§5. Analysis of fixed pts / attracting sets

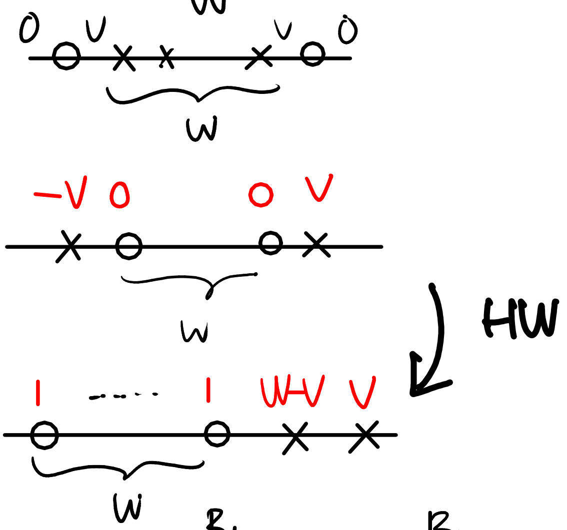
5.1 Warm Up

type A_1 quiver variety $V \underset{}{\overset{}{\rightleftarrows}} W$

cobalanced

$O \leftrightarrow X$

balanced



i.e.,

$$\begin{tikzcd} \mathbb{C} \arrow[r, shift left] & \mathbb{C}^2 \arrow[l, shift left] \arrow[r, shift left] & \cdots \arrow[l, shift left] \arrow[r, shift left] & \mathbb{C}^W \arrow[l, shift left] \arrow[loop above, "B_1"] \arrow[rr, "A"] \arrow[dr, "b_1"'] & & \mathbb{C}^V \arrow[loop above, "B_2"] \arrow[dr, "b_2"] & \\ & & & & \mathbb{C} \arrow[ur, "a_1"'] & & \mathbb{C} \end{tikzcd}$$

$(\mathbb{C}^\times)^2 \ni t_1$ (acting on the first $\mathbb{C}$), t_2 (acting on the second $\mathbb{C}$)

$\cup$

$\{t_1 = t_2\}$ acts trivially $\Rightarrow t_2 = 1$, $t_1 = t$

Prop $\mathcal{A} \overset{\text{attracting set}}{=} \phi$ if $V > W$

$\because$

$b_2 a_1 \mapsto t^{-1} b_2 a_1 \quad \therefore b_2 a_1 = 0$

$b_2 \neq 0$ by (S1) $\quad \therefore a_1 = 0 \quad \therefore B_2 A = A B_1$

$\therefore \operatorname{Im} A + \underbrace{\operatorname{Im} a_1}_{=0}$ is B_2-inv.

$\therefore$ (S2) $\Rightarrow A$: surjective //

Assume $W \geqq V$.

$$\begin{tikzcd}
\mathbb{C} \arrow[r, shift left] & \mathbb{C}^2 \arrow[l, shift left] \arrow[r, shift left] & \cdots \arrow[l, shift left] \arrow[r, shift left] & \mathbb{C}^W \arrow[l, shift left] \arrow[loop above, "B_1"] \arrow[rr, "A"', "\text{surjective}"] \arrow[dr, "b_1"'] & & \mathbb{C}^V \arrow[loop above, "B_2"] \arrow[dr, "b_2"] & \\
& & & & \mathbb{C} \arrow[ur, "a_1"'] & & \mathbb{C}
\end{tikzcd}$$

$a_1 = 0$

fn. attracting

B_1: nilpotent
$B_2 A = A B_1$
$\Rightarrow B_2$: nilpotent

$\therefore$ by $GL(V)$ action we can normalise

$$b_2 = [0 \cdots 0\, 1] \qquad B_2 = \begin{bmatrix} 0 & & 0 \\ 1 & \ddots & \\ 0 & \ddots & 1 \end{bmatrix}$$

basis of $(\mathbb{C}^W)^*$ $\quad b_2 B_2^{V-1} A,\ b_2 B^{V-2} A, \ldots, b_2 A$

