w/ Braverman, Finkelberg

($\mathbb{G}$: cpx reductive grp $\mathbb{G}_c$: max. cpt subgrp
$\mathbb{N}$: finite dim. rep. of $\mathbb{G}$
$\mathbb{M} = \mathbb{N} \oplus \mathbb{N}^*$: symplectic rep. of $\mathbb{G}$
(cotangent type)

$\rightsquigarrow$ 3d $N=4$ SUSY gauge theory
based on (• a connection A on $P \xrightarrow[\mathbb{G}_c]{} M^3_g$
• spinor / forms
with values in $P \times^{\mathbb{G}_c} \mathbb{M}$

$\rightsquigarrow$ Higgs / Coulomb branches

$\mathcal{M}_H \equiv \mathcal{M}_H(\mathbb{G}, \mathbb{N})$ $\mathcal{M}_C \equiv \mathcal{M}_C(\mathbb{G}, \mathbb{N})$

hyperKähler manifolds
possibly with singularities

Roughly gauge theory
"equiv." σ-model $M^3 \longrightarrow \#$
whose
target space
$\mathcal{M}_C \cup \mathcal{M}_H$

$\mathcal{M}_H$ <u>does not</u> receive "quantum correction"

$\mathcal{M}_C$ <u>does</u> receive "

BFN : mathematically rigorous definition of $\mathcal{M}_C$ as an algebraic variety

---

✲ FI parameter

$$\zeta \in \mathrm{Hom}(\mathbb{C}^\times, \pi_1(\mathbb{G})^\wedge)$$

↳ Pontryagin dual

$$\cong \mathrm{Hom}_{grp}(\mathbb{G}, \underset{\pi_1(\mathbb{C}^\times)^\wedge = (\mathbb{C}^\times)^\vee}{\overset{\parallel}{\mathbb{C}^\times}})$$

dual

ex. • $\mathbb{G} = GL_n(\mathbb{C})$ & its products $\longrightarrow \pi_1(\mathbb{G}) = \mathbb{Z}$, $\mathbb{Z}^N$

• $\mathbb{G} = SL_n(\mathbb{C}) \longrightarrow \pi_1(\mathbb{G}) = 1$

✲ Flavor symmetry

$\tilde{\mathbb{G}} \triangleright \mathbb{G}$ normal subgrp $\qquad \tilde{\mathbb{G}} \curvearrowright \mathbb{N}$, $\curvearrowleft \mathbb{G}$

$\mathbb{G}_F = \tilde{\mathbb{G}}/\mathbb{G}$ flavor symmetry group

$\cup$

T_F max torus

mass parameter $m \in \mathrm{Hom}(\mathbb{C}^\times, T_F)$

§ Higgs branch

$\mathcal{M}_H \equiv \mathbb{M} /\!/\!/ \mathbb{G}$ symplectic reduction

$\mu : \mathbb{M} \longrightarrow \mathfrak{g}^*$ moment map

$$\langle \underset{\mathbb{M}}{\mu(\underset{\uparrow}{x})}, \underset{\mathfrak{g}}{\underset{\uparrow}{\xi}} \rangle = \frac{1}{2}\, \omega(\xi x, x)$$

ω: symp. form on $\mathbb{M}$.

$$\mathbb{M} /\!/\!/ \mathbb{G} = \mu^{-1}(0) /\!/ \mathbb{G}$$ categorical quotient

$$= \operatorname{Spec}(\mathbb{C}[\mu^{-1}(0)]^{\mathbb{G}})$$

$G_F \curvearrowright \mathcal{M}_H$

$G_F \cong \widetilde{\mathbb{G}}/\mathbb{G}$, $\quad T_F \subset G_F$, $\quad \mathbb{C}^\times \xrightarrow{m} T_F$

✲ FI parameter $\quad \zeta \in \mathrm{Hom}_{grp}(\mathbb{G}, \mathbb{C}^\times)$

$$\mathbb{C}[\mu^{-1}(0)]^{\mathbb{G}, \zeta^n} = \{ f \in \mathbb{C}[\mu^{-1}(0)] \mid g^* f = \zeta^n(g) f \}$$

relative invariants

$$\mathbb{C}[\mu^{-1}(0)]^{\mathbb{G}} \subset \bigoplus_{n \geq 0} \mathbb{C}[\mu^{-1}(0)]^{\mathbb{G}, \zeta^n}$$ graded algebra

$$\mathcal{M}_H^\zeta := \mu^{-1}(0) /\!/_{\zeta} \mathbb{G} = \operatorname{Proj}(\quad\uparrow\quad) \simeq \mu^{-1}(0)^{\zeta\text{-ss}} / \sim \ (\zeta\text{-equiv.})$$

$\mathcal{M}_H^\zeta \longrightarrow \mathcal{M}_H$ proj. morphism

$\mathcal{M}_H^\zeta = \mu^{-1}(0)^{\zeta\text{-st}} / \mathbb{G}$ (nice case)

(Kähler parameter)

$$d\zeta \in \mathrm{Hom}_{\text{Lie alg}}(\mathfrak{g}, \mathbb{C}) \subset \mathfrak{g}^*$$

