$\underline{\text{Cor 2}}$ $\quad \tilde{\mathcal{M}}(\underline{v}) \times_{\mathbb{C}^{\underline{v}}} (\mathbb{C}^{\underline{v}'} \times \mathbb{C}^{\underline{v}''})_{\text{disj.}}$

$$\cong \tilde{\mathcal{M}}(\underline{v}') \times \tilde{\mathcal{M}}(\underline{v}'') \times_{\mathbb{C}^{\underline{v}'} \times \mathbb{C}^{\underline{v}''}} (\mathbb{C}^{\underline{v}'} \times \mathbb{C}^{\underline{v}''})_{\text{disj}} \times \frac{GL(\underline{v})}{GL(\underline{v}') \times GL(\underline{v}'')}$$

Recall $\quad \dim \tilde{\mathcal{M}}(\underline{v}) = v_1^2 + v_2^2 + v_1 + v_2 = \dim GL(\underline{v}) + \underbrace{v_1 + v_2}_{\text{additive}}$

$\mathring{\mathbb{C}}^{\underline{v}} \overset{\text{def.}}{=}$ all eigenvalues are distinct

$$\begin{tikzcd} \mathbb{C} \arrow[loop above, "B_1"] \arrow[d, "b \neq 0"] & \mathbb{C} \arrow[loop above, "B_2"] \\ \mathbb{C} & \mathbb{C} \arrow[u, "a \neq 0"] \end{tikzcd}$$

Both are $\mathbb{C} \times \mathbb{C}^\times$ $\leftarrow$ (S1,2)

$\underline{\text{Cor 3}}$ $\quad \tilde{\mathcal{M}}(\underline{v}) \times_{\mathbb{C}^{\underline{v}}} \mathring{\mathbb{C}}^{\underline{v}} \cong \mathring{\mathbb{C}}^{\underline{v}} \times (\mathbb{C}^\times \times \cdots \times \mathbb{C}^\times) \times \dfrac{GL(\underline{v})}{(\underbrace{\mathbb{C}^\times \times \cdots \times \mathbb{C}^\times}_{v_1 + v_2 \text{ times}})}$

$$= \mathring{\mathbb{C}}^{\underline{v}} \times GL(\underline{v})$$

$\Phi : \tilde{\mathcal{M}}(\underline{v}) \longrightarrow \mathbb{C}^{\underline{v}} \qquad$ generic fiber $= GL(\underline{v})$

$\underline{\text{Prop 4}}$. dim. of any fiber $= \dim GL(\underline{v})$

(sketch) enough to check this when B_1, B_2 are nilpotent

compute dim when B_1, B_2 : regular nilpotent

general case $\longrightarrow$ smaller dimension $\leftarrow$ Exercise

$B_1 = J_{v_1} \qquad \dim \mathcal{O}(J_{v_1}) = v_1^2 - v_1$

$B_2 = J_{v_2} \qquad \dim \mathcal{O}(J_{v_2}) = v_2^2 - v_2$

$$J_{v_2} = \begin{bmatrix} 0 & & \\ 1 & \ddots & \\ & \ddots & \\ & & 1 \end{bmatrix}$$

$v_1 \leqq v_2$ $\qquad A = \begin{bmatrix} a_{11} & \cdots & a_{1v_1} \\ \vdots & & \vdots \\ a_{v_2 1} & \cdots & a_{v_2 v_1} \end{bmatrix}$

$\varphi : A \mapsto B_2 A - A B_1 \qquad \dim \mathrm{Ker}\,\varphi = \min(v_1, v_2) = v_1$

$$B_2 A - AB_1 = \begin{bmatrix} 0 & \cdots & 0 \\ a_{11} & \cdots & a_{1v_1} \\ \vdots & & \vdots \\ a_{v_2-1,1} & \cdots & a_{v_2-1,v_1} \end{bmatrix} - \begin{bmatrix} a_{12} & \cdots & a_{1v_1} & 0 \\ \vdots & & \vdots & \vdots \\ a_{v_2 2} & \cdots & a_{v_2 v_1} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} & -a_{1,v_1-1} & -a_{1v_1} & 0 \\ & a_{1v_1-1} - a_{2v_1} & a_{1v_1} & \\ & & a_{2v_1} & \\ \iddots & & & \end{bmatrix}$$

sum

0

0

0

0

v_1 equations

$\therefore (a, b, B_1, B_2)$: fixed $\Rightarrow \dim\{A\} = v_1$

$$\varphi(A) = -ab = -\begin{bmatrix} a_1 b_{v_1} & \cdots & a_1 b_2 & a_1 b_1 \\ \vdots & & & a_2 b_1 \\ \vdots & & & \vdots \\ a_{v_2} b_1 & \cdots & & a_{v_2} b_1 \end{bmatrix}$$