$\qquad\qquad\qquad\qquad b_1 B_1^{W-V-1}, \ldots, b_1$

- $V<W$

$$B_1 = \begin{array}{c c} & \overbrace{\qquad\qquad}^{V} \\ V & \left[\begin{array}{ccc|ccc} \boxed{\begin{matrix} 0 & & \\ 1 & \ddots & \\ & \ddots & 1 \end{matrix}} & & & & 0 & * \\ * & \cdots & * & 0 & & \vdots \\ & 0 & & 1 & \ddots & 0 \end{array}\right] \end{array} \quad B_2$$

$* = 0$

$\therefore \mathcal{A} =$ attracting set $= \mathbb{C}^V$

- $V=W$ $\quad B_1 = \begin{bmatrix} 0 & & 0 \\ 1 & \ddots & \\ 0 & \ddots & 1 \end{bmatrix} \quad b_1 = [* \cdots *]$

$\therefore$

	$V=0$	1	2		W		
$\mathcal{A}$	pt	$\mathbb{C}$	$\mathbb{C}^2$		$\mathbb{C}^W$	ϕ	$\cdots$

$$\rightsquigarrow \bigoplus_V H^{BM}_{top}(\mathcal{A}(V,W)) = (W+1) \text{ dim irrep of } \mathfrak{sl}_2$$

Next we put stability condition:

$$
\begin{tikzcd}
\mathbb{C} \arrow[r, shift left, "C"] & \mathbb{C}^2 \arrow[l, shift left, "D"] \arrow[r, shift left] & \cdots \arrow[l, shift left] \arrow[r, shift left] & \mathbb{C}^W \arrow[l, shift left] \arrow[loop above, "B_1"] \arrow[rr, "A"] \arrow[dr, "b_1"'] & & \mathbb{C}^V \arrow[loop above, "B_2"] \arrow[dr, "b_2"] & \\
& & & & \mathbb{C} \arrow[ur, "a_1"'] \arrow[loop below, "t"] & & \mathbb{C}
\end{tikzcd}
$$

$\exists !$ $g_1(t), g_2(t), \quad , \quad g_W(t), \; g_{W+1}(t)$

s.t.
$$
\left\{
\begin{aligned}
g_{i+1}(t)\, C_i\, g_i(t)^{-1} &= C_i \\
g_i(t)\, D_i\, g_{i+1}(t)^{-1} &= D_i \\
b_1\, g_W(t)^{-1} &= t\, b_1 \\
g_{W+1}(t)\, a_1 &= t\, a_1 \\
b_2\, g_{W+1}(t)^{-1} &= b_2
\end{aligned}
\right.
\qquad
\begin{aligned}
g_W(t) B_1 g_W(t)^{-1} &= B_1 \\
g_{W+1}(t) A\, g_W(t)^{-1} &= A
\end{aligned}
$$

Decompose v. spaces according to eigenvalues of $g(t)$

$$
\begin{tikzcd}
V_i(m) \arrow[r, shift left, "C"] & V_{i+1}(m) \arrow[l, shift left, "D"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
V_W(m) \arrow[dr, "b_1"] & \\
& \mathbb{C}
\end{tikzcd}
\qquad
\begin{tikzcd}
V_{W+1}(m) \arrow[dr] & \\
& \mathbb{C}
\end{tikzcd}
$$

0 " unless $m=-1$ $\qquad$ $b_2 = 0$ unless $m=0$

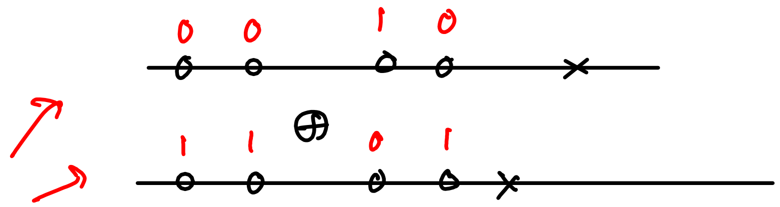
(S1) $\Rightarrow$ only $m=0$

$m = 0$ & -1

$$
\begin{tikzcd}
V_1(0) \arrow[r, shift left] & V_2(0) \arrow[l, shift left] \arrow[r, shift left] & \cdots \arrow[l, shift left] \arrow[r, shift left] & V_W(0) \arrow[l, shift left] \arrow[loop above, "B_1"] \arrow[r, "A", "\cong"'] & V_{W+1}(0) \arrow[loop above, "B_2"] \arrow[dr, "b_2"] & \\
V_1(-1) \arrow[r, shift left] & \cdots \arrow[l, shift left] \arrow[r, shift left] & V_W(-1) \arrow[l, shift left] \arrow[loop above, "B_1"] \arrow[dr] & & & \mathbb{C} \\
& & & \mathbb{C} & &
\end{tikzcd}
$$

