	$\mathcal{M}_H$	$\mathcal{M}_C$
FI parameter	Kähler parameter	
mass parameter	equivariant param.	

<u>Example</u> (of convolution algebra)

Fix w
$$\underbrace{\underbrace{\coprod_{0\leq v\leq w} T^*Gr(v,w)}_{=\mathcal{M}_H^{\zeta}}}_{=M} \longrightarrow \underbrace{\underbrace{\{X\in \mathrm{End}(W)\mid X^2=0\}}_{=\mathcal{M}_H}}_{=X}$$

$$\underbrace{GL(W)\times \mathbb{C}^\times}$$

scaling in cotangent direction

$$Z = M\times_X M$$
$$= \coprod_{v_1,v_2} \underbrace{T^*Gr(v_1,w)\times T^*Gr(v_2,w)}_{X}$$

lagrangian subvar. in $T^*Gr(v_1,w)\times T^*Gr(v_2,w)$

$$H_{top}(Z) = \bigoplus_{v_1,v_2} H_{top}(\quad)$$

$$top = 2(\dim Gr(v_1,w) + \dim Gr(v_2,w))$$

<u>Th</u> [Ginzburg]
$$U(sl_2) \longrightarrow H_{top}(Z)$$
$\exists$ alg. hom.

E, F, H

$\sum$ multiple of $\Delta_{T^*Gr(v,w)}$

(Higgs side of geometric Satake correspond. for the simplest case of sl_2

E, F are defined by correspondences

$$T^*Gr(v,w) \times T^*Gr(v+1,w)$$

$\cup$ smooth subv.

Correspondence $\{(S_1, X, S_2) \mid S_1 \subset S_2\}$ smooth subvariety

$S_1 \subset W$, $S_2 \subset W$

$$F \mapsto \sum_v \pm [\text{fund. class of } \ ]$$

$$E \mapsto \sum \pm [\text{exchange } v \leftrightarrow v+1]$$

$G_F = GL(W)$

$\mathbb{C}^\times \curvearrowright$ scaling on fibers

$\mathcal{M}_H^s$, $\mathcal{M}_H$ has symplectic forms ω on the open locus of free orbits

G_F preserves ω

$\mathbb{C}^\times$ scales by wt 1

$H_*^{G_F \times \mathbb{C}^\times}(Z)$ convolution algebra

$H^*_{\mathbb{C}^\times}(pt) = \mathbb{C}[\hbar]$

Th [Ginzburg–Vasserot]

$Y_\hbar(\mathfrak{sl}_2) \xrightarrow{\cong} H_*^{G_F \times \mathbb{C}^\times}(Z)$ alg. hom

$U(\mathfrak{sl}_2) \subset$ Yangian of $\mathfrak{sl}_2$

$\cup\ H_{top}(Z)$

(deformation of $U(\mathfrak{sl}_2[z])$

generator E_k + relations

F_k

H_k

$E \otimes z^k$

$F \otimes z^k$ $(k = 0, 1, \cdots)$

$H \otimes z^k$

$E_k \mapsto \pm c_1(\underbrace{S_2/S_1}_{})^k \cap [\text{fund. class}]$

line b'dle over the correspondence

$F_k \mapsto v \leftrightarrow v+1$

H_k = some Chern class of natural bundles of $\Delta_{T^*Gr(v,w)}$

cf. Maulik–Okounkov

geometric construction of R-matrix

$\xrightarrow[RTT]{}$ Yangian (Hopf algebra)