$a = \begin{bmatrix} a_1 \\ \vdots \\ a_{v_2} \end{bmatrix}$

$b = [b_{v_1} \cdots b_1]$

$$\begin{cases} a_1 b_1 = 0 \\ a_1 b_2 + a_2 b_1 = 0 \\ a_1 b_3 + a_2 b_2 + a_3 b_1 = 0 \\ \quad\vdots \\ a_1 b_{v_1} + \cdots + a_{v_1} b_1 = 0 \end{cases}$$

- $a_1 \neq 0 \Rightarrow b_1 = \cdots = b_{v_1} = 0$
- $a_1 = 0,\ a_2 \neq 0$
  $\Rightarrow b_1 = \cdots = b_{v_1-1} = 0$
  $\vdots$

$\therefore \dim\{(a, b)\} = v_2$

$\therefore \quad v_1 + v_2 + v_1^2 - v_1 + v_2^2 - v_2 = v_1^2 + v_2^2$ //

∘ Two way part

$$V_1 \underset{D}{\overset{C}{\rightleftarrows}} V_2 \qquad B_1 = -DC \quad B_2 = -CD$$

$$\operatorname{tr}_{V_1}(t-CD)^n = \operatorname{tr}_{V_2}(t-DC)^n + t^n(\dim V_1 - \dim V_2)$$

$\dim V_1 \leqq \dim V_2$

$$\therefore \quad \operatorname{eigen}(B_2) = \operatorname{eigen}(B_1) + \underbrace{(V_2 - V_1)}_{\text{mult. of eigenvalue } 0}\, 0$$

$$\operatorname{eigen}(B_1) = C_1' + C_1'' \qquad C_1' \cap C_1'' = \phi$$
$$0 \notin C_1''$$

$$V_1 = V_1' \oplus V_1''$$
$$V_2 = V_2' \oplus V_2''$$

<u>Prop 5.</u> (1)

$$C(V_1') \subset V_2'$$
$$C(V_1'') \subset V_2''$$
$$D(V_2') \subset V_1'$$
$$D(V_2'') \subset V_1''$$

$$\begin{array}{ccc} V_1' & \underset{D'}{\overset{C'}{\rightleftarrows}} & V_2' \\ \oplus & & \oplus \\ V_1'' & \underset{D''}{\overset{C''}{\rightleftarrows}} & V_2'' \end{array}$$

(2) D'' $\cong$

proof) $B_2 C = -CDC = CB_1$
$B_1 D = -DCD = DB_2$ //

<u>Remark 6.</u>
Example in Warm Up

$$\begin{array}{l} V_1 \rightleftarrows \cdots \rightleftarrows V_n \\ \downarrow\uparrow \\ W \end{array} \qquad \xi \sim \begin{bmatrix} \xi_1 \mathrm{id} & 0 \\ 0 & \xi_n \mathrm{id} \end{bmatrix}$$

follows from the factorization

combine triangle + two way
$\longrightarrow$ factorization for $M \xrightarrow{\Phi} \mathbb{C}^V \quad \prod_x \mathbb{C}^{V_{i(x)}} / \mathfrak{S}_{V_{i(x)}}$

$$o(x) \xrightarrow{x} i(x)$$

Th 7. If the balanced condition is satisfied

$\Rightarrow \mathcal{M} \xrightarrow{\Phi} \mathbb{C}^{V}$ is an integrable system

