Ginzburg-Riche in geometric Satake for affine Lie algebras

@zoom 20200512

joint work in progress with Dinakar Muthiah

Plan

§1 geometric Satake (usual & Braverman-Finkelberg)

§2. Coulomb branch & geometric Satake for Kac-Moody

§3. Ginzburg-Riche

§4. Application to higher level AGT (instantons on $\mathbb{C}^2/\mathbb{Z}/\ell$)

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§1. geometric Satake (usual & Braverman-Finkelberg)

$K = \mathbb{C}((z)) \supset \mathcal{O} = \mathbb{C}[[z]]$

G: complex reductive group

$\mathrm{Gr}_G = G_K/G_\mathcal{O}$: affine Grassmannian

$\circlearrowleft$

$G_\mathcal{O}$ orbits $\overset{1:1}{\longleftrightarrow}$ dominant coweights Λ^+

$\overline{\mathrm{Gr}^\lambda_G}$ = closure of $G_\mathcal{O}$-orbit through z^λ

$$(\overline{\mathrm{Gr}^\lambda_G})^T = \coprod_{\mu \le \lambda} \{z^\mu\}$$

Choose a dominant cocharacter χ

$\Phi = \Phi_\chi$: hyperbolic restriction functor

$$\overset{\shortparallel}{\oplus \Phi_\mu} \quad \mathrm{Perv}_{G_{\mathcal{O}}}(\mathrm{Gr}_G) \longrightarrow \bigoplus_\mu \mathrm{Vect} = \mathrm{Rep}(T^\vee)$$

<u>Rem</u>. hyperbolic restriction sends pure to pure

In this situation, a stronger assertion holds:

perverse to perverse.

<u>Th</u> [Mirkovic-Vilonen]

$\exists$ canonical isom

$$\oplus \Phi_\mu : \Phi(IC(\overline{\mathrm{Gr}^\lambda_G})) \cong V_{G^\vee}(\lambda) = \bigoplus_\mu V_\mu(\lambda)$$

$i_\mu : \{z^\mu\} \hookrightarrow \mathrm{Gr}_G$

Consider costalk

<u>Th</u> [Lusztig, Brylinski, Ginzburg, ...]

(1) $H^*(i_\mu^! IC(\overline{\mathrm{Gr}^\lambda_G})) \cong \mathrm{gr}^*_F V_\mu(\lambda)$

graded v. sp.

where F = Brylinski filtration

e : principal nilpotent $F^i V(\lambda) = \{ e^i v = 0 \}$

(2) Poincaré polynomial of RHS = q-analog of weight multiplicity

combinatorial expression $\nearrow$

in terms of Kostant partition function

In order to generalise this to affine Lie algebras, let us give a slightly different formulation:

Suppose μ: dominant $\quad Gr_G^\mu \subset \overline{Gr_G^\lambda}$

a standard construction gives a transversal slice

$$\mathcal{M}(\lambda,\mu) \circlearrowleft \mathbb{T} = T \times \mathbb{C}^\times$$

└ loop rotation

- affine algebraic variety
- conical (e.g. fixed pt $= \{z^\mu\}$)

<u>Notation</u> $\quad IC_\mu^\lambda := IC(\mathcal{M}(\lambda,\mu))$

The hyperbolic restriction $\Phi \equiv \Phi_\chi$ can be defined in this setting and

$$\Phi(IC_\mu^\lambda) \cong V_\mu(\lambda)$$

<u>Braverman–Finkelberg (07)</u>

The same should hold for affine Kac-Moody group if we replace $\mathcal{M}(\lambda,\mu)$ by moduli space of G_{cpt}-instantons on $\mathbb{C}^2/\mathbb{Z}_\ell$

- ℓ = level
- appropriate b'dry condition corresponding to λ
- appropriate monodromy at $0 \in \mathbb{C}^2/\mathbb{Z}_\ell$ cor. to μ
- partial compactification (due to bubbling)

(– $\otimes$ can be realized as in the usual geometric Satake)

§2. Coulomb branch & geometric Satake for Kac-Moody

We omit the definition of Coulomb branches
([N 15], [Braverman-Finkelberg-N 16], [N-Weekes 19])

Upshot : $\mathfrak{g}$: symmetrizable Kac-Moody Lie algebra
λ : dominant integral weight
μ : another wt (not necessarily dominant)

$\rightsquigarrow \mathcal{M}(\lambda, \mu) \curvearrowleft \mathbb{T} = T \times \mathbb{C}^\times_\hbar$

If you know the [BFN] definition : $\pi_1(G)^\wedge$ homological grading

generalise
- affine Grassmannian slices [BFN], [NW]
- instanton moduli spaces [N-Takayama 17] (at least type A)

Choose χ : dominant cocharacter $\mathbb{C}^\times \to T$

Conjecture [BFN'17] $\Phi(IC^\lambda_\mu) \overset{\exists}{\cong} V_\mu(\lambda)$

for $\mathfrak{g}^\vee$ = Langlands dual of $\mathfrak{g}_Q$

[N'18]
- recipe to define the action of e_i, f_i (Chevalley generators) on LHS assuming reasonble symplectic geometric properties of $\mathcal{M}(\lambda, \mu)$
- this recipe works for affine type A thanks [NT17].