collapse

B_1, B_2 irregular nilpotent



Only 1 or 0 & $\begin{matrix}0 & 1\\ 0 & 1\end{matrix}$ are not possible

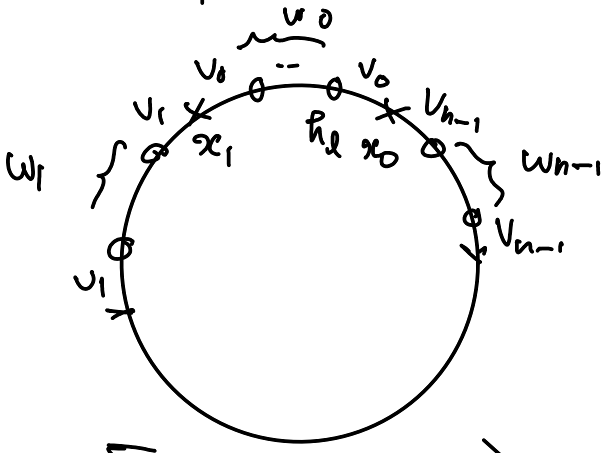
$$\therefore \ \# \text{fixed pts} = \binom{W}{V}$$

$$= \text{dim wt space of } \underbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{W}$$

weight $W - 2V$

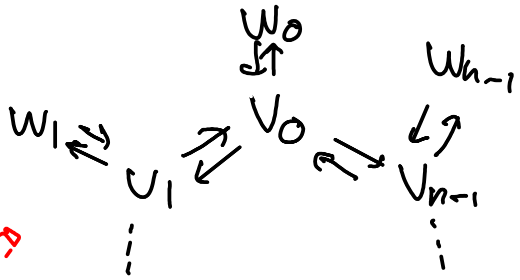
Compatible with $\bigoplus_{V} H^{BM}_{top}(\mathfrak{L}(V,W)) \cong \underbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{W} \supset S^W(\mathbb{C}^2)$

5.2 $\hat{A}_{n-1}$ case

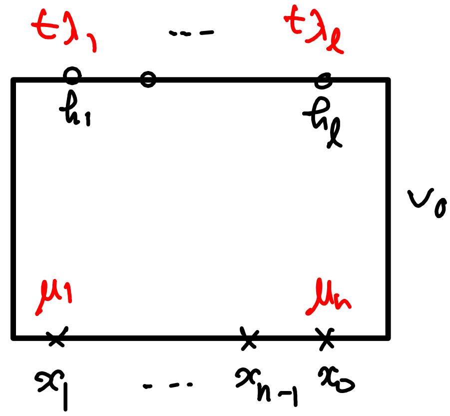


$l = \#\circ = \sum w_i$

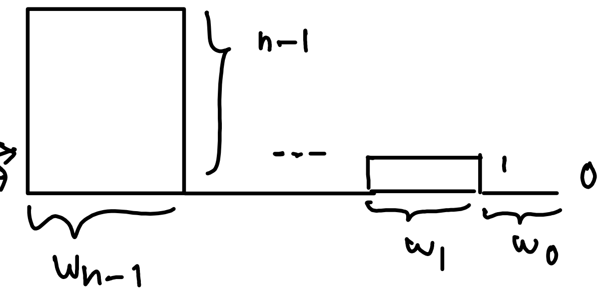
$\circ \longleftrightarrow \times$



HW keeping



$(t\lambda_1 \cdots t\lambda_l) =$



$$\begin{cases} \lambda_1 = w_1 + w_2 + \cdots + w_{n-1} \\ \lambda_2 = w_2 + \cdots + w_{n-1} \\ \vdots \\ \lambda_{n-1} = w_{n-1} \\ \lambda_n = 0 \\ \lambda_1 - l = -w_0 \end{cases}$$

$\therefore \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq \lambda_1 - l$

