§ Arkihpov – Bezrukavnikov – Ginzburg [ABG]

G : cpx semisimple Lie group, simply-conn
e.g. $SL_n(\mathbb{C})$

$\mathfrak{g} = \mathrm{Lie}\, G$

$G \supset B \supset T$ $\quad \mathfrak{b}, \mathfrak{t}$

Borel torus

G/B : flag variety

$T^*(G/B) = G \times_B (\mathfrak{g}/\mathfrak{b})^*$: cotangent b'dle

μ : dominant weight of G $\quad T \xrightarrow{\mu} \mathbb{C}^*$

$\mathcal{O}(\mu)$: line b'dle over G/B $\quad B \nearrow$

$T^*(G/B)$ pull back

$G^\vee$: Langlands dual group $\supset T^\vee$ dual torus of T
e.g. $PGL_n(\mathbb{C})$

$Gr_{G^\vee}$: affine Grassmannian (∞-dim'l partial flag variety)

$\quad \| $

$G^\vee(K)/G^\vee(\mathcal{O})$ $\quad K = \mathbb{C}((z))$, $\mathcal{O} = \mathbb{C}[[z]]$

$\quad \cong$ homeo

(polynomial) based loop group $\quad S^1 \longrightarrow G_c^\vee$

max. cpt subgroup of $G^\vee$

$\lambda \in \mathrm{Hom}(\mathbb{C}^\times, T^\vee) \cong \mathrm{Hom}(T, \mathbb{C}^\times)$

coweight of $T^\vee$ $\quad$ weight of T

$$T^\vee(K) \longrightarrow G^\vee(K) \longrightarrow Gr_{G^\vee}$$

$$\lambda \longmapsto z^\lambda$$

$G^\vee(\mathbb{C}) \curvearrowright Gr_{G^\vee}$

$Gr^\lambda_{G^\vee} = G^\vee(\mathcal{O}) \cdot z^\lambda$: $G^\vee(\mathcal{O})$-orbit through z^λ

$\Uparrow$

$\overline{Gr^\lambda_{G^\vee}} = \bigsqcup_{\mu \leq \lambda} Gr^\mu_{G^\vee}$ "analog" of Schubert cell $\parallel$ variety

$\mu \leq \lambda$ dominance order

Fact. $\overline{Gr^\lambda_{G^\vee}}$: (finite dim'l) proj. variety

$$Gr_{G^\vee} = \lim \overline{Gr^\lambda_{G}}$$

$$\left(\overline{Gr^\lambda_{G^\vee}} \supset \overline{Gr^\mu_{G^\vee}} \qquad \lambda \geq \mu \right)$$

$IC(\overline{Gr^\lambda_{G^\vee}})$: intersection cohomology complex.

$$i_\mu : z^\mu \hookrightarrow Gr^\mu_{G^\vee} \subset \overline{Gr^\lambda_{G^\vee}} \qquad (\lambda \geq \mu)$$

Th [ABG 2004]

$\exists$ A G-equiv. graded vectr space isomorphism

μ : dominant wt

$$H^0(T^*(G/B), \mathcal{O}(\mu)) \underset{(\star)}{\cong} \bigoplus_{\lambda} V(\lambda)^* \otimes H^*(i^!_\mu IC(\overline{Gr^\lambda_{G^\vee}}))$$

graded by $\mathbb{C}^\times$ $\qquad$ dom. weight $\qquad$ [shift]

$V(\lambda)$ = rep. of G with highest weight $= \lambda$
irr. fin. dim.

Moreover $(\star)$ is compatible with multiplication
LHS : $\mathcal{O}(\mu_1) \otimes \mathcal{O}(\mu_2) \cong \mathcal{O}(\mu_1+\mu_2)$
RHS : Convolution product on $Gr_{G^\vee}$

((equiv) cpx of sheaves on $Gr_{G^\vee} \times Gr_{G^\vee}$ $\rightsquigarrow$ equiv cpx of sheaves on $Gr_{G^\vee}$)

We regard this result as

$$\boxed{\text{a construction of } T^*(G/B) \text{ via topology of } Gr_{G^\vee}}$$

cf. Coulomb branch

§2. singular monopoles

$$G^\vee[z]_1 = \{ g \in G^\vee[z^{-1}] \mid g(z=\infty) = 1 \}$$

$$(G^\vee(\mathcal{O}) = G^\vee[[z]])$$

$G^\vee[z]_1 \cdot z^\mu$ transverse to $Gr^\mu_{G^\vee}$

$$\overline{W}^\lambda_{G^\vee,\mu} = G^\vee[z^{-1}]_1 \cdot z^\mu \cap \overline{Gr^\lambda_{G^\vee}} \qquad \mu \leq \lambda$$

$z^\mu \overset{\mathrm{i}_\mu}{\hookrightarrow}$ affine Grassmannian slice

Fact: $H^*(i_\mu^! IC(\overline{Gr}^\lambda_{G^\vee}))[\text{shift}]$

$$\cong H^*(i_\mu^! IC(\overline{W}^\lambda_{G^\vee,\mu}))$$

$$\cong IH^*_c(\overline{W}^\lambda_{G^\vee,\mu})$$

$G^\vee_c \subset G^\vee$ max. cpt

Th [Braverman-Finkelberg 2010]

$\overline{W}^\lambda_{G^\vee,\mu} \cong$ Uhlenbeck partial cptification of moduli of singular $G^\vee_c$-monopoles on $\mathbb{R}^3$

- $*F_A = d_A\Phi$
- $\mathbb{R}^4 = \mathbb{C}^2 \longrightarrow \mathbb{R}^4/S^1 \cong \mathbb{R}^3$

