m : mass parameter

$$\mathrm{Hom}(\mathbb{C}^\times, T_F) \cong \mathrm{Hom}(T_F^\vee, \mathbb{C}^\times)$$

$$\tilde{G} \curvearrowright \mathbb{N} \qquad \tilde{G} \vartriangleright G \qquad \tilde{G}/G = T_F$$

$$\pi_1(\tilde{G})^\wedge \leftarrow \pi_1(T_F)^\wedge = T_F^\vee$$

<u>Prop</u> $\mathcal{M}_C(G, \mathbb{N}) \cong \mathcal{M}_C(\tilde{G}, \mathbb{N}) /\!/\!/ T_F^\vee$

(symplectic reduction)

$$\rightsquigarrow \quad \mathcal{M}_C(\tilde{G}, \mathbb{N}) /\!/\!/_{m} T_F^\vee =: \mathcal{M}_C^{m}(G, \mathbb{N})$$

$$\mu^{-1}(\underbrace{dm}_{= m_{\mathbb{C}}}) /\!/_{m} T_F^\vee =: \mathcal{M}_C^{m_{\mathbb{C}}, m}(G, \mathbb{N})$$

(Sketch of proof)

$$\mathcal{M}_C(\tilde{G}, \mathbb{N}) \quad \text{vs} \quad \mathcal{M}_C(G, \mathbb{N})$$

$$H_*^{\tilde{G}(\mathcal{K})}(\mathcal{R}_{\tilde{G}, \mathbb{N}}) \qquad H_*^{G(\mathcal{O})}(\mathcal{R}_{G, \mathbb{N}})$$

<u>1st diff.</u> $\mathrm{Gr}_{\tilde{G}}$ vs Gr_G

$$\left(\mathrm{Gr}_{T_F} = \underbrace{\mathrm{Hom}(\mathbb{C}^\times, T_F)}_{\text{discrete}} \right)$$

$$\mathrm{Gr}_{\tilde{G}} = \coprod_{\mu \in \mathrm{Hom}(\mathbb{C}^\times, T_F)} \underbrace{\mathrm{Gr}_G \cdot [z^\mu]}_{\text{copy of } \mathrm{Gr}_G}$$

Taking quotient by $T_F^\vee$ $\longleftrightarrow$ Taking the identity comp. Gr_G

$\underline{2^{nd} \text{ diff.}}$ $\quad H^*_{\tilde{G}}$ vs H^*_G differ by $H^*_{T_F}$

$$H^*_{T_F}(pt) = \mathbb{C}[\underbrace{\mathrm{Lie}\, T_F}_{\substack{\| \\ (\mathrm{Lie}\, T_F^\vee)^*}}]$$

$\nwarrow$ target space of the moment map

$\rightsquigarrow$ taking $\mu = 0$

<u>Examples</u>

(1) $G = \mathbb{C}^\times$, $\mathbb{N} = 0$

$$G(\mathcal{O}) \curvearrowright \mathcal{R} = Gr_G \cong \coprod_{\lambda \in \mathbb{Z}} \{z^\lambda\}$$

$$M \supset \mathcal{R} = \{[g(z), s(z)] \mid g(z)s(z) \in \mathbb{N}(\mathcal{O})\}$$

$$Gr_G \swarrow \qquad [g(z), s(z)] \in G(\mathcal{K}) \times^{G(\mathcal{O})} \mathbb{N}(\mathcal{O})$$

$$H_*^{G(\mathcal{O})}(\mathcal{R}) = \bigoplus_{\lambda \in \mathbb{Z}} \underbrace{H^*_{G(\mathcal{O})}(pt)}_{\substack{\| \wr \\ H^*_{\mathbb{C}^\times}(pt) \cong \mathbb{C}[\xi]}} [z^\lambda]$$

fund. class

$$[z^\lambda] * [z^\mu] = ?$$

naive definition of the convolution product

$$Gr_G \times Gr_G \times Gr_G \xrightarrow[p_{12}\ p_{23}\ p_{13}]{} Gr_G \times Gr_G$$

$$p_{13*}\big(\underbrace{p_{12}^*[z^\lambda]}_{\coprod_\nu (z^\nu,\, z^{\lambda+\nu})} \cap \underbrace{p_{23}^*[z^\mu]}_{\coprod_{\nu'} (z^{\nu'},\, z^{\nu'+\mu})}\big)$$