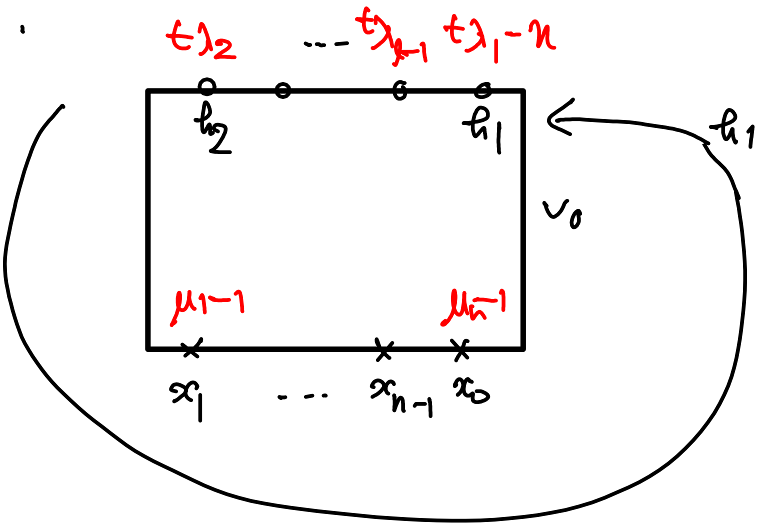
dominance cond. for affine wts

$\mu_n = v_{n-1} - v_0$ (because it is unchanged).

Then $\mu_i = v_{n-1} - v_0 + \sum_{j=i}^{n-1} u_j \qquad u_j = w_j + v_{j-1} + v_{j+1} - 2v_j$

as $\mu_i - \mu_{i+1} = u_i$

Remark 1.



${}^t\lambda_2 \geqq \cdots \geqq {}^t\lambda_{\ell-1} \geqq {}^t\lambda_1 - n$ ← from now on we don't need to assume the balanced cond.

Torus action $\mathbb{C}^\times$ on each $\mathbb{C}$ for $\times$

$\rightsquigarrow (\mathbb{C}^\times)^n$ but the Δ acts trivially

One additional

$$\begin{tikzcd} V_{n-1} \arrow[r, "t_* A_{x_0}"] \arrow[dr, "t_* b_{x_0}"'] & V_0 \\ & \mathbb{C}_{x_0} \arrow[u, "a_{x_0}"'] \end{tikzcd} \qquad T^n$$

$\widetilde{\mathcal{M}}$: smooth $\curvearrowleft T^n$

$[(A, B, a, b, C, D)]$ is fixed

$\iff \exists_1\ \rho : T^n \longrightarrow \prod GL(V_3)$ as before

$\rho(t) \curvearrowright V$ eigenvalue : $\lambda = t_0^{m_0} t_1^{m_1} \cdots t_{n-1}^{m_{n-1}} t_*^{m_*}$

$V(\lambda) =$ eigenspace for eigenvalue λ

But $b_{x_i}=0$ on $V(\lambda)$ with $\lambda \neq t_i$

$$\begin{tikzcd}
\circlearrowleft V_{i-1}(\lambda) \arrow[r, "\cong"] & V_i(\lambda) \circlearrowright \\
\oplus & \oplus \\
\circlearrowleft V_{i-1}(t_i) \arrow[r] \arrow[dr] & V_i(t_i) \circlearrowright \\
& \mathbb{C}_{x_i} \arrow[u]
\end{tikzcd}$$

by (S1,2)

$x_i \neq x_0$

$x_i = x_0$

$$\begin{tikzcd}
V_{n-1}(\lambda t_*) \arrow[r, "\cong"'] & V_0(\lambda) \\
\oplus & \oplus \\
V_{n-1}(t_0 t_*) \arrow[r] \arrow[dr] & V_0(t_0) \\
& \mathbb{C}_{x_0} \arrow[u]
\end{tikzcd}$$

stability condition $\Rightarrow \bigoplus_{n\in\mathbb{Z}} V(\lambda t_*^n) = 0$ if $\lambda \neq t_0, t_1, \dots, t_{n-1}$

