by $GL(V_{n+1})\times\cdots\times GL(V_m)$

$\parallel$ $\quad\quad$ $\parallel$

$\mathfrak{M}$ $\quad\quad$ $\widetilde{\mathcal{M}}_{tri}$

impose $CD+B=0$
and take the quotient by $GL(V_n)$

$\mathfrak{M} \curvearrowleft GL(V_n)$ $\quad\quad$ $CD =$ moment map
symplectic $\quad$ hamiltonian
$\widetilde{\mathcal{M}}_{tri} \curvearrowleft GL(V_n)$ $\quad\quad$ $B =$ moment map
" $\quad$ "

$\therefore$ $\mathcal{M} =$ hamiltonian reduction of $\mathfrak{M}\times\widetilde{\mathcal{M}}_{tri}$ by the diagonal $GL(V_n)$-action

Examples $\quad \mathfrak{M} = \mathcal{N}(J_\lambda) \hookrightarrow \mathfrak{gl}(V_n)$

$\widetilde{\mathcal{M}}_{tri} = GL(V_n)\times S(J_\mu) \xrightarrow[\text{2nd proj.}]{} \mathfrak{gl}(V_n)$

$\therefore$ $\mathcal{M} = \mathcal{N}(J_\lambda)\cap S(J_\mu)$

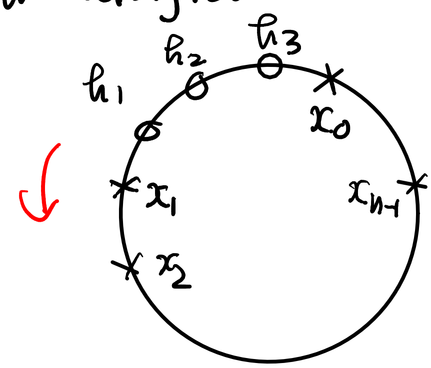
Here $\lambda$ and $\mu$ are given as follows

$$\underset{\substack{\| \\ {}^t\lambda_n}}{\dim V_1} \leqq \underset{\substack{\| \\ {}^t\lambda_{n-1}}}{\dim V_2 - \dim V_1} \leqq \cdots \leqq \underset{\substack{\| \\ {}^t\lambda_1}}{\dim V_n - \dim V_{n-1}}$$

$$\underset{\substack{\| \\ \mu_1}}{\dim V_n - \dim V_{n+1}} \geqq \underset{\substack{\| \\ \mu_2}}{\dim V_{n+1} - \dim V_{n+2}} \geqq \cdots \geqq \underset{\substack{\| \\ \mu_{m-n+1}}}{\dim V_m}$$

Def. 2

○ bow diagram




$x_i$ : anti-clockwise $0 \leqq i \leqq n-1$

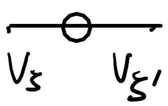
$h_\sigma$ : clockwise order $1 \leqq \sigma \leqq \ell$

$\zeta$ : segment x—x o—x

$v_\zeta \in \mathbb{Z}_{\geqq 0}$ $\quad V_\zeta$ : cpx vector sp $\dim = v_\zeta$

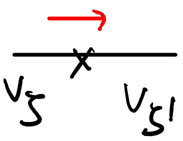
$\underline{v} = (v_\zeta)$

We devide the circle at the middle of segments




$$\rightsquigarrow \widetilde{M}_{tw} = \mathrm{Hom}(V_\zeta, V_{\zeta'}) \oplus \mathrm{Hom}(V_{\zeta'}, V_\zeta)$$

$$V_\zeta \rightleftarrows V_{\zeta'} \qquad \curvearrowleft GL(V_\zeta) \times GL(V_{\zeta'})$$



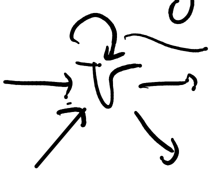

$$\rightsquigarrow \mathcal{M}_{tri}$$

$$= \left\{ \begin{tikzcd} V_\zeta \arrow[loop above] \arrow[rr] \arrow[dr] & & V_{\zeta'} \arrow[loop above] \\ & \mathbb{C} \arrow[ur] & \end{tikzcd} \right\} \curvearrowleft \qquad //$$

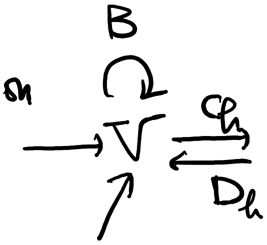
$\prod \tilde{\mathcal{M}}_{tw}$ or $\tilde{\mathcal{M}}_{tri}$ $\curvearrowleft$ hamiltonian reduction by $\prod GL(V_S)$