$$G(\mathcal{K}) \curvearrowright \mathrm{Gr}_G \times \mathrm{Gr}_G \quad \longleftrightarrow \quad G(\mathcal{O}) \curvearrowright \mathrm{Gr}_G$$

$$p_{12}^*[z^\lambda] \cap p_{23}^*[z^\mu]$$

$$= \Big[\coprod_\nu (z^\nu, z^{\lambda+\nu}, z^{\lambda+\mu+\nu})\Big]$$

$$\xrightarrow{p_{13}}$$

$$\therefore \quad [z^\lambda] * [z^\mu] = [z^{\lambda+\mu}]$$

$$H_*^{G(\mathcal{O})}(\mathcal{R}) = \bigoplus_{\lambda \in \mathbb{Z}} \mathbb{C}[\xi][z^\lambda]$$

$$= \mathbb{C}[\xi, \underbrace{[z^1]}_{\underset{x}{\shortparallel}}, \underbrace{[z^{-1}]}_{\underset{x^{-1}}{\shortparallel}}] \qquad x \cdot x^{-1} = 1.$$

$$\therefore \quad \underset{\mathcal{M}_C}{\overset{\mathrm{Spec}}{\shortparallel}} \cong \underset{\xi}{\mathbb{C}} \times \underset{x}{\mathbb{C}^\times}$$

(Rem. quantization $\mathcal{A}_\hbar = H_*^{G(\mathcal{O}) \rtimes \mathbb{C}^\times}(\mathcal{R})$

$$\xi * [z^{\pm 1}] - [z^{\pm 1}] * \xi = \pm \hbar [z^{\pm 1}]$$

$(z^{\pm 1})$ is a difference operator of ξ

Remark classical Coulomb branch

$(G, \mathbb{N})$: general

$\underset{T}{\cup}$ max. torus $\quad W$: Weyl group

$$\mathcal{M}_C^{\text{classical}}(G,\mathbb{N}) \cong \big(\underbrace{\mathfrak{t} \times T^\vee}_{\text{cotangent bdle to } T^\vee}\big)/W$$

$\mathfrak{t} = \operatorname{Lie} T$

$T^\vee$ = dual of T

$\mathcal{M}_C^{\text{class}} = \mathcal{M}_C$ in this example.

(2) $G = \mathbb{C}^\times$
$\mathbb{N} = \mathbb{C}$ (wt 1)

$$\begin{tikzcd} \textcircled{1} \arrow[d, no head] & v \\ \boxed{1} & w \end{tikzcd}$$

$\mathcal{M}_H = \{0\}$

$\mathcal{M}_H^{\zeta} = pt$

$$\mathcal{R} = \{ [g(z), s(z)] \mid \underset{\mathbb{N}(\mathcal{O})}{g(z)s(z) \in} \} \qquad \begin{cases} G = GL(v) \\ \mathbb{N} = \operatorname{Hom}(\mathbb{C}^w, \mathbb{C}^v) \end{cases}$$

$$= \bigsqcup_{\lambda \in \mathbb{Z}} \underbrace{\{z^\lambda\} \times \left(\mathbb{C}[[z]] \cap z^{-\lambda}\mathbb{C}[[z]]\right)}_{= R_\lambda}$$

$$\underset{\mathcal{M}}{\cap} = G(\mathcal{K}) \times^{G(\mathcal{O})} \mathbb{N}(\mathcal{O}) = \bigsqcup_{\lambda \in \mathbb{Z}} \{z^\lambda\} \times \mathbb{C}[[z]]$$

$\lambda \geq 0 \qquad \mathbb{C}[[z]] \cap z^{-\lambda}\mathbb{C}[[z]] = \mathbb{C}[[z]]$

$\lambda < 0 \qquad \mathbb{C}[[z]] \cap z^{-\lambda}\mathbb{C}[[z]] \subsetneq \mathbb{C}[[z]]$

codim $= |\lambda|$

$$[\mathcal{R}_\lambda] = \text{fund. class of } \mathcal{R}_\lambda$$

$$\Vert$$

$$\xi^{\max(-\lambda,0)} \cap [\underbrace{Z^\lambda}_{\text{previous examples}}]$$