We decompose according to $\lambda = t_i$

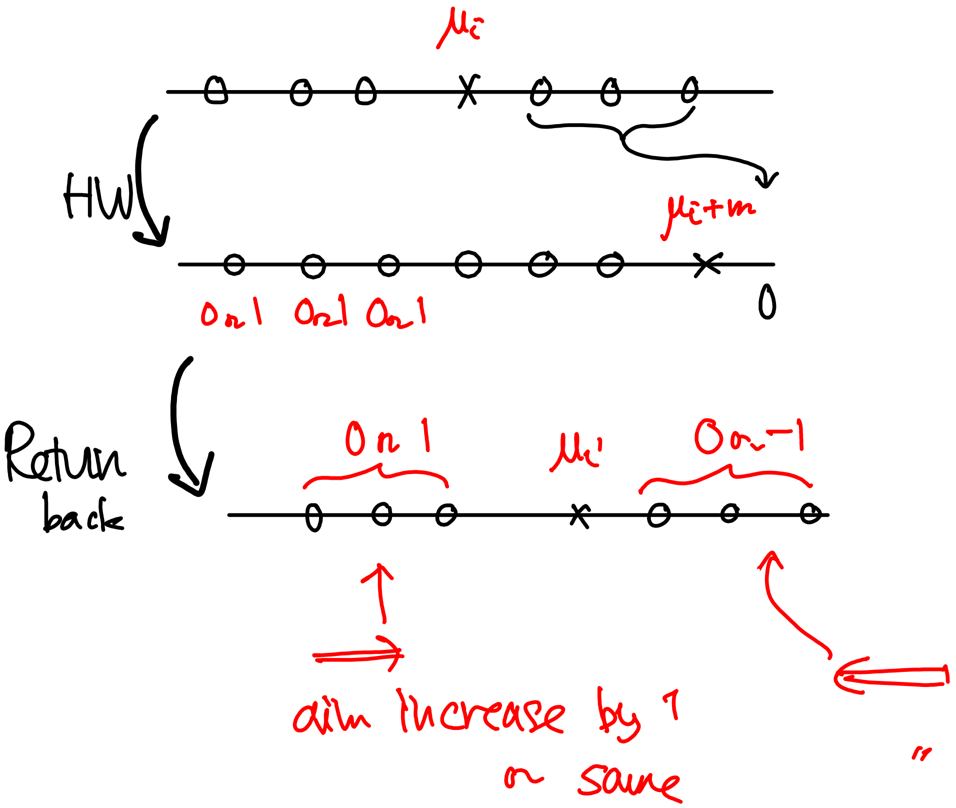
$$\begin{tikzcd}
& V_{i-1}(t_i t_*^m) \arrow[r, "\cong"] & V_i(t_i t_*^m) & (m \neq 0) \\
& \oplus & \oplus & \\
\arrow[r] & V_{i-2}(t_i) \arrow[r, "\cong"] & V_{i-1}(t_i) \arrow[r] \arrow[dr] & V_i(t_i) \arrow[r, "\cong"] & V_{i+1}(t_i) \\
& & & \mathbb{C}_{x_i} \arrow[u] &
\end{tikzcd}$$

We collapse the triangular part except

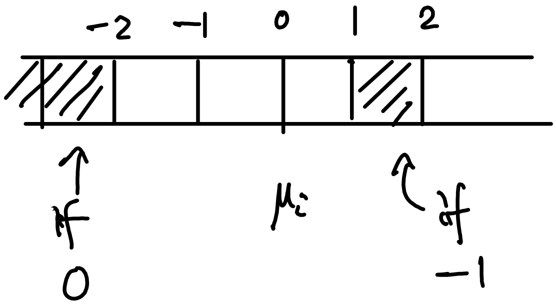
We unfold the circle to a line

$$\begin{tikzcd} \rightleftarrows \cdots\cdots \rightleftarrows & V_{i-1}(t_i) \arrow[loop above] \arrow[rr] \arrow[dr] & & V_i(t_i) \arrow[loop above] \rightleftarrows \cdots\cdots \rightleftarrows \\ & & \mathbb{C}_{x_i} \arrow[ur] & \end{tikzcd}$$

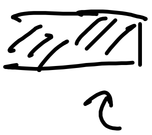
$\mu_i = \dim V_{i-1}(t_i) - V_i(t_i)$ as other summands have $A_{x_i} \xrightarrow{\cong}$



