3d $\mathcal{N}=4$ gauge theory $\longrightarrow$ 3d TQFT Z

$\mathcal{M}$ = "moduli space" of solutions of equ. of motion

the space of all fields

$F_A = 0$, SW equ, ⋯

closed $M^3 \mapsto Z(M^3) : \mathbb{Q}.(\mathbb{C}) = $ "#" $\mathcal{M}(M^3)$

$\Sigma^2 \rightsquigarrow Z(\Sigma)$: vectn space = "(co)homology" of $\mathcal{M}(\Sigma^2)$

M with ∂M $\rightsquigarrow Z(M) \in Z(\partial M)$

$$\mathcal{M}(M^3) \xrightarrow{\text{"bdry value"}} \mathcal{M}(\Sigma^2)$$

$\left(\begin{array}{l} \Sigma = \Sigma_1 \sqcup \Sigma_2 \\ Z(\Sigma) = Z(\Sigma_1) \otimes Z(\Sigma_2) \end{array} \right.$

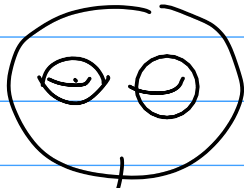
$= \text{image}[\mathcal{M}(M^3)] \in H_*(\mathcal{M}(\Sigma))$

<u>Warning</u> : • This picture is <u>not</u> mathematically realized in any example.

• Casson inv. is the best example,
↳ $G_c = SU(2)$ or $SO(3)$
$\mathcal{N}=0$
but ~~not~~ TQFT is <u>not</u> constructed even this case.

$Z(S^2) = Z(\partial \underset{\sim}{B^3})$ small ball

commutative algebra

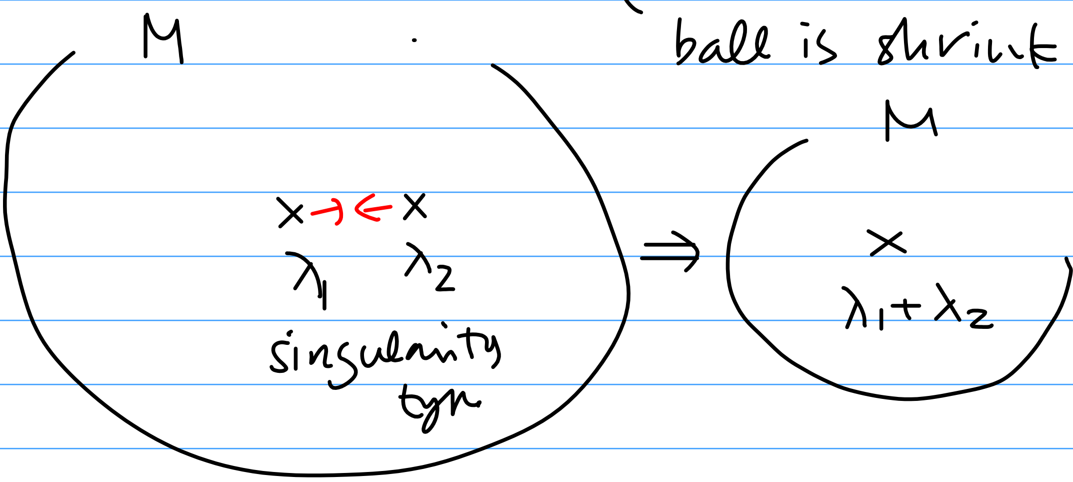


$Z(S^2) \otimes Z(S^2) \longrightarrow Z(S^2)$

$B \setminus \underbrace{B' \sqcup B''}_{\text{small balls}}$

Better to replace S^2 by pt singularity

ball is shrink to a pt



singular monopoles W_μ^λ $\left(\subset \overline{W}_\mu^\lambda \right)$ affine Grassmannian slice

λ : singularity type at 0

μ : asymptotic behavior at ∞

$W_\mu^{\lambda_1, \lambda_2}(x_1, x_2)$

λ_1: singularity at $x_1 \in \mathbb{R}^3$

λ_2 // at $x_2 \in$

μ : asymp. cond.

- BD grassmannian $\quad x_1, x_2 \in \mathbb{C}$ $\quad \mathbb{R}^3$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \shortparallel$

- convolution diagram $\quad x_1, x_2 \in \mathbb{R}$ $\quad \mathbb{C} \times \mathbb{R}$

  (↑ used to define convolution product

Recall

$$\begin{tikzcd}
\mathcal{Q} \arrow[r] & \mathrm{Gr}_G \\
 & \overline{W}^\lambda_\mu \arrow[u, phantom, "\cup"] \\
\mathcal{Q}|_{W^\lambda_\mu} \arrow[r, "\text{vectr bdle}"'] & W^\lambda_\mu \arrow[u, phantom, "\subset_{\text{open}}"]
\end{tikzcd}$$