(more precisely fiber over Z^λ in $\underset{\sim}{M}$

$$[\mathcal{R}_1] * [\mathcal{R}_{-1}] = \xi\, [z^1] * [z^{-1}]$$
$$= \xi$$

$$H_*^{G(\mathcal{O})}(\mathcal{R}) = \mathbb{C}[\xi, \underset{x}{\underset{\Vert}{[\mathcal{R}_1]}}, \underset{y}{\underset{\Vert}{[\mathcal{R}_{-1}]}}] / xy = \xi$$

$$= \mathbb{C}[x, y]$$

$$\therefore\ \mathcal{M}_C \cong \mathbb{C}^2$$

more generally

$$\begin{array}{c} \textcircled{1} \\ | \\ \boxed{w} \end{array} \qquad \begin{array}{l} G = \mathbb{C}^\times \\ \mathbf{N} = \mathbb{C}^{\oplus w} \\ \quad (\text{wt } 1) \end{array}$$

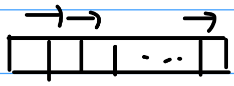
$$H_*^{G(\mathcal{O})}(\mathcal{R}) = \mathbb{C}[\xi, x, y] / xy = \xi^w$$

type A_{w-1} singularity.

$$\mathcal{M}_H = \{ \underline{X \in \operatorname{End}(\mathbb{C}^w) \mid X^2 = 0,\ \operatorname{rk} X \leq 1} \}$$

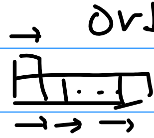
$$= \text{minimal nilpotent orbit}$$

Brieskorn-Slodowy $\mathcal{N} \cap S \cong$ type A_{w-1} singularity



$\mathcal{N}$: nilpotent variety in $\mathfrak{sl}_w$

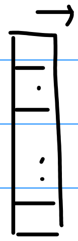
S: slice to subregular orbit

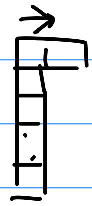


minimal nilpotent orbit $\cong S' \cap \mathcal{N}'$

S' = slice to 0-orbit = $\mathfrak{g} = \mathfrak{sl}_w$

$\mathcal{N}'$: minimal nilpoten orbit





taking transposes of Young diagrams and changing orbit and slice

$$\mathcal{M}_H \longleftrightarrow \mathcal{M}_C$$

This holds for all nilpotent orbit $\cap$ slice in type A

(3) toric hyperkähler manifold

$$1 \to T \to \tilde{T} \to T_F \to 1$$

exact seq of tori

$\tilde{T} = (\mathbb{C}^\times)^n$

$\mathbf{N} = \mathbb{C}^n$ std rep. of $\tilde{T}$

$$\mathcal{M}_H = \mathbb{C}^n \oplus (\mathbb{C}^n)^* /\!\!/\!\!/ T$$

$$\mathcal{M}_C(T,\mathbf{N}) = \mathcal{M}_C(\tilde{T},\mathbf{N}) /\!\!/\!\!/ T_F^\vee$$

$$\underset{\text{By }(2)}{=} (\mathbb{C}_{\oplus \mathbb{C}^{\text{dual}}})^n /\!\!/\!\!/ T_F^\vee = \mathbb{C}^n \oplus (\mathbb{C}^n)^* /\!\!/\!\!/ T_F^\vee$$

$$1 \to T_F^\vee \to (\tilde{T})^* \to T^\vee \to 1$$

$$(\tilde{T})^* \,\|\, \tilde{T},\quad \tilde{T} \curvearrowright \mathbf{N}$$

☆ classical Coulomb branch and localization of equiv. homology

$(G,\mathbf{N})$: general

- $H_*^{G(\mathcal{O})}(\mathcal{R}) \cong H_*^{T(\mathcal{O})}(\mathcal{R})^W$

$$H_G^*(pt) = H_T^*(pt)^W$$

- $H_*^{T(\mathcal{O})}(\mathcal{R}) \otimes_{H_T^*(pt)} \mathrm{Frac}\,H_T^*(pt)$

$$\underset{\text{localization}}{\cong} H_*^{T(\mathcal{O})}(\underbrace{\mathcal{R}^T}_{\text{fixed pt set.}}) \otimes_{H_T^*(pt)} \mathrm{Frac}\,H_T^*(pt)$$