$\circlearrowleft S^1$ $\quad (z_1, z_2) \mapsto (tz_1, t^{-1}z_2)$

S^1-equiv. instanton $\qquad$ monopole singular at 0

$\lambda : S^1 \to G^\vee_c$

$\sim$ fiber at $0 \in \mathbb{R}^4$

$\mu : S^1 \to G^\vee_c$

$\sim$ fiber at ∞

$\mathbb{R}^4 \subset S^4$

Uhlenbeck cptification $\sim$ bubble $\leftarrow$ appear only at $0 \in \mathbb{R}^4$

$$\overline{W}^{\lambda}_{G^\vee,\mu} = \bigsqcup_{\lambda \geq \kappa \geq \mu} W^{\kappa}_{G^\vee,\mu}$$

moduli space of singular monopole

§4. Affine case

G : affine Kac-Moody ("dual" to untwisted $G^\vee$)

e.g. $SL_n(K)$ $\qquad$ $PGL_n(K)$

$G^\vee_{fin,c}$ $\qquad$ e.g. $PU_n = U_n/S^1$

finite dim'l group

λ, μ : dominant weights of G.

with level $\ell \in \mathbb{Z}_{>0}$

$\longleftrightarrow \quad \mathrm{Hom}_{grp}(\mathbb{Z}/\ell\mathbb{Z}, T^\vee_{fin,c})$

+ additional combinatorial information $\in \mathbb{Z}$

Replace $\underset{\mathbb{R}^3}{\overset{\parallel}{\mathbb{R}^4}}/S^1$ by $\mathbb{R}^4/\mathbb{Z}/\ell\mathbb{Z}$

$\mathcal{M}(\lambda,\mu)$ = Uhlenbeck partial compactification of $\mathbb{Z}/\ell\mathbb{Z}$ - equiv. $G_{c,\mathrm{cpt}}$ instantons on $\mathbb{R}^4 (\subset S^4)$

$0 \in \mathbb{R}^4$ fixed by $\mathbb{Z}/\ell\mathbb{Z}$
$\rightsquigarrow \lambda$

$\infty \in S^4 \rightsquigarrow \mu$

add. information of $(\lambda,\mu) \longrightarrow$ instanton number

bubble $\in \cdots S^n(\mathbb{R}^4 \setminus 0 / \mathbb{Z}/\ell\mathbb{Z})$

<u>Conjecture</u> & <u>Th</u> for $G = SL_n(\mathbb{K})$ (work in progress D. Muthiah)

μ: dominant

$$H^0(T^*(G/B), \mathcal{O}(\mu)) \cong \bigoplus_{\lambda \text{ dominant}} V(\lambda)^* \otimes \underbrace{H^*(i_\mu^! IC(\mathcal{M}(\lambda,\mu)))}_{= IH_c^*(\mathcal{M}(\lambda,\mu))}$$

∞-dim'l affine flag variety

constructed from topology of instanton moduli space

LHS : $H^0(G \times_B (\mathfrak{b}/\mathfrak{c})^*, \mathcal{O}(\mu))$

$$= \mathrm{Ind}_B^G\left(\mathbb{C}[(\mathfrak{n}/_{\mathfrak{b}})^*] \otimes \mathbb{C}_{-\mu} \right)$$

↖ B-module

§ Coulomb branch

Braverman-Finkelberg-N

$$\bigoplus_\lambda V(\lambda)^* \otimes H^*(i_\mu^! IC(\overline{Gr_{G^\vee}^\lambda}))$$

$$H^*(i_\mu^!, \underbrace{\bigoplus_\lambda V(\lambda)^* \otimes IC(\overline{Gr_{G^\vee}^\lambda})}_{= \mathcal{A}_R})$$

$\mathcal{A}_R$: ring object of $\mathrm{Perv}_{G^\vee(\mathcal{O})}(Gr_{G^\vee})$

$\cong$ geom. Satake

tensor category

$\mathbb{C}[G]$: ring object of $(\mathrm{Rep}\, G, \otimes)$

$\| $

$\bigoplus_\lambda V(\lambda)^* \otimes V(\lambda)$ $\mathbb{C}[G]$: regular rep.

$$\mathbb{C}[G] \otimes \mathbb{C}[G] \xrightarrow{\text{mult.}} \mathbb{C}[G]$$

$$\mathcal{A}_R \ \underset{\uparrow \text{conv.}}{*} \ \mathcal{A}_R \xrightarrow{\text{mult}} \mathcal{A}_R$$

comm. ass.

$$
\begin{tikzcd}
(G, N) \arrow[r, squiggly] & \mathcal{R} \arrow[r, squiggly] \arrow[dr] \arrow[d, squiggly] & H^{G(\mathcal{O})}_*(\mathcal{R}) \arrow[d, "\mathrm{Spec}"] \\
& \text{ring object in } D_{G(\mathcal{O})}(\mathrm{Gr}_G) \arrow[r, squiggly] & \text{Coulomb branch}
\end{tikzcd}
$$

G: f.d. cpx red. group

N: rep. of G

$\mathcal{R}$: certain ∞-dim'l variety

$\mathrm{Gr}_G \circlearrowleft$

$D_{G(\mathcal{O})}(\mathrm{Gr}_G) \supset \mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_G)$

ABG is an example of this construction, but start from a ring object

Conj. & Th

can be regarded as an example of Coulomb branch construction

for $G \rightsquigarrow$ affine KM group

$\mathrm{Gr}_{\text{affine KM group}}$ may not be defined.

double affine Grassmanian $\longrightarrow$ Use instanton moduli sp instead!