S : segment

More concretely



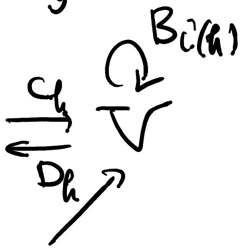
B : common



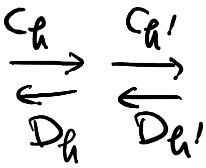

deformed version

$D_h C_h + B_{o(h)} = 0$ on $\nu_h^{\mathbb{C}}$

or




$B_{i(h)} + C_h D_h = 0$ $\nu_h^{\mathbb{C}}$




$C_h D_h - \nu_h^{\mathbb{C}} = D_{h'} C_{h'} - \nu_{h'}^{\mathbb{C}}$

We can deform the defining equation by introducing deformation parameters $\nu_h^{\mathbb{C}}$ for each $h = 0$

$\nu_*^{\mathbb{C}}$

$(B + \nu_*^{\mathbb{C}}) A - AB + ab = 0$

Similarly we can introduce a stability condition when we take quotients.

But only for two way parts except one

$\nu_h^{\mathbb{R}}$ $(h = 0)$, $\nu_*^{\mathbb{R}}$

overall shift is irrelevant

$\ell$ parameter in total

We will not write down a general stability cond.

give only 2 kinds of stability conditions :

ⓐ $\nu^{\mathbb{R}}=0$ $\cdots$ affine algebro-geometric quotient

ⓑ positive $\nexists_{0\neq} S=\bigoplus S_\xi \subset \bigoplus V_\xi$ ) closed orbits

s.t. invariant under $A, C, D$

$b(S)=0$

and
$$\begin{array}{ccc} S_{o(x)} & \xrightarrow[\cong]{A_x} & S_{i(x)} \\ \cap & & \cap \\ V_{o(x)} & \xrightarrow{A} & A_{i(x)} \end{array}$$

These give different quotient spaces

$\mathcal{M}$ : type ⓐ $\quad \tilde{\mathcal{M}}$ : type ⓑ $\quad \mathcal{M}^{\nu^{\mathbb{C}}}$ : parameter 付き $\tilde{\mathcal{M}}^{\nu^{\mathbb{C}}}$

affine algebraic variety $\quad$ quasi projective variety smooth

We have a natural map (projective morphism)

$\pi : \tilde{\mathcal{M}} \longrightarrow \mathcal{M}$ $\quad$ smallest orbit in the orbit closure.

Prop3 type ⓑ : stabilizer is trivial

$$\begin{array}{ccc} V_{o(x)} & \xrightarrow{A_x} & V_{i(x)} \\ & {}_{b_x}\searrow \quad \nearrow_{a_x} & \\ & \mathbb{C} & \end{array}$$

☺ Suppose $g_{i(x)} A_x = A_x g_{o(x)}$

$g_\xi B_\xi = B_\xi g_\xi$

$g_{i(x)} a_x = a_x$

$b_x = b_x g_{o(x)}$

$S_\xi := \mathrm{Im}(g_\xi - \mathrm{id}) \leftarrow$ invariant

$T_\xi := \mathrm{Ker}(g_\xi - \mathrm{id}) \leftarrow$ under $A, B, C, D$

$b(S)=0$

$T \supset \mathrm{Im}\, a$

Lemma4 $S_{o(x)} \xrightarrow[\cong]{A|_S} S_{i(x)}$, $T_{o(x)} \xrightarrow[\cong]{A|_S} T_{i(x)}$ for

$$\begin{array}{ccc} \circlearrowleft & & \circlearrowleft \\ V_{o(x)} & \longrightarrow & V_{i(x)} \\ & \searrow \quad \nearrow & \\ & \mathbb{C} & \end{array}$$

☺ $\dim S_{o(x)} \leq \dim S_{i(x)}$

as if $>$ $\operatorname{Ker} A|_{\mathfrak{Z}} \neq 0$

$\operatorname{codim} T_{o(x)} \geq \operatorname{codim} T_{i(x)}$

$$\begin{array}{ccc} T_{i(x)}^{\perp} & \xrightarrow{{}^{t}A|_{\mathfrak{Z}}} & T_{o(x)}^{\perp} \\ \cap & & \\ V_{i(x)}^{*} & & \end{array}$$

But $\dim S_{\mathfrak{Z}} = \dim V_{\mathfrak{Z}} - \dim T_{\mathfrak{Z}}$
$= \operatorname{codim} T_{\mathfrak{Z}}$

$\therefore$ both are $=$ //

Lemma 4 $\Longrightarrow$ Prop 3

GIT theory $\rightsquigarrow$ $\tilde{\mathcal{M}}$ is smooth
$\pi : \tilde{\mathcal{M}} \to \mathcal{M}$ is a proper map.