$\mathcal{R}^T$?

$$\mathrm{Gr}_G^T \cong (\Omega G_c)^{T_c} \cong \mathrm{Hom}(S^1, T_c) \cong \mathrm{Gr}_T$$

$$\mathbf{N}(\mathcal{O})^T = (\mathbf{N}^T)(\mathcal{O}) \qquad \mathbf{N}^T = T\text{-fixed pts in } \mathbf{N}$$

$$\begin{array}{c} M^T = (G(\mathcal{K})\times^{G(\mathcal{O})}\mathbf{N}(\mathcal{O}))^T = Gr_T \times (\mathbf{N}^T)(\mathcal{O}) \\ \cup \qquad\qquad\qquad\qquad\qquad\qquad T(\mathcal{O}) \circlearrowleft \text{trivially} \\ \mathcal{R}^T = M^T \end{array}$$

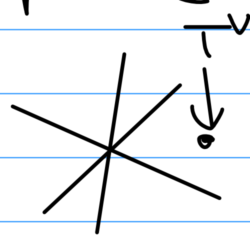
$$H_*^{T(\mathcal{O})}(\mathcal{R}^T) \cong H_T^*(pt)\otimes H_*(Gr_T)$$

$$\operatorname{Spec} = \mathcal{M}_C(T, \underset{\downarrow \atop \{0\}}{\mathbf{N}^T}) = \mathfrak{t}\times T^\vee$$

Although two isom's do not respect convolution product,

$$\begin{tikzcd} \mathcal{M}_C(G,\mathbf{N}) \arrow[rr, dashed, "\approx", "\text{birational}"'] \arrow[dr, "\varpi"'] \arrow[rr, phantom, shift right=6, "\circlearrowright"] & & \mathcal{M}_C(T,\mathbf{N}^T)/W = \mathfrak{t}\times T^\vee/W \arrow[dl] \\ & \mathfrak{t}/W = \operatorname{Spec} H_G^*(pt) = \mathbb{C}^\ell & \end{tikzcd}$$

$\Rightarrow$ generic fiber of $\varpi = T^\vee$



$G \circlearrowleft \mathbf{N} = \mathbf{N}' \oplus$ trivial representation $\Rightarrow \mathcal{M}_C(G,\mathbf{N}) \cong \mathcal{M}_C(G,\mathbf{N}')$

Examples (without explanation)
why

(1) $\mathbb{N} = \mathfrak{g}$: adjoint repr.

$$\mathcal{M}_C(G, \underset{\mathfrak{g}}{\underset{\parallel}{\mathbb{N}}}) \cong \mathfrak{t} \times T^\vee / W = \mathcal{M}_C^{\text{classical}}(G, \mathbb{N}).$$

3d $N=8$ SUSY

quantization $\mathcal{A}_\hbar(G, \mathbb{N}) \cong$ spherical part of trigonometric DAHA

(cf. Vasserot)

$W_{aff} = W \ltimes Q \rightsquigarrow H_{aff}$
deformation

$W_{aff} \ltimes Q' \rightsquigarrow$ DAHA

(2) $\mathbb{N} = 0$, G : arbitrary

$G^\vee$ = Langlands dual group

$\mathcal{M}_C$ = universal centralizer for $G^\vee$

$H_*^{G(\mathcal{O})}(Gr_G)$

$S^\vee$ = Kostant-Slodowy slice to regular nilpotent orbit in $\mathfrak{g}^\vee$

$$S^\vee \subset \mathfrak{g}^\vee \longrightarrow \mathfrak{g}^\vee /\!\!/ G^\vee \cong \mathfrak{t}^\vee / W$$

comp. = isom

$$\begin{tikzcd} \mathfrak{g}^\vee \arrow[r] & \mathfrak{t}^\vee / W \arrow[l, bend left] \end{tikzcd}$$

$\mathfrak{t}^\vee/W = S^\vee$

section to the adjoint quotient

elements in $S^\vee$ are <u>regular</u>
i.e. stabilizer is of maximal dimension

$$\text{univ. centralizer} = \{ (g,s) \in G^\vee \times S^\vee \mid \mathrm{Ad}_g(s) = s \}$$

Bezrukavnikov - Finkelberg - Mirković
(before $\mathcal{M}_C$ general is introduced.)