---

monopole $\quad * F_A = d_A \Phi$

A : G_c-conn

Φ : $\mathfrak{g}$-valued section

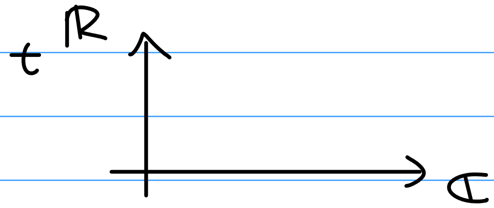
$A_{3d} = A_1 dx_1 + A_2 dx_2 + A_3 dx_3$

Φ

$A_{4d} = A_{3d} + \Phi \, d\theta$

$\mathbb{R}^4 \circlearrowleft S^1$

$\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$

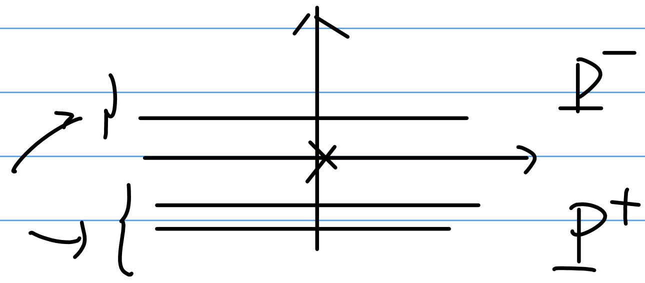


$$[\bar{\partial}_{\mathbb{C}}, \partial_t + \Phi] = 0$$

$(P, \bar{\partial}_{\mathbb{C}})$: holomorphic b'dle over $\mathbb{C} \times \{t\}$

It is constant in t-direction

singular monopole



holo. str. constant outside the origin

$P^+ \cong P^-$ outside the origin $\mathbb{C} \setminus \{0\}$

$G(K)/G(\mathcal{O})$

(Gaiotto-Witten)
Kapustin-Witten

Hecke correspondence = singular monopole

$$Z(S^2) = H_*^{G(\mathcal{O})}(\mathcal{R})$$

σ-model side

3d gauge theory $\overset{\text{tg}}{\simeq}$ σ-model with target $\mathcal{M}_H \cup \mathcal{M}_C$
equiv.

top. twist

$$Z_{gauge} = Z_{\sigma\text{-model}}$$

(Rozansky-Witten theory)

$M^3 \xrightarrow{\varphi} X$ hyperkähler mfd

$\mathcal{M}$ = moduli space = {constant maps} = X

$Z(\Sigma) = \mathrm{Func}(\text{fields on } \Sigma)$

$\xrightarrow[\text{SUSY}]{}$ cohomology of $\mathcal{M}(\Sigma)$ $= X$

Dolbeault

$\uparrow$ depends on $\Sigma_1 = \Sigma_g$

(cf. Rozansky-Witten)

($\Sigma = S^2$

X : affine algebraic variety

$H^0_{Dol}(X) \cong \mathbb{C}[X]$ polynomial functions

$$Z_{gauge}(S^2) = Z_{\sigma\text{-model}}(S^2) = \mathbb{C}[X]$$

$Z_{gauge}(S^2) = H_*^{G(\mathcal{O})}(\mathcal{R})$

$\mathbb{C}[X] \leftarrow \mathcal{M}_C$

$X = \mathrm{Spec}\, H_*^{G(\mathcal{O})}(\mathcal{R})$

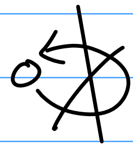
<u>Rem</u> • $\mathcal{M}_C$: singular, $Z_{\sigma\text{-mode}}(S^2) = \mathbb{C}[\mathcal{M}_C]$ might be dangerous.

Braverman-Finkelberg
1807.09038
Dimofte-Garner-Geracie-Hilburn
1908.00013

more general (singular) target

$\mathcal{M}(M^3_\Sigma) = \underline{\text{locally}}$ constant maps

§ geometric Satake



$Q = (Q_0, Q_1)$: quiver
V, W : quiver gauge theory

$\mathfrak{g}^\vee \equiv \mathfrak{g}^\vee_Q$: Kac–Moody Lie alg.
symmetrizable
└ if we consider $\mathcal{M}_C$

$\mathcal{M}_H$ Need to assume symmetric.

FI param. $\zeta \in \mathrm{Hom}(\mathbb{C}^\times, \pi_1(G)^\wedge)$

$\lambda = \sum \dim W_i \Lambda_i^\vee$, $\mu = \lambda - \sum \dim V_i \alpha_i^\vee$

• Higgs $\mathfrak{g} = \mathfrak{g}^\vee$

ζ : regular dominant

$$\begin{array}{c} \mathcal{M}_H^\zeta(\lambda,\mu) \quad \text{nonsingular} \\ \pi \big\downarrow \\ 0 \in \mathcal{M}_H(\lambda,\mu) \equiv \mathbf{M} /\!/\!/ G \\ \equiv \mathbf{N} \oplus \mathbf{N}^* /\!/\!/ G \end{array}$$

$\pi^{-1}(0) =: \mathcal{L}_H^\zeta(\lambda,\mu)$
└ lagrangian subvariety

$\bigoplus_\mu H_{top}(\mathcal{L}_H^\zeta(\lambda,\mu))$ is a representation of $\mathfrak{g}$ with highest wt $= \lambda$

[N, 1994]

$\bigoplus_{\mu} H_*^{GF}(\mathcal{L}_H^{\zeta}(\lambda,\mu))$: rep. of Yangian for KM (Varagnolo)
quantum loop for KM (N)

Coulomb side

Q : finite type $\mathfrak{g}^\vee$ is a cpx simple Lie alg.