## Torus action

$\mathbb{C}^{\times}$ acting for $\mathbb{C}$ in each triangle $(\mathbb{C}^{\times})^{n}$

But the diagonal $\mathbb{C}^{\times} \underset{\Delta}{\subset} (\mathbb{C}^{\times})^{n}$ acts trivially

as it can be absorbed to $\prod_{\mathfrak{Z}} GL(V_{\mathfrak{Z}})$

$+$
$$\begin{tikzcd} V \arrow[loop above] \arrow[rr, "A_{x_0}"] \arrow[dr, "b_{x_0}"'] & & V \arrow[loop above] \\ & \mathbb{C} \arrow[ur] & \end{tikzcd}$$

scalar倍する。

**Def** A bow diagram is cobalanced

$\iff \forall\ V_\xi \overset{}{—\!\circ\!—} V_{\xi'}$ $\quad V_\xi = V_{\xi'}$

$$\mathbb{C}^{V_\xi} \rightleftarrows \mathbb{C}^{V_{\xi'}}$$

$\forall\ V_\xi \overset{}{—\!\times\!—} V_{\xi'}$ $\quad V_\xi = V_{\xi'}$

$$\begin{tikzcd} \mathbb{C}^{V_\xi} \arrow[loop above] \arrow[rr] \arrow[dr] & & \mathbb{C}^{V_{\xi'}} \arrow[loop above] \\ & \mathbb{C} \arrow[ur] & \end{tikzcd}$$

**Prop5.** If the cobalanced cond. is satisfied, $\mathcal{M} \cong$ a quiver variety of affine type A

☺ Recall

$$\begin{tikzcd} V_1 \arrow[loop above] \arrow[rr] \arrow[dr] & & V_2 \arrow[loop above] \\ & \mathbb{C} \arrow[ur] & \end{tikzcd} \cong GL(V_1) \times \mathfrak{gl}(V_1) \times V_1 \times V_1^*$$

if $\dim V_1 = \dim V_2$

This factor is killed by the hamiltonian reduction by $GL(V_1)$

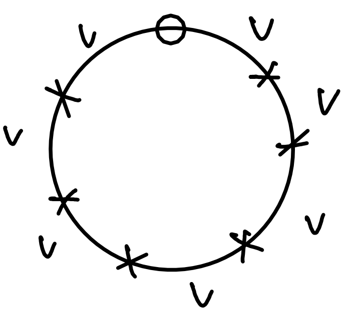
So we can replace this by $\begin{tikzcd} V_2 \arrow[d, shift right] \\ \mathbb{C} \arrow[u, shift right] \end{tikzcd}$ $\circlearrowleft GL(V_2)$

$$\begin{tikzcd} V_1 \arrow[loop above] \arrow[rr] \arrow[dr] & & V_2 \arrow[loop above] \arrow[rr] \arrow[dr] & & V_3 \arrow[loop above] \\ & \mathbb{C} \arrow[ur] & & \mathbb{C} \arrow[ur] & \end{tikzcd} \quad \dim V_1 = \dim V_2 = \dim V_3 \rightsquigarrow \begin{tikzcd} V_3 \arrow[d, shift right] \\ \mathbb{C}^2 \arrow[u, shift right] \end{tikzcd}$$

//

**Example**

$n=1$




$$\cong \begin{tikzcd} V \arrow[loop left] \arrow[loop right] \arrow[d, shift right] \\ \mathbb{C}^n \arrow[u, shift right] \end{tikzcd}$$

ADHM descr. of $SU(n)$ inst on $\mathbb{R}^4$

or rk $n$ framed sheaves on $\mathbb{CP}^2$

## 3.2 factorization morphism

- triangle

$$\begin{tikzcd} V_1 \arrow[loop above, "B_1"] \arrow[rr, "A"] \arrow[dr, "b"'] & & V_2 \arrow[loop above, "B_2"] \\ & \mathbb{C} \arrow[ur, "a"'] & \end{tikzcd}$$