This example is a hyperKähler manifold constructed by using sol's of Nahm's equation

ODE reduction of 1d ASD equation

☆ degeneration and monopole formula

Recall $\overline{\mathrm{Gr}_G^\lambda} \longleftarrow \mathcal{R}_{\leq\lambda}$

$$H_*^{G(\mathcal{O})}(\mathcal{R}_{\leq\lambda}) \subset H_*^{G(\mathcal{O})}(\mathcal{R}) = \mathbb{C}[\mathcal{M}_C]$$
$$\overset{\text{def}}{=}: F_{\leq\lambda} \quad (\lambda : \text{dominant coweight})$$

This gives a filtration compatible with conv. product. $F_{\leq\lambda} \cdot F_{\leq\mu} \subset F_{\leq\lambda+\mu}$

$$\mathrm{gr}_F\,\mathbb{C}[\mathcal{M}_C] \cong \bigoplus H^{G(\mathcal{O})}_*(\mathcal{R}_\lambda)$$

$$= \bigoplus_\lambda F_{\le\lambda}/F_{<\lambda}$$

$$\mathcal{R}_\lambda = \mathcal{R}_{\le\lambda} \setminus \mathcal{R}_{<\lambda}$$

$$\begin{array}{c} \mathcal{R}_\lambda \\ \downarrow {\scriptstyle \text{vector bdle}} \\ \mathrm{Gr}_G^\lambda = G(\mathcal{O})z^\lambda \\ \searrow {\scriptstyle \text{vector bdle}} \\ G/P_\lambda \end{array}$$

($\underline{\text{Rem}}$ $\mathcal{R}_\lambda$: smooth)

$$H^{G(\mathcal{O})}_*(\mathcal{R}_\lambda) \cong H_G^{*+\text{shift}}(G/P_\lambda)$$

partial flag.

$$\cong H_{L_\lambda}^{*+\text{shift}}(\mathrm{pt})$$

polynomial ring

L_λ = Levi subgroup of P_λ

$\rightsquigarrow$ Poincaré polynomial of $H^{G(\mathcal{O})}_*(\mathcal{R}_\lambda)$ has an explicit combinatorial formula

$\rightsquigarrow$ graded dim of $\mathrm{gr}_F\,\mathbb{C}[\mathcal{M}_C]$

monopole formula (Cremonesi-Hanany-Zaffaroni)

As a graded vector space $\mathrm{gr}_F\,\mathbb{C}[\mathcal{M}_C] \cong \mathbb{C}[\mathcal{M}_C]$

As a commutative ring $\mathrm{gr}_F\,\mathbb{C}[\mathcal{M}_C] \not\cong \mathbb{C}[\mathcal{M}_C]$

$\mathrm{Spec}(\mathrm{gr}_F\, \mathbb{C}[\mathcal{M}_C])$: degenerate version of $\mathcal{M}_C$.

bad singularities

nevertheless $\mathrm{gr}_F\, \mathbb{C}[\mathcal{M}_C]$ has an explicit presentation

Atiyah–Bott–Lefschetz (Berline–Vergne)

explicit localization of equiv. cohomology

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§ quiver gauge theory

$Q = (Q_0, Q_1)$

Q_0: vertices (finite)

Q_1: oriented edges

$$o(h) \xrightarrow{h} i(h)$$

V, W : Q_0-graded f.d. cpx vector spaces

$V = \bigoplus V_i$, $W = \bigoplus W_i$ ← framing vector spaces

$w_i = \dim W_i$

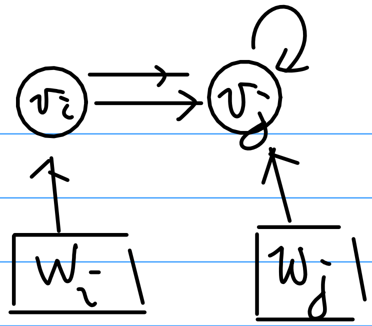
$v_i = \dim V_i$

$$G = \prod_i GL(V_i)$$

$G \curvearrowright \mathbf{N}$

$$\mathbf{N} = \bigoplus_{h \in Q_1} \mathrm{Hom}(V_{o(h)}, V_{i(h)}) \oplus \bigoplus_{i \in Q_0} \mathrm{Hom}(W_i, V_i)$$

rep. of G.