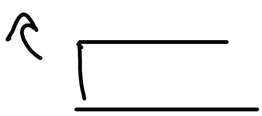
We assign Maya diagram



$i = 0, \cdots, n-1$



at left end

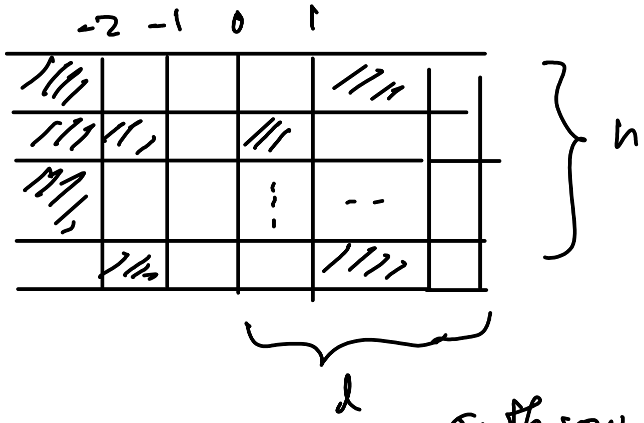


right end

$\# \square$ in $\longleftarrow -2 \ -1 \ 0$

$- \ \# \boxed{/\!/}$ in $0 \ 1 \ 2 \longrightarrow \quad = \mu_i$

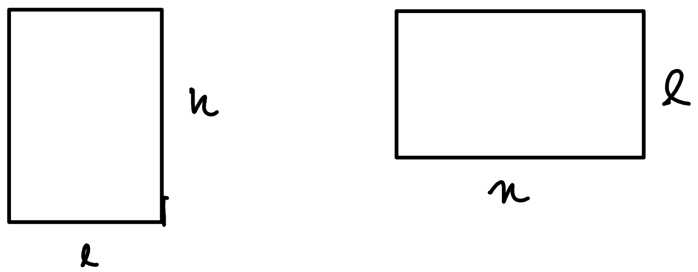
Combine $i = 0, \sim n-1$



σ-th row

$\Uparrow \ t\lambda_\sigma$

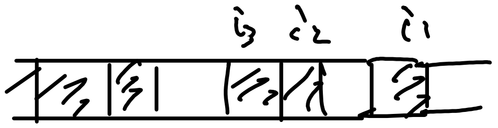
<u>Prop</u> $e(\tilde{M}) = e(\tilde{M}(0 \leftrightarrow x))$



Maya diagram appears in representation theory of $\widehat{gl}(\infty)$

$$\mathbb{C}^\infty = \langle v_{\frac{1}{2}+m} \mid m \in \mathbb{Z} \rangle$$

$$\bigwedge^{\frac{\infty}{2}} \mathbb{C}^\infty = \langle v_{i_1} \wedge v_{i_2} \wedge \cdots \mid i_1 > i_2 > \cdots \rangle$$



$\cdots \wedge v_N \wedge v_{N-1} \wedge v_{N-2} \wedge \cdots$ at right end

$$\mathfrak{gl}(\infty) = \left\{ \begin{array}{c|c} & \cdots\ -\tfrac{3}{2}\ \ -\tfrac{1}{2}\ \ \tfrac{1}{2}\ \ \tfrac{3}{2}\ \cdots \\ \hline \begin{array}{c} \vdots \\ -\tfrac{1}{2} \\ \tfrac{1}{2} \\ \vdots \end{array} & \begin{pmatrix} \ddots & \ddots & \ddots & & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & & \ddots & \ddots & \ddots \end{pmatrix} \end{array} \;\middle|\; \right\}$$

for sufficiently large

$\widehat{\mathfrak{gl}}(\infty)$: central extension of $\mathfrak{gl}(\infty)$ s.t. $\begin{pmatrix} \ddots & & & 0 \\ & \ddots & & \\ & & \ddots & \\ 0 & & & \ddots \end{pmatrix}$

diagonal

$$\widehat{\mathfrak{sl}}(n) \hookrightarrow \widehat{\mathfrak{gl}}(\infty)$$

Fock rep. $\leftsquigarrow \Lambda^{\frac{\infty}{2}} \mathbb{C}^{\infty}$