(a) f, g : functions on $\mathbb{C}^{V}$

$\{\Phi^* f, \Phi^* g\} = 0$

(b) generic fiber of Φ $= (\mathbb{C}^{\times})^{\frac{1}{2}\dim \mathcal{M}}$

(c) Moreover all fibers of Φ have $\dim = \frac{1}{2}\dim\mathcal{M}$

Remark. balanced cond. $\Rightarrow$ $\dim\mathcal{M} = 2\sum_{x} v_{i(x)}$

$$\begin{tikzcd}
\mathbb{C}^{v_1} \arrow[loop above] \arrow[rr] \arrow[dr] & & \mathbb{C}^{v_2} \arrow[loop above] \arrow[r, shift left] & \cdots \arrow[l, shift left] \arrow[r, shift left] & \mathbb{C}^{v_2} \arrow[loop above] \arrow[l, shift left] \arrow[rr] \arrow[dr] & & \mathbb{C}^{v_3} \arrow[loop above] \\
& \mathbb{C} \arrow[ur] & & & & \mathbb{C} \arrow[ur] &
\end{tikzcd}$$

hamiltonian reduction

$/\!/\!/ GL(\mathbb{C}^{v_2})^{w+1}$

$$\begin{array}{c} v_1^2 + \underline{v_2^2} \\ + v_1 + v_2 \end{array} + \underline{2 w v_2^2} + \begin{array}{c} \underline{v_2^2} + v_3^2 \\ + v_2 + v_3 \end{array} \qquad - 2(w+1) v_2^2$$

(sketch of the proof)

(a) $\leftarrow$ calculation

(b) $\leftarrow$ factorization

(c) $DC \circlearrowleft$ $\quad C \circlearrowleft$ $\leftarrow$ CD : nilpotent $\begin{bmatrix} 0 & & 0 \\ 1 & \ddots & \\ 0 & \ddots 1 & \end{bmatrix}$

$\begin{bmatrix} 0 & & 0 \\ 1 & \ddots & \\ 0 & \ddots 1 & \end{bmatrix}$ $\quad \mathbb{C}^{v_2} \underset{D}{\overset{C}{\rightleftarrows}} \mathbb{C}^{v_2}$

suppose

$$\Rightarrow \quad C = \begin{bmatrix} c_1 & & 0 \\ c_2 & \ddots & \\ \vdots & \ddots c_2 & c_1 \end{bmatrix} \qquad D = \begin{bmatrix} d_1 & & \\ d_2 & \ddots & \\ \vdots & \ddots d_2 & d_1 \end{bmatrix}$$

$$CD = \begin{bmatrix} c_1 d_1 & & \\ c_2 d_1 + c_1 d_2 & \ddots & \\ & \ddots & c_1 d_1 \end{bmatrix} \qquad \begin{cases} c_1 d_1 = 0 \\ c_2 d_1 + c_1 d_2 = 1 \\ c_3 d_1 + c_2 d_2 + c_1 d_3 = 0 \\ \vdots \end{cases}$$

$c_1 \neq 0 \Rightarrow d_1 = 0, \quad d_2 = 1/c_1$

$\rightsquigarrow$ v_2 dim'l

$$\begin{aligned} & w \cdot v_2 + (w+1)(v_2^2 - v_2) \\ & \quad + v_1^2 + v_2 \quad + v_2 + v_3^2 - (w+1)v_2^2 \\ & = v_2 + v_1^2 + v_3^2 \end{aligned}$$

§4. Hanany–Witten transition

Th1

$$
\begin{tikzcd}
V_1 \arrow[loop above, "B_1"] \arrow[r, shift left, "C"] & V_2 \arrow[loop above, "B_2"] \arrow[l, shift left, "D"] \arrow[r, "A"] \arrow[dr, "b"'] & V_3 \arrow[loop above, "B_3"] \\
& & \mathbb{C} \arrow[u, "a"']
\end{tikzcd}
\quad \overset{\text{bijection}}{\longleftrightarrow} \quad
\begin{tikzcd}
V_1 \arrow[loop above, "B_1"] \arrow[r, "A^{new}"] \arrow[dr, "b^{new}"'] & V_2^{new} \arrow[loop above, "B_2^{new}"] \arrow[r, shift left, "C^{new}"] & V_3 \arrow[loop above, "B^3"] \arrow[l, shift left, "D^{new}"] \\
& \mathbb{C} \arrow[u, "a^{new}"'] &
\end{tikzcd}
$$