$v_i = \dim V_i$

$\underline{v} = (v_1, v_2)$

Define $\Phi : \widetilde{\mathcal{M}}_{tri}(\underline{v}) \longrightarrow \mathbb{C}^{\underline{v}} := \mathbb{C}^{v_1}/\mathfrak{S}_{v_1} \times \mathbb{C}^{v_2}/\mathfrak{S}_2$

by eigenvalues of $B_1, B_2$ counted with multiplicities

We consider a point of $\mathbb{C}^{\underline{v}}$ as a colored configuation in $\mathbb{C}$ (colored by 1 & 2)

This is invariant under $GL(\underline{v}) = GL(V_1) \times GL(V_2)$

$\Phi : [\widetilde{\mathcal{M}}_{tri}(\underline{v}) / GL(\underline{v})] \longrightarrow \mathbb{C}^{\underline{v}}$

( I emphasised that a naive quotient space does not behave well. This is a stack.

Suppose Eigenvalues of $B_1 = \mathcal{C}_1' + \mathcal{C}_1''$

$B_2 = \mathcal{C}_2' + \mathcal{C}_2''$

s.t. $\mathcal{C}_i' \cap \mathcal{C}_j'' = \phi$ $(i,j) = (1,1), (2,2), (1,2), (2,1)$

Consider

$V_1 = V_1' \oplus V_1''$

$V_2 = V_2' \oplus V_2''$

$B_i = \begin{bmatrix} B_i' & 0 \\ 0 & B_i'' \end{bmatrix}$ $i = 1, 2$

$a = \begin{bmatrix} a' \\ a'' \end{bmatrix}$, $b = [b' \ b'']$ $A = \left[\begin{array}{c|c} A' & A_{12} \\ \hline A_{21} & A'' \end{array}\right]$

$\left\{\begin{array}{l}(A', B_1', B_2', a', b') \\ (A'', B_1'', B_2'', a'', b'')\end{array}\right.$ are regarded as new data for $\widetilde{\mathcal{M}}_{\mathrm{tri}}(\underline{v}')$, $\widetilde{\mathcal{M}}_{\mathrm{tri}}(\underline{v}'')$ (for different dimensions)

<u>Prop 1</u> (1) (S1, 2) are satisfied. ← <u>exercise</u> $A_{12}(S_1') = 0$

(2) $\exists 1$ $A_{12}, A_{21}$ for given other components s.t. $B_2 A - A B_1 + ab = 0$

(proof) $B_1' A_{12} - A_{12} B_2'' + a' b'' = 0$

$$\begin{array}{ccc} \mathrm{Mat}(V_1', V_2'') & \longrightarrow & \mathrm{Mat}(V_1', V_2'') \\ A_{12} & \longmapsto & B_1' A_{12} - A_{12} B_2'' \end{array}$$

is invertible as $\mathrm{eigen}(B_1') \cap \mathrm{eigen}(B_2'') = \phi$ //

$$(\mathbb{C}^{\underline{v}'} \times \mathbb{C}^{\underline{v}''})_{\mathrm{disj}} = \{ (c_1', c_2', c_1'', c_2'') \in \mathbb{C}^{\underline{v}'} \times \mathbb{C}^{\underline{v}''} \mid \text{above cond.} \}$$

<u>Cor 2</u> $\widetilde{\mathcal{M}}(\underline{v}) \times_{\mathbb{C}^{\underline{v}}} (\mathbb{C}^{\underline{v}'} \times \mathbb{C}^{\underline{v}''})_{\mathrm{disj}}$

$$\cong \widetilde{\mathcal{M}}(\underline{v}') \times \widetilde{\mathcal{M}}(\underline{v}'') \times_{\mathbb{C}^{\underline{v}'} \times \mathbb{C}^{\underline{v}''}} (\mathbb{C}^{\underline{v}'} \times \mathbb{C}^{\underline{v}''})_{\mathrm{disj}} \times_{GL(\underline{v}') \times GL(\underline{v}'')} GL(\underline{v})$$

Recall $\dim \widetilde{\mathcal{M}}(\underline{v}) = v_1^2 + v_2^2 + v_1 + v_2 = \dim GL(\underline{v}) + \underbrace{v_1 + v_2}_{\text{additive}}$