Open Problem Construction of Yangian in Coulomb branch side

equivariant parameter in $\mathcal{M}_H$ — Kähler parameter in $\mathcal{M}_C$

§ affine Grassmannian

<u>Ref</u> Xinwen Zhu 1603.05593

$\mathbb{G} = \mathbb{G}$: cpx reductive group

$\mathcal{O} = \mathbb{C}[[z]] \subset K = \mathbb{C}((z))$

$\operatorname{Spec} \mathcal{O} = D \supset \operatorname{Spec} K = D^\times$

formal disk punctured disk

$G(\mathcal{O}) \subset G(K)$

$Gr_G = G(K)/G(\mathcal{O})$ affine Grassmannian

∞-dim'l paratial flag variety

G/P

$G(K) \curvearrowright G(\mathcal{O})$ $\quad$ $G(K)$ $\quad$ $G(\mathcal{O})$

$$\{G(\mathcal{O})\text{-orbits}\} = G(\mathcal{O})\backslash G(K)/G(\mathcal{O})$$

$$\xleftrightarrow[1:1]{} \{z^\lambda \mid \lambda : \text{dominant coweight}\}$$

<u>Ex</u> ∘ $G = GL_n$ elementary divisors

$$G(\mathcal{O})\backslash G(K)/G(\mathcal{O}) \simeq G(\mathcal{O})\begin{bmatrix} z^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & z^{\lambda_n}\end{bmatrix} G(\mathcal{O})$$

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$$

∘ $n = 1$ $\quad Gr_{\mathbb{C}^\times} = \coprod_{\lambda \in \mathbb{Z}} \{z^\lambda\} \simeq \mathbb{Z}$

(discrete set)

$G_c \subset G$ max. cpt

$\Omega G_c = \{\phi : S^1 \longrightarrow G_c$ polynomial map, $\phi(1) = e\}$

Fact. $Gr_G \simeq \Omega G_c$ (homeo)

$z^\lambda \in Gr_G$ $\qquad$ $Gr_G^\lambda = G(\mathcal{O}) z^\lambda$ orbit

$$Gr_G^\lambda \xrightarrow[\text{vectr bdle}]{\text{Fact}} G/P_\lambda \leftarrow \text{partial flag variety}$$

$\overline{Gr_G^\lambda} = \bigcup_{\mu \leq \lambda} Gr_G^\mu$ $\qquad$ $\left(\mu \leq \lambda \Leftrightarrow \lambda - \mu \in Q_+^\vee = \sum_i \mathbb{Z}_{\geq 0} \alpha_i^\vee \right.$ simple coroots

"closure" of Gr_G^λ

Fact. $\overline{Gr_G^\lambda}$ is a projective variety (finite dimensional)

Gr_G should be understood as $\lim_{\lambda \to \infty} \overline{Gr_G^\lambda}$

$\overline{Gr_G^\mu} \subset \overline{Gr_G^\lambda}$ if $\lambda \geq \mu$

closed embedding

§ Coulomb branch

$\mathbb{M} = \mathbf{N} \oplus \mathbf{N}^* = \mathbf{N}' \oplus \mathbf{N}'^*$

$\left(\begin{array}{l} G \\ \mathbf{N} : \text{representation} \end{array} \right.$ $\mathbf{N}(\mathcal{K}), \mathbf{N}(\mathcal{O})$ same $\mathcal{M}_C$

$$\begin{array}{l} \mathcal{M} := G(\mathcal{K}) \overset{G(\mathcal{O})}{\times} \mathbf{N}(\mathcal{O}) \ni [g(z), s(z)] \\ \pi \downarrow \qquad\qquad\qquad\qquad \downarrow \\ \mathcal{X} := \mathbf{N}(\mathcal{K}) \ni g(z)s(z) \end{array}$$

vector b'dle over $G(\mathcal{K})/G(\mathcal{O})$ with fiber $= \mathbf{N}(\mathcal{O})$

analog

$\mathcal{M} = T^*Gr(v,w) = \{ (S, X) \mid X(S)=0, \ \mathrm{Im} X \subset S \}$ (S : sub sp)