(sometimes drop $\to \oslash$)

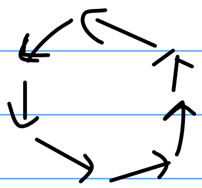
- FI parameters $\zeta \in \mathrm{Hom}(\mathbb{C}^\times, \underbrace{\pi_1(G)}_{\substack{\| \\ (\mathbb{C}^\times)^{Q_0}}}{}^\wedge)$

- $\tilde{G} = G \times \prod_i \underset{\substack{\cap \\ GL(W_i)/\mathbb{C}^\times}}{T(W_i)} \times (\mathbb{C}^\times)^{\underbrace{b_1(Q)}}$

$b_1(Q)$: cycles in Q (forgetting orientation)

$$T_F = \tilde{G}/G = \prod \frac{T(W_i)}{\mathbb{C}^\times} \times (\mathbb{C}^\times)^{b_1(Q)}$$

(affine type A

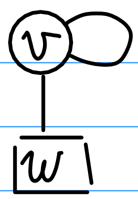

)

Higgs branch $\quad \mathcal{M}_H, \ \mathcal{M}_H^\zeta$

$\mathbb{M} = \mathbb{N} \oplus \mathbb{N}^* \qquad \mathcal{M}_H \cong \mathbb{M} /\!/\!/ G, \qquad \mathcal{M}_H^\zeta = \mathbb{M} /\!/\!/_\zeta G$

v = instanton charge

example



moduli space of framed $U(w)$-instantons on $\mathbb{R}^4$
Atiyah–Drinfeld–Hitchin–Manin

$\mathcal{M}_H$ // (Uhlenbeck partial compactification of

$$\begin{array}{l} \mathcal{M}_H^{\zeta} = \text{moduli sp. of framed torsion-free sheaves on } \mathbb{P}^2 \\ \quad \downarrow {\scriptstyle \zeta \to 0} \\ \mathcal{M}_H \end{array}$$

— affine quiver $\qquad \ell = \text{level}$

$\mathcal{M}_H^{\zeta}$ = moduli space $U(\ell)$ of framed instantons on $\mathbb{R}^4/\Gamma$ or resolution

Kronheimer-N

Γ: finite subgrp of SL_2

$\mathcal{M}_H, \mathcal{M}_H^{\zeta}$ are called quiver varieties

---

Th [BFN]

$$\mathcal{M}_C\left(\begin{array}{c} v \circlearrowright \\ | \\ \boxed{w} \end{array} \right) \cong \mathrm{Sym}^v(S_w) \qquad S_w = \{xy = z^w\} \subset \mathbb{C}^3$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| \; \mathbb{C}^2/\mathbb{Z}/_{w-1}$$

$$\mathcal{M}_C^{m} = \mathrm{Hilb}^v(\text{minimal resolution of } S_w)$$

quantization $\quad$ Kodera–N

cyclotomic rational DAHA.

Q : type ADE (finite)

(Rem variant of construction
for quiver with symmetrizer
$\rightsquigarrow_{\text{e.g.}}$ Q : type BCFG. (N-Weekes)

$\rightsquigarrow$ $\mathfrak{g}_Q^\vee$: cpx simple Lie algebra
$\quad \| $
$\quad \mathfrak{g}^\vee$

Th [BFN, NW]

$\begin{cases} \lambda = \sum \dim W_i \cdot \Lambda_i^\vee \\ \mu = \lambda - \sum \dim V_i \cdot \alpha_i^\vee \end{cases}$

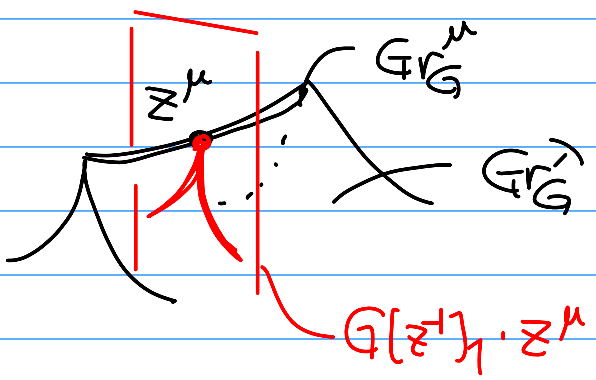
$\Lambda_i^\vee$ = fund. coweight
$\alpha_i^\vee$ = simple coroot of $\mathfrak{g}^\vee$