$$
\begin{cases}
CD + B_1 = 0 \\
DC + B_2 = 0 \\
B_3A - AB_2 + ab = 0
\end{cases}
$$

similar

$$\dim V_2 + \dim V_2^{new} = \dim V_1 + \dim V_3 + 1$$

(Sketch of proof

$$
\begin{tikzcd}[column sep=large]
0 \arrow[r, "(S1)"'] & V_2 \arrow[r, "\alpha = \begin{bmatrix} D \\ A \\ b \end{bmatrix}"] \arrow[d, equal] & V_1 \oplus V_3 \oplus \mathbb{C} \arrow[r, "\beta'' \, {[AC \;\; B_3 \;\; a]}"] \arrow[d, equal] & V_3 & \\
0 \arrow[r] & V_2 \arrow[r] & {''} \arrow[r] \arrow[d, equal] & V_2^{new} \arrow[r] \arrow[u, "C^{new}"'] \arrow[d, equal] & 0 \\
& V_1 \arrow[r] \arrow[u, "C"] & {''} \arrow[r] & V_2^{new} \arrow[r, "(S2)"] & 0
\end{tikzcd}
$$

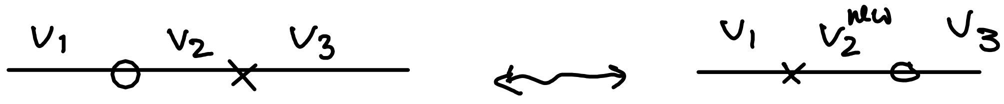
$$\alpha^{new} = \begin{bmatrix} -B_1 \\ C^{new} A^{new} \\ b^{new} \end{bmatrix} \qquad [A^{new} \;\; -D^{new} \;\; a^{new}] = \beta^{new}$$

i.e. $V_2^{new} = \operatorname{Coker} \alpha$ $\quad \beta^{new} = $ proj.

C^{new} induced morphism

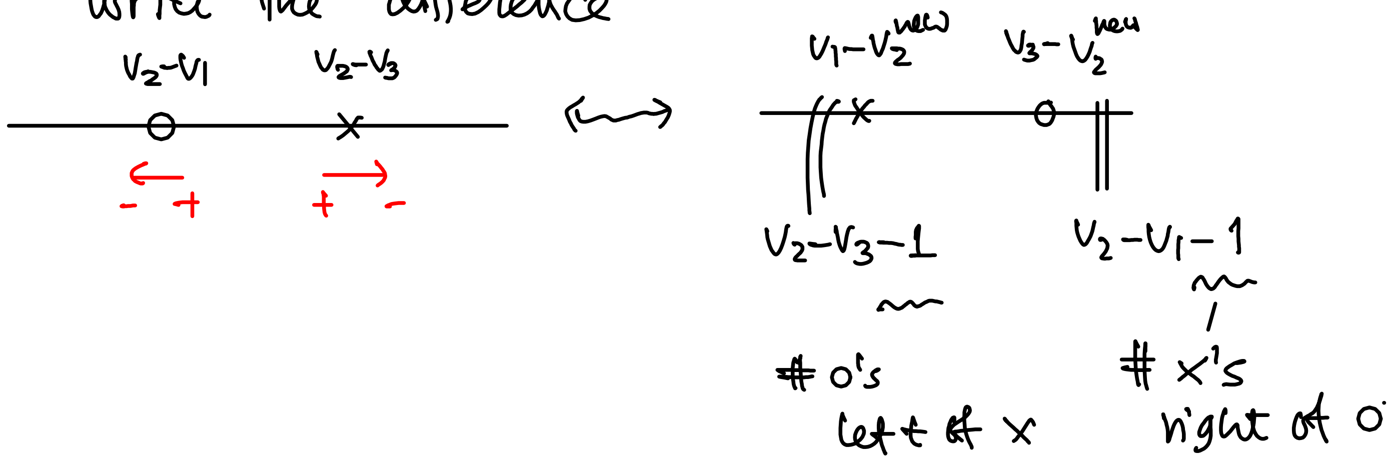
$b^{new} = bC$

Check (S1, 2)