$\downarrow$

$\mathcal{X} = \{ X \mid X^2 = 0, \ \mathrm{rk} X \leq v \}$, $X \in \mathrm{End}(\mathbb{C}^w)$

$$\begin{array}{c} GL(w) \times_{GL(v)} \mathrm{Hom}(W/V, V) \\ \downarrow \\ \mathrm{Hom}(W, W) \end{array}$$

$\mathcal{Z} = \mathcal{M} \times_{\mathcal{X}} \mathcal{M} \curvearrowleft G(\mathcal{K})$

naively we can consider the convolution algebra $H_*^{G(\mathcal{K})}(\mathcal{Z})$

technical issue all ∞-dim'l.

not regarded as limit of finite dim'l variety.

modification $\mathcal{Z} = G(\mathcal{K}) \times_{G(\mathcal{O})} \mathcal{R}$

$$\{[g_1(z), s_1(z)], [g_2(z), s_2(z)] \mid g_1(z)s_1(z) = \underbrace{g_2(z)}_{= id} s_2(z)\}$$

$$\mathcal{R} = \{[g(z), s(z)] \in \mathcal{M} \mid g(z)\underbrace{s(z)}_{= s_2(z)} \in \overline{N(\mathcal{O})}\}$$

normalise $g_2(z) = id$ by the action of $G(\mathcal{K})$

induction thm in equiv. homology: $H_*^{G(\mathcal{K})}(\mathcal{Z}) = H_*^{G(\mathcal{O})}(\mathcal{R})$

$$\begin{tikzcd}
\overline{Gr_G^\lambda} & \mathcal{R}_{\le\lambda} = \pi^{-1}(\overline{Gr_G^\lambda}) \arrow[l] \\
Gr_G \arrow[u, phantom, "\cap" rotate=90] & \mathcal{R} \arrow[l, "\pi"] \arrow[u, phantom, "\cap" rotate=90]
\end{tikzcd}$$

$\mathcal{R}^d_{\le\lambda}$ truncate by z^d

$\leftarrow$ finite dim'l variety

$G(\mathcal{O})$ factor through finite dim grp.

$$H_*^{G(\mathcal{O})}(\mathcal{R}) = \varinjlim_{\lambda} \varprojlim_{d} H_*^{\text{f.d. quotient of } G(\mathcal{O})}(\mathcal{R}^d_{\le\lambda})$$

degree $\in \mathbb{Z}$

normalised so that [fiber over $1 = z^0 \in Gr_G$ of π] fund. class has degree 0

Th [BFN]
(1) $H_*^{G(\mathcal{O})}(\mathcal{R})$ has a convolution product.

graded associative algebra.
(homological degree)

(2) It is a <u>commutative</u> algebra.

There are two proofs
a) reduction to the case $G=T$
T-case : explicit computation

b) Use Beilinson-Drinfeld Grassmannian
(Lonergan)

introduced to show the commutativity of tensor products in geometric Satake

$$\mathrm{Perv}_{G(\mathcal{O})} Gr_G \underset{\otimes\text{-equiv}}{\cong} \mathrm{Rep}(G^\vee) \ni V_1, V_2$$

$$\ni S_1, S_2 \qquad V_1 \otimes V_2 \cong V_2 \otimes V_1$$

$$S_1 * S_2 \cong S_2 * S_1$$

Def. $\mathcal{M}_C \overset{\text{def.}}{=} \mathrm{Spec}\, H_*^{G(\mathcal{O})}(\mathcal{R})$

affine scheme
One can show it is irreducible affine algebraic variety.

☆ $\mathbb{Z}$-graded algebra $H_*^{G(\mathcal{O})}(\mathcal{R})$

$\longrightarrow \mathcal{M}_C \curvearrowleft (\mathbb{Z})^\wedge = \mathbb{C}^\times$

☆ quantization

$\mathbb{C}^\times \curvearrowright \mathbb{C}[[z]], \mathbb{C}((z)) \quad z \mapsto \lambda z \quad \lambda \in \mathbb{C}^\times$

loop rotation $G(\mathcal{O}) \ \mathcal{R}$

$H_*^{G(\mathcal{O}) \rtimes \mathbb{C}^\times}(\mathcal{R})$ also has a convolution product

noncommutative

$\mathcal{M}_C$ has a deformation to a noncommu alg.