$\mathcal{M}_C(\lambda,\mu)$: (generalized) affine Grassmannian slice

$Gr_{G^\vee}$ ($G^\vee$: adj. $\mathrm{Lie}\,G^\vee = \mathfrak{g}^\vee$)

geometric Satake realises repr. of G (simply-conn.) in terms of geom. top. of $Gr_{G^\vee}$.

<u>Proposal</u> (Braverman-Finkelberg-N)

Rep. of $\mathfrak{g}$ can be also realised by $\mathcal{M}_C(\lambda,\mu)$.

Q: finite type Representation ~~ca~~ are realized explicitly by Mirkovic–Vilonen

$\mathcal{M}_C(\lambda,\mu)$ regular dominant ξ (FI)

$\circlearrowleft \ T^\vee (= \pi_1(G)^\wedge)$

$\xi \nearrow$ max. torus of $G^\vee$

$\mathbb{C}^\times$

$\mathcal{M}_C(\lambda,\mu)^{T^\vee} = \mathcal{M}_C(\lambda,\mu)^{\xi=0} = \phi$ or $\{z^\mu\}$ ← fixed pt

single pt

$\mathcal{M}_C(\lambda,\mu)^{\xi\leq 0}$ repelling set

$:= \{x \in \mathcal{M}_C(\lambda,\mu) \mid \lim_{t\to\infty} \xi(t)\cdot x$ exists $\}$

$\cap$

$\mathcal{M}_C(\lambda,\mu)$ ϕ or lagrangian subvariety

<u>MV</u> $\bigoplus_\mu H_{top}(\mathcal{M}_C(\lambda,\mu)^{\xi\leq 0})$ is a representation of G with highest wt $=\lambda$.

<u>Conj</u>. This is true for any symmetrizable $\mathfrak{g} = \mathfrak{g}_Q$

more correctly

H^0(hyperbolic restriction $IC(\mathcal{M}_C(\lambda,\mu))$)

$\uparrow$

$Perv_{G^\vee(\mathcal{O})}(Gr_{G^\vee})$

$$\Longrightarrow \quad H_{top}\left(\mathcal{M}_C(\lambda,\mu)^{\xi\leq 0}\right) \cong H_{top}\left(\mathcal{L}_H^{\xi}(\lambda,\mu)\right)$$

$\exists$ canonical isomorphism.

This canonical isom. seem to be exist more general Higgs/Coulomb branches

<u>Example</u> A_1 $\quad$ $\bigcirc\!\!\!\!v$ — $\boxed{w}$

$$\begin{array}{ccl} \mathcal{M}_H^{\xi}(v,w) & = & T^*Gr(v,w) \\ & & \cup \\ \downarrow & & Gr(v,w) \\ \mathcal{M}_H(v,w) & \ni & 0 \swarrow \end{array}$$

$$\bigoplus_{v=0}^{w} H_{top}(Gr(v,w)) \longleftarrow$$ $(w+1)$-dim'l irr. rep. of $\mathfrak{sl}_2$

$$\mathcal{M}_C(\underset{\overset{\|}{1}}{v},w) = S_w = \{xy = z^w\} \subset \mathbb{C}^3$$

$$\circlearrowleft \ \underset{\pi_1(\mathbb{C}^\times)^\wedge}{\overset{\|}{\mathbb{C}^\times}} \qquad \begin{array}{l} x \mapsto tx \\ y \mapsto t^{-1}y \end{array}$$

m: generic $\quad \mathcal{M}_C^m(1,w) = \widetilde{S_w}$: minimal resolution of S_w

$t \to \infty$ exists

$\Updownarrow$

$x = 0$

$H_{top}(\widetilde{S_w}^{\zeta \leqq 0})$:

w – dimensional space

$H_{top}(S_w^{\zeta\leqq 0})$: 1 – dim. space

$H_{top}(\mathcal{M}_C(v,w)^{\zeta\leqq 0})$ is 1-dim. if $0 \leqq v \leqq w$

0 (no fixed pt) otherwise

$\Longrightarrow \bigoplus_v H_{top}(\mathcal{M}_C(v,w)^{\zeta\leqq 0})$: $(w+1)$-dim. irr. rep. of $\mathfrak{sl}_2$.

$$\bigoplus_v H_{top}(\mathcal{M}_C^m(v,w)^{\zeta\leqq 0}) \cong (\mathbb{C}^2)^{\otimes w}$$

2-dim. rep. of $\mathfrak{sl}_2$

<u>Conj'</u> is checked for Q : affine type A.

$$\left(\begin{array}{l} G = \text{loop group} \\ \mathbb{N} = \cdots\cdots \end{array}\right.$$

$$\mathcal{M}_C(G, \mathbb{N}) = T^*(\text{affine flag variety})$$

∞-dim'l

using instanton moduli space instead of aff. Grassm.