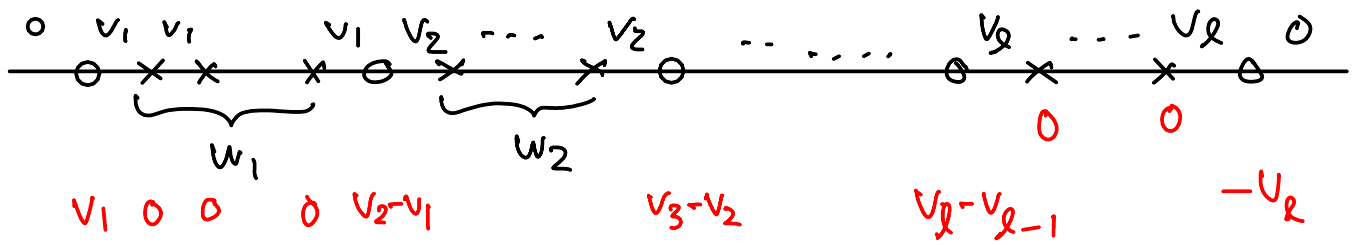


$$v_1 + v_3 + 1 = v_2 + v_2^{new}$$

Reminding relation to partitions for nilpotent orbits
write the difference

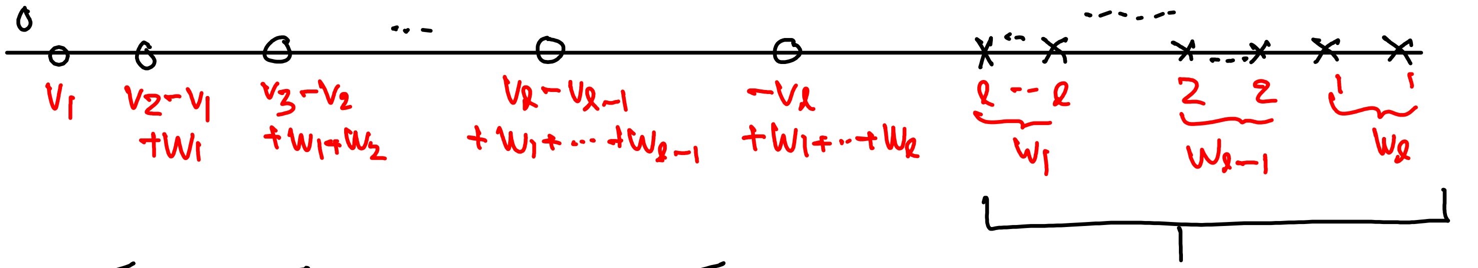


Consider a cobalanced bow diagram



$\mathcal{M}$ = quiver var. of type A

$$\begin{tikzcd} V_1 \arrow[r, shift left] \arrow[d, shift left] & V_2 \arrow[l, shift left] \arrow[r, shift left] \arrow[d, shift left] & \cdots \arrow[l, shift left] \arrow[r, shift left] & V_\ell \arrow[l, shift left] \arrow[d, shift left] \\ W_1 \arrow[u, shift left] & W_2 \arrow[u, shift left] & & W_\ell \arrow[u, shift left] \end{tikzcd}$$



$\leqq \qquad \leqq \qquad \leqq$

dominance cond. is automatically satisfied

$$\Updownarrow$$

$$\underbrace{w_1 + v_2 - 2v_1 \geqq 0, \quad w_2 + v_3 - 2v_2 \geqq 0, \quad \cdots \quad w_\ell + v_{\ell-1} - 2v_\ell \geqq 0}_{\mu}$$

λ

<u>Prop2</u> quiver variety of type A with the dominance cond. $\cong \overline{\mathcal{O}(J_\mu)} \cap S(J_\lambda)$

This result was proved in [N, 94] by a completely different method. The proof was much more involved.

This proof works even for $\widetilde{\mathfrak{M}}$ originally proved by Maffei

$0 \longleftrightarrow \times$ $\overline{\mathcal{O}(J_{{}^t\lambda})} \cap S(J_{{}^t\mu})$

balanced $\leftarrow$ Coulomb branch