$\mathcal{M}_C$ = generalised affine Grassmannian slice in $\mathrm{Gr}_{G^\vee}$ ($G^\vee$: $\mathrm{Lie}\, G^\vee = \mathfrak{g}^\vee$ adjoint type)

Assume μ : dominant (λ : dominant always)

~~generalised~~ affine Grass

$$\overline{\mathrm{Gr}_G^\lambda} = \overline{G(\mathcal{O}) z^\lambda} \supset \mathrm{Gr}_G^\mu$$



$G[z^{-1}]_1 = \mathrm{Ker}(G[z^{-1}] \xrightarrow[\text{at } z=\infty]{\text{ev}} G)$

$G(\mathcal{O}) = G[[z]]$

$G[z^{-1}]_1 \cdot z^\mu$: transversal to $G(\mathcal{O}) \cdot z^\mu$

$$G[z^{-1}]_1 z^\mu \cap \overline{Gr_G^\lambda}$$
$$\mathrel{\|}\!\!\!\!\cdot\!\cdot$$
$$\overline{W}^\lambda_\mu \cong \mathcal{M}_C$$

affine Grassmannian slice

$$\begin{tikzcd} \mathcal{M}_C \arrow[rr, "\overset{☆}{\cong}"] \arrow[dr, "\varpi"'] & \circlearrowleft & \mathfrak{M} \arrow[dl, "\pi"] \\ & \mathfrak{t}/W & \end{tikzcd}$$

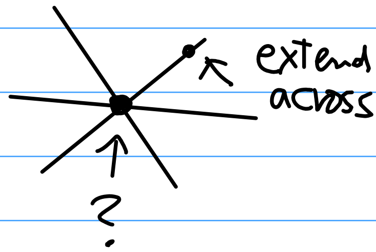
Want to show ☆

1^{st} step const. this

2^{nd} show generic fiber $\cong T^\vee$

3^{rd} construct birational isom. $☆_{rat}$.

4^{th}. $☆_{rat}$ extends in codim 2



5^{th}, show that

($\mathfrak{M}$ is normal
and π is flat

$\Longrightarrow$ $☆_{rat}$ extends everywhere

Another differential geometric description

moduli space of framed S^1-equivariant G_c-inst. $\curvearrowleft G$

on $\mathbb{R}^4 \circlearrowleft$

$\quad \mathrel{/\!/}$

$\mathbb{C}^2$ $\quad (z_1, z_2) \mapsto (tz_1, t^{-1}z_2)$

$S^4 = \mathbb{C}^2 \cup \infty \circlearrowleft S^1$ $\quad 0, \infty$: fixed pts

P : G_c-b'dle over S^4

$P_0, P_\infty \cong G_c \leftarrow S^1$ group hom

up to conjugacy $\Downarrow$

λ, μ : dominant coweights

$\overline{W}^\lambda_\mu \supset_{\text{open}} W^\lambda_\mu$ = moduli of S^1-equiv G_c-inst. on $\mathbb{R}^4$ with $P_0, P_\infty \leftrightarrow \lambda, \mu$

$$\left(\begin{array}{ccc} \mathbb{R}^4/S^1 & \longrightarrow & \mathbb{R}^3 \\ \cup & & \cup \\ 0 & & 0 \end{array} \right.$$

S^1-equiv inst. $\qquad$ $\underline{\text{singular monopole}}$ $\leftarrow$ Dirac type singular