$\mathfrak{g} = \mathrm{Lie}\,\mathbb{G}$

$d\zeta = \zeta_{\mathbb{C}}$ fixed by coadjoint action

$$\mu^{-1}(\zeta_{\mathbb{C}}) /\!/ \mathbb{G} = \mathcal{M}_H^{\zeta_{\mathbb{C}}}$$

$$\mu^{-1}(\zeta_{\mathbb{C}}) /\!/_{\zeta} \mathbb{G} = \mathcal{M}_H^{\zeta, \zeta_{\mathbb{C}}}$$

<u>Remarks</u> (1) hyperkähler quotient
(instead of GIT quotient)

(2) If $\mathbb{G} \curvearrowright \mu^{-1}(0)^{\zeta\text{-stable point (semi)}}$
is free,

$\mu^{-1}(\zeta_{\mathbb{C}})^{\zeta\text{-st.}}$ and $\mu^{-1}(\zeta_{\mathbb{C}}) /\!/_{\zeta} \mathbb{G}$
are of expected dim,
$\dim \mathbb{M} - 2\dim \mathbb{G}$.

Example $\begin{array}{c} \textcircled{v} \\ | \\ \boxed{w} \end{array}$ $\qquad v, w \in \mathbb{Z}_{\geq 0}$

(quiver gauge theory)

$$V = \mathbb{C}^v \quad , W = \mathbb{C}^w$$

$$G = GL(V)$$
$$N = \mathrm{Hom}(W, V) \curvearrowleft G \triangleleft \tilde{G} = G \times GL(W)$$

$$M = N \oplus N^* \qquad \begin{tikzcd} V \arrow[d, shift right, "b"'] \\ W \arrow[u, shift right, "a"'] \end{tikzcd} \qquad \mu = ab \in \mathfrak{gl}(V) \cong \mathfrak{gl}(V)^*$$

$$\mathcal{M}_H = \mu^{-1}(0) /\!/ GL(V) = \mathrm{Spec}\left(\mathbb{C}[\mu^{-1}(0)]^G\right)$$

$$X := ba \in \mathrm{End}(W) \quad (G\text{-invariant})$$

$$X^2 = b\underbrace{ab}_{=0}a = 0$$

$$(a,b) \bmod GL(V) \mapsto ba$$

Fact. $\mathcal{M}_H \cong \{ X \in \mathrm{End}(W) \mid X^2 = 0, \mathrm{rk}\, X \leq v \}$

If $v \geq w/2$, $\mathrm{rk}\, X \leq v$ is automatically satisfied

$$\left(X \sim (22 \cdots 1 \cdots 1) \right)$$

$\zeta \in \mathrm{Hom}(G, \mathbb{C}^\times)$
$\quad \| $
$\quad \det$

$\Downarrow$ b: injective

GIT $\qquad \mu^{-1}(0) /\!\!/_{\zeta}\, G = \mu^{-1}(0)^{\zeta\text{-st}} / G$

$$\mathcal{M}_H^{\zeta} \cong T^* Gr(v, w)$$

Grassmannian of v-dim. subsp. in $W = \mathbb{C}^w$

$$[a,b] \mapsto (\mathrm{Im}\, b,\ X = ba)$$

$\mathrm{Im}\, b \subset W$

- $X(\mathrm{Im}\, b) = 0$
- $\mathrm{Im}\, X \subset \mathrm{Im}\, b$

If $v > w \Rightarrow \mathcal{M}_H^{\zeta} = \emptyset$

$v = w \quad \mathcal{M}_H^{\zeta} = pt \longrightarrow \mathcal{M}_H = \{X \mid X^2 = 0\}$

<u>Fact.</u> If $v \leq w/2 \Rightarrow \mathcal{M}_H^{\zeta} \to \mathcal{M}_H$ resol. of singularities

$\zeta_{\mathbb{C}} = d\zeta \in \mathrm{Hom}_{\text{Lie alg}}(\mathfrak{g}, \mathbb{C})$

scalar matrix

$$\mathcal{M}_H^{\zeta_{\mathbb{C}}} = \mu^{-1}(\zeta_{\mathbb{C}})/G \cong \text{coadjoint orbit through } \begin{bmatrix} \overbrace{0 \ \ddots \ 0}^{w-v} & 0 \\ 0 & \overbrace{\zeta_{\mathbb{C}} \ \ddots \ \zeta_{\mathbb{C}}}^{v} \end{bmatrix}$$

§ Convolution algebra

X : finite set

$\mathcal{F}(X\times X) = \mathbb{C}$-valued fcts on $X\times X$

$\ni$ K, K' $\qquad K*K'(x,z) \overset{\text{def}}{=} \sum_{y\in X} K(x,y)K'(y,z)$