$\mathcal{A}_\hbar$ $\quad H^*_{\mathbb{C}^\times}(pt) = \mathbb{C}[\hbar]$

quantized Coulomb branch (in some cases close to Yangian)

$$\{f, g\} = \left. \frac{\tilde{f}\tilde{g} - \tilde{g}\tilde{f}}{\hbar} \right|_{\hbar = 0} \qquad \tilde{f}, \tilde{g} : \text{lifts to } \mathcal{A}_\hbar$$

$f, g \in \mathbb{C}[\mathcal{M}_C]$

Poisson bracket

<u>Fact.</u> $\{\ ,\ \}$ gives a symplectic form on the regular locus of $\mathcal{M}_C$

☆ (Poisson) commutative subalgebra

$$H^*_G(pt) \xrightarrow{\text{alg. hom}} H_*^{G(\mathcal{O})}(\mathcal{R}) \ , \ H_*^{G(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathcal{R})$$

$$f \longmapsto f \cdot \mathrm{id} \qquad\qquad = \mathcal{A}_\hbar$$

commutative

$$\mathbb{C}[\mathfrak{g}]^G = \mathbb{C}[\mathfrak{t}]^W \qquad \mathfrak{t} = \mathrm{Lie}\, T \qquad T \subset G$$

max torus

W : Weyl group

- commutative subalg in $\mathcal{A}_\hbar$
- Poisson. comm. subalg in $\mathbb{C}[\mathcal{M}_C]$.

$$\varpi : \mathcal{M}_C \longrightarrow \mathfrak{t}/W \cong \mathbb{C}^\ell \qquad \ell = \mathrm{rank}\, G.$$

integrable system.

Later : generic fiber of $\varpi = T^\vee$

dual torus of T

<u>Open Problem</u>
(solved by C. Teleman?)

Definition of $\mathcal{M}_C$ for $\mathbb{M} = N \oplus N^*$

✱ FI parameter

$\zeta \in \mathrm{Hom}(\mathbb{C}^\times, \pi_1(G)^\wedge)$

$$\pi_0(\mathcal{R}) = \pi_0(\underbrace{Gr_G}_{\Omega G_c}) \cong \pi_1(G_c) = \pi_1(G)$$

$$\mathcal{R} = \coprod_{\gamma \in \pi_1(G)} \mathcal{R}^\gamma \qquad H_*^{G(\mathcal{O})}(\mathcal{R}) = \bigoplus_{\gamma \in \pi_1(G)} H_*^{G(\mathcal{O})}(\mathcal{R}^\gamma)$$

$$\therefore \quad \mathcal{M}_C \curvearrowleft \pi_1(G)^\wedge \xleftarrow{\zeta} \mathbb{C}^\times$$

equivariant parameters

✱ mass parameter $m \in \mathrm{Hom}(\mathbb{C}^\times, T_F)$

$\tilde{G} \curvearrowright N$, $G \triangleleft \tilde{G}$

$G_F = \tilde{G}/G$

Assume $\| T_F$

$$\pi_1(\tilde{G})^\wedge \longleftarrow \pi_1(T_F)^\wedge = T_F^\vee$$

dual torus

$$\mathcal{M}_C(\tilde{G}, N) \curvearrowleft \pi_1(\tilde{G})^\wedge \longleftarrow T_F^\vee$$

Prop. $\mathcal{M}_C(G, N) = \mathcal{M}_C(\tilde{G}, N) /\!\!/ T_F^\vee$

symplectic reduction.

$m \in \mathrm{Hom}(T_F^\vee, \mathbb{C}^\times)$

$\rightsquigarrow$ GIT quotient

Kähler parameter