A, Φ

$*F_A = d_A \Phi$

W^λ_μ = moduli space of singular G_c-monopoles on $\mathbb{R}^3$

λ .... type of singularity

μ - - behavior at ∞

§ intuition of the definition of Coulomb branch

3d SUSY gauge theory $\underset{\text{topological twist}}{\rightsquigarrow}$ 3d TQFT
$\uparrow$ topological

$$(M^3, g) \rightsquigarrow Z(M^3) = \int_{\mathcal{F}} D\Phi \; e^{-S(\Phi)}$$

$\uparrow$ independent of Riemannian metric g

S: lagrangian

Suppose M^3 has a boundary $\partial M = \Sigma$ $\leftarrow$ 2-mfd

φ : field on Σ

$$Z(M_3) = \int_{\text{"}\Phi|_{\partial M} = \varphi\text{"}} D\Phi \; e^{-S(\Phi)}$$

$\uparrow$ function in φ

$Z(M_3) \in Z(\Sigma) =$ "Func" on the space of fields on Σ

(Axiomized by Atiyah-Segal)

$\mathcal{N}=4$ SUSY $\rightsquigarrow$ "Func" is replaced by cohomology
(Atiyah-Jeffrey)
also space of fields on Σ is replaced by

moduli spaces of solutions of equations of motion

e.g. $\mathcal{N}=0$ $\quad F_A = 0$

flat G_c-connection

finite dim'l

usually singular

Better to remember ∞-dim'l origin in some way.

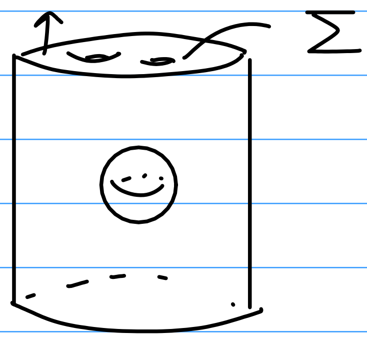
Need appropriate treatment of (co)homology of moduli space

Consider $\Sigma = S^2$ $\qquad \partial(M^3 \setminus B) = S^2$

closed $\quad$ small ball

bdry of $\underline{\underline{\text{small}}}$ nbd of a point

$M = \Sigma \times [0,1] \setminus B$



$Z(M) \in Z(\partial M)$

$= Z(\Sigma) \otimes Z(S^2)$

$\longrightarrow Z(\Sigma)$

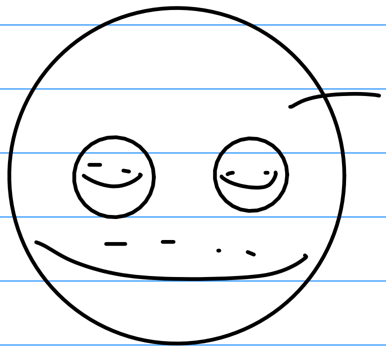
element of $Z(S^2) \Rightarrow \mathrm{End}(Z(\Sigma))$

$\Sigma = S^2$ $\qquad Z(S^2) \otimes Z(S^2) \longrightarrow Z(S^2)$

multiplication on $Z(S^2)$

$Z(S^2)$: algebra

commutative!



$D \setminus D' \cup D''$

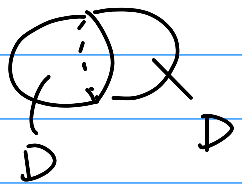
small disks

$$H_*^{G(\mathcal{O})}(\mathcal{R}) = Z(S^2)$$

$Z(M^3)$: number
$Z(\Sigma)$: vector sp
$Z(S^1)$: category of line operators

$D^2 \qquad \partial D^2 = S^1$

$$\underset{\text{object}}{Z(D^2)} \in \underset{\text{cat.}}{Z(S^1)}$$

$S^2 =$ 

$$\underbrace{Z(S^2)} = \mathrm{End}_{Z(S^1)}(Z(D^2))$$
$$\| $$
$$H_*^{G(\mathcal{O})}(\mathcal{R}) = H_*^{G(K)}(M \underset{X}{\times} M)$$

$\pi : M \longrightarrow X$ proper

smooth

$Z = M \times_X M$

$$H_*^G(Z) \cong \mathrm{Ext}^*_{D_G(X)}(\pi_* \mathbb{C}_M, \pi_* \mathbb{C}_M)$$

Ginzburg

$$Z(S^1) = D_G(X)$$

G, constructible

$$\begin{pmatrix} G = G(K) \\ X = N(K) \end{pmatrix}$$

$$\pi_* \mathbb{C}_M = Z(D^2)$$

$$M = G(K) \times^{G(\mathcal{O})} N(\mathcal{O})$$