$\mathcal{F}(X\times X)$ is an associative algebra

$\cong \mathrm{Mat}_{n\times n}(\mathbb{C}) \qquad n=|X|$

G finite group $\curvearrowright X$

$G \curvearrowright X\times X$ diagonal action

$\mathcal{F}(X\times X)^G$ G-inv. functions on $X\times X$

$\llcorner$ subalgebra

$G = GL_n(\mathbb{F}_q) \supset B =$ upper triangular part

$X = G/B$ flag variety over $\mathbb{F}_q$

$G \curvearrowright X\times X$ $\qquad$ G-orbits in $X\times X$ $\overset{1:1}{\longleftrightarrow}$ B-orbits in X

$\longleftrightarrow \mathfrak{S}_n$: symmetric grp

Th (Iwahori)

$\mathbb{F}(X \times X)^G$ generators T_i $i=1,\dots,n-1$

relations $(T_i+1)(T_i-q)=0$

$T_i T_j = T_j T_i$ if $|i-j|>1$

$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$

deformation of $\mathbb{C}[S_n]$

Iwahori – Hecke algebra

For the definition of Coulomb branches we use convolution algebra on equivariant Borel – Moore homology group.

(cf (non-equiv.) Borel – Moore homology Fulton : Young tableaux

M : smooth and cpt, oriented

$G \curvearrowright$

$H_*^G(M) \cong H_G^{\dim M - *}(M)$ (Poincaré duality)

M : noncpt replaced homology of by locally finite chains

functorial under proper morphisms

$M \xrightarrow{\pi} X$ proper

smooth

$\circlearrowleft G \circlearrowleft$ - equivariant

$$Z = M \times_X M = \{(x_1, x_2) \in M\times M \mid \pi(x_1) = \pi(x_2)\}$$

$\cap$ closed

$M \times M$

$$M\times M\times M \xrightarrow{p_{ij}} M\times M \qquad (i,j) = (1,2),(2,3),(1,3)$$

$$\begin{array}{l} p_{12}^{-1}(Z) \cap p_{23}^{-1}(Z) = \{(x_1,x_2,x_3) \mid \pi(x_1) = \pi(x_2) = \pi(x_3)\} \\ \quad \Big\downarrow p_{13} \qquad \text{proper} \\ \quad Z \end{array}$$

$H_*^G(Z)$ convolution product

$\ni$

K, K'

$$p_{13*}\Big(\underbrace{p_{12}^*(K)}_{\substack{\uparrow \\ H_*^G(\underbrace{p_{12}^{-1}Z}_{= Z\times M})}} \cap_{M\times M\times M} \underbrace{p_{23}^*(K')}_{\substack{\uparrow \\ H_*^G(M\times Z)}}\Big) =: K * K'$$

$H_*^G(Z)$ with $*$ is an associative algebra.

(convolution algebra)

usually noncommutative.

$H^*_G(M)$ with $\cup$

$M \xrightarrow[\text{diagonal embedding}]{} Z = M \times_X M$

$H^G_*(M) \longrightarrow H^G_*(Z)$

$\cup \qquad\qquad *$

alg. hom.

<u>Example</u>

Fix w

$$\underbrace{\coprod_{0 \le v \le w} T^*Gr(v,w)}_{\rotatebox{90}{=} \, \mathcal{M}_H^{\xi}} \longrightarrow \underbrace{\{X \in End(W) \mid X^2 = 0\}}_{\rotatebox{90}{=} \, \mathcal{M}_H}$$

$$\underbrace{\phantom{\coprod T^*Gr(v,w) = \mathcal{M}_H^{\xi}}}_{= M} \qquad \underbrace{\phantom{\{X \in End(W)\}}}_{= X}$$

$GL(W) \times \underbrace{\mathbb{C}^\times}_{\text{scaling in cotangent direction}}$

$$Z = M \times_X M$$

$$= \coprod_{v_1, v_2} \underbrace{T^*Gr(v_1,w) \times_X T^*Gr(v_2,w)}$$

$$H_{top}(Z) = \bigoplus_{v_1,v_2} H_{top}(\qquad)$$

lagrangian subvar. in $T^*Gr(v_1,w) \times T^*Gr(v_2,w)$

Th [Ginzburg]

$$U(\mathfrak{sl}_2) \longrightarrow H_{top}(Z)$$

$\exists$ alg. hom.

E, F, H

$\uparrow \nearrow$ $\sum$ multiple of $\Delta_{T^*Gr(v,w)}$

correspondences

(Higgs side of geometric Satake correspond. for the simplest case of $\mathfrak{sl}_2$)

$$T^*Gr(v,w) \times T^*Gr(v+1,w)$$

$\cup$ smooth subv.

$$\{(S_1, X, S_2) \mid S_1 \subset S_2\}$$

$S_1 \subset W$, $S_2 \subset W$

$$F \mapsto \sum_v \pm [\text{fund. class of } \ ]$$

$$E \mapsto \sum \pm [\text{exchange } v \leftrightarrow v+1]$$

$$\left(\begin{array}{l} H_*^{G\times\mathbb{C}^*}(Z) \longleftarrow \text{Yangian} \\ K^{G\times\mathbb{C}^*}(Z) \longleftarrow U_q(\text{affine } \mathfrak{sl}_2) \end{array} \right.$$