Remark. $\mathcal{M}(\lambda,\mu) \neq \phi \quad \forall \mu \leq \lambda$

Conjecture includes $\mathcal{M}(\lambda,\mu)^T \neq \phi \Leftrightarrow V_\mu(\lambda) \neq 0$

and $\mathcal{M}(\lambda,\mu)^T = \{z^\mu\}$ single point if $\neq \phi$

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§3. Ginzburg–Riche

Consider equivariant costalk $H^*_{\mathbb{T}}(i^!_\mu IC^\lambda_\mu)$

where $i_\mu : \{z^\mu\} \hookrightarrow \mathcal{M}(\lambda,\mu)$

[GR13] description of costalk in terms of Langlands dual G

Note $H^*_{\mathbb{T}}(i^!_\mu IC^\lambda_\mu) \hookrightarrow H^*_{\mathbb{T}}(\Phi(IC^\lambda_\mu)) \cong V_\mu(\lambda) \otimes_{\mathbb{C}} H^*_{\mathbb{T}}(pt)$

$H^*_{\mathbb{T}}(pt) = S(\mathfrak{t}^*)[\hbar] = S(\mathfrak{t}^\vee)[\hbar]$

This embedding become $\cong$ after $\otimes_{H^*_{\mathbb{T}}(pt)}$ Frac $H^*_{\mathbb{T}}(pt)$

Recall $\mathbb{T} = T \times \mathbb{C}^\times_\hbar$

For brevity, let us explain the precise statement only for $\hbar = 0$, and mention general $\hbar$ briefly.

Take $G^\vee \supset B^\vee \supset T^\vee$, $\supset U^\vee$, $\mathfrak{g}^\vee \supset \mathfrak{b}^\vee \supset \mathfrak{t}^\vee$, $\supset \mathfrak{n}^\vee$

$$T^*\text{flag} = G^\vee \times_{B^\vee} (\mathfrak{g}^\vee/\mathfrak{b}^\vee)^* \subset G^\vee \times_{B^\vee} (\mathfrak{g}^\vee/\mathfrak{n}^\vee)^* =: \tilde{\mathfrak{g}}^\vee$$

universal deformation of T^*flag parametrized by $(\mathfrak{t}^\vee)^* = \mathfrak{t}$

Both are vector b'dles over flag variety.
In particular, we can pull back $\mathcal{O}(\mu)$ on flag.

Thm [GR13] $H^*_T(i_\mu^! IC_\mu^\lambda) \cong \left(V(\lambda)\otimes H^0(\tilde{\mathfrak{g}}^\vee, \mathcal{O}_{\tilde{\mathfrak{g}}^\vee}(\mu))\right)^{G^\vee}$

graded $H^*_T(pt)$-module isomorphism; $H^*_T(pt) = S(\mathfrak{t}^\vee)$; the right side is graded by $\mathbb{C}^\times$-action mult. on fibers

(n=0 version follows from [Arkhipov-Bezrukavnikov-Ginzburg 04])
usually stated for the ring object $\mathcal{A} = \mathrm{Satake}^{-1}(\mathbb{C}[G^\vee])$

$a : G^\vee/T^\vee \times \mathfrak{t}^{\vee *} \longrightarrow \tilde{\mathfrak{g}}$; $(gT, \xi) \mapsto g^*{}_B\xi$ birational

$\therefore$ RHS $\xrightarrow{a^*} \left(V(\lambda)\otimes \mathrm{Ind}_{T^\vee}^{G^\vee}(\mathbb{C}_{-\mu})\otimes S(\mathfrak{t}^\vee)\right)^{G^\vee} = V_\mu(\lambda)\otimes S(\mathfrak{t}^\vee)$

This corresponds to LHS: $H^*_T(i_\mu^! IC_\mu^\lambda) \hookrightarrow H^*_T(\Phi(IC_\mu^\lambda))$