$\overset{\circ}{\mathbb{C}}{}^{\underline{v}} \overset{\text{def.}}{=}$ all eigenvalues are distinct

$$\begin{tikzcd} \mathbb{C} \arrow[loop above, "B_1"] \arrow[d, "b \neq 0"] \\ \mathbb{C} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{C} \arrow[loop above, "B_2"] \\ \mathbb{C} \arrow[u, "a \neq 0"'] \end{tikzcd}$$

← (S1, 2)

Both are $\mathbb{C} \times \mathbb{C}^\times$

Cor 3 $\widetilde{\mathcal{M}}(\underline{v}) \times_{\mathbb{C}^{\underline{v}}} \mathring{\mathbb{C}}^{\underline{v}} \cong \mathring{\mathbb{C}}^{\underline{v}} \times (\mathbb{C}^{\times} \times \cdots \times \mathbb{C}^{\times}) \times_{\underbrace{(\mathbb{C}^{\times} \times \cdots \times \mathbb{C}^{\times})}_{v_1+v_2 \text{ times}}} GL(\underline{v})$

$$= \mathring{\mathbb{C}}^{\underline{v}} \times GL(\underline{v})$$

$\Phi : \widetilde{\mathcal{M}}(\underline{v}) \longrightarrow \mathbb{C}^{\underline{v}}$ generic fiber $= GL(\underline{v})$

Prop 4. dim. of any fiber $= \dim GL(\underline{v})$

(sketch) enough to check this when $B_1, B_2$ are nilpotent
compute dim when $B_1, B_2$ : regular nilpotent
general case $\longrightarrow$ smaller dimension //

∘ Two way part

$V_1 \underset{D}{\overset{C}{\rightleftarrows}} V_2$ $B_1 = -DC \quad B_2 = -CD$

$\dim V_1 \leqq \dim V_2$

$$\mathrm{tr}_{V_1}(t - CD)^n = \mathrm{tr}_{V_2}(t - DC)^n + t^n(\dim V_1 - \dim V_2)$$

$\therefore$ $\mathrm{eigen}(B_2) = \mathrm{eigen}(B_1) + \underbrace{(v_2 - v_1)}_{\text{mult. of eigenvalue } 0} 0$

$\mathrm{eigen}(B_1) = \mathcal{C}_1' + \mathcal{C}_1''$ $\mathcal{C}_1' \cap \mathcal{C}_1'' = \phi$

$0 \notin \mathcal{C}_1''$

$V_1 = V_1' \oplus V_1''$
$V_2 = V_2' \oplus V_2''$

Prop 5. (1) $C(V_1') \subset V_2'$
$C(V_1'') \subset V_2''$
$D(V_2') \subset V_1'$
$D(V_2'') \subset V_1''$

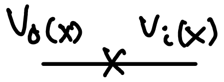
$$\begin{array}{ccc} V_1' & \underset{D'}{\overset{C'}{\rightleftarrows}} & V_2' \\ \oplus & & \oplus \\ V_1'' & \underset{D''}{\overset{C''}{\rightleftarrows}} & V_2'' \end{array}$$

(2) $D''$ $\cong$

proof) $B_2 C = -CDC = CB_1$
$B_1 D = -DCD = DB_2$ //

$$\Phi : \underset{\Psi}{\mathcal{M}} \longrightarrow \prod_{x} \mathbb{C}^{v_{i(x)}} / \mathfrak{S}_{v_{i(x)}} \cong \mathbb{C}^r \qquad r = \sum_x v_{i(x)}$$

$[(A, B, C, D, a, b)] \mapsto$ eigenvalues of $B_{i(x)}$
counted with multiplicities




$$\begin{tikzcd} V_{0(x)} \arrow[loop above] \arrow[r] \arrow[dr] & V_{i(x)} \arrow[loop above, "B_{i(x)}"] \\ & \mathbb{C} \arrow[u] \end{tikzcd}$$

$$V_{o(h)} \underset{D_h}{\overset{C_h}{\rightleftarrows}} V_{i(h)} \qquad \mathrm{tr}_{V_{i(h)}}(t - C_h D_h)^n$$

$$= \mathrm{tr}_{V_{o(h)}}(t - D_h C_h)^n + t^n(\dim V_{i(h)} - \dim V_{o(h)})$$

$\therefore$ enough to consider $B_{i(x)}$

We consider a point of $\mathbb{C}^r$
as a colored configuation in $\mathbb{C}$
colored by $x$