We also have purely algebraic formulation of RHS:

$$\mathrm{RHS} = \left(V(\lambda)\otimes \mathrm{Ind}_{B^\vee}^{G^\vee}(S(\mathfrak{g}^\vee/\mathfrak{n}^\vee)\otimes \mathbb{C}_{-\mu})\right)^{G}$$
$$= \left(V(\lambda)\otimes S(\mathfrak{g}^\vee/\mathfrak{n}^\vee)\otimes \mathbb{C}_{-\mu}\right)^{B}$$

For $\hbar$ replace $S(\mathfrak{g}^\vee/\mathfrak{n}^\vee)\otimes \mathbb{C}_{-\mu}$ by a suitable version of Verma $M(\mu)$

Remarks (1) Together with cohomology vanishing $H^{>0}(\tilde{\mathfrak{g}}, \mathcal{O}(\mu)) = 0$
one can prove Poincaré poly. of RHS = q-weight multiplicity [Lusztig, ...]

(2) Considering $\otimes_{H^*_T(pt) \uparrow \text{eval. at } 0} \mathbb{C} = \otimes_{S(\mathfrak{t}^\vee)} \mathbb{C}$, one can prove

$$\text{RHS} \otimes_{S(\mathfrak{t}^\vee)} \mathbb{C} \cong gr^\bullet_F V_\mu(\lambda)$$ [Brylinski, Broer, Ginzburg]

↖ Brylinski filtration

Note (1), (2) are proved by [Slofstra 10] for affine Lie algebras once Brylinski filtration is replaced by one defined by principal Heisenberg subalgebra. (correction to [BF07] conjecture)

Conjecture [Muthiah-N'20]

$$H^*_{\mathbb{T}}(i^!_\mu IC^\lambda_\mu) \cong (V(\lambda) \otimes M(\mu))^B$$

Main Th (in progress)
Conjecture is true for affine type A

Remarks (1) The strategy of the proof is same as [GR]: reduction to "rank 1" — $\mathfrak{sl}(2)$
↘ Heisenberg alg.

(2) Our proof should work once geometric Satake is established (for affine case)

§4. Application to higher level AGT.

(learn from Arakawa)

Assume $\mathfrak{g}$: affine type ADE ($\Rightarrow \mathfrak{g}^\vee = \mathfrak{g}$)

Take $\lambda = \Lambda_0$ (basic representation of $\mathfrak{g}^\vee$)

$\mu = \lambda - N\delta \qquad N \in \mathbb{Z}_{\geq 0}$

$\Rightarrow \mathcal{M}(\lambda,\mu) \underset{\substack{\text{Conj.} \\ \text{true for type A}}}{=}$ moduli of G_{cpt} instantons on $\mathbb{C}^2$ with $c_2 = N$

Th. [BFN14] (a version of AGT)

$\bigoplus_N IH^*_{\mathbb{T},C}\left(\substack{\text{inst.} \\ \text{moduli} \\ c_2=N}\right)$: "universal" Verma module of "integral form" of W-algebra

General case :

fix λ , $\mu = \mu_{max} - N\delta \quad (N \in \mathbb{Z}_{\geq 0})$

$$\bigoplus_N H^*_{\mathbb{T}}(i_\mu^! IC_\mu^\lambda) = \bigoplus_N (V(\lambda) \otimes M(\mu))^B$$

We can ask which vertex algebra acts on the RHS.

Answer : the coset $(V_\ell(\mathfrak{g}) \otimes V^k(\mathfrak{g}))^{\hat{\mathfrak{g}}[z]}$

(Conj. by Belavin-Feigin $\mathfrak{g} = \mathfrak{sl}(2), \ell = 2$ Nishioka-Tachikawa general

ℓ = level of λ

$k = k(\varepsilon_1, \varepsilon_2, \mathfrak{g}) \qquad (\varepsilon_1 + \varepsilon_2 = \hbar \ , \ \varepsilon_1 - \varepsilon_2 = \delta)$

$V^k(\mathfrak{g})$ = universal affine vertex algebra at level k

$V_\ell(\mathfrak{g})$: simple quotient at level $\ell \in \mathbb{Z}_{>0}$

This recover the [BFN14] as
coset for $\ell = 1$ $\cong$ W-algebra

by [Arakawa-Creutzig-Linshaw 19]

Rem. If μ : dominant $\mathcal{M}(\lambda, \mu) =$ quiver variety of affine type A

$\therefore$ affine Yangian acts on equiv. cohomology [Varagnolo-Vasserot 99] [N '99], [Varagnolo 00]

Thus affine Yangian $\xrightarrow[?]{}$ coset $\mathfrak{g} =$ type A

(affine analog of Brundan-Kleshchev)