Instanton moduli spaces and W-algebras

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§0. Overview

Vague Expectation: (cf. Gukov's lectures)

4d gauge theories, in particular instanton moduli spaces

$\xrightarrow[\text{theory (convolution algebra)}]{\text{technique of geometric representation}}$ VOA's (2d CFT's)

We discuss examples of this expectation (Heisenberg, cosets, W-alg), which can be mathematically rigorously established.

old work, AGT

Remark This is superficially similar to a topic discussed in Arakawa's lectures. But it is different.

Question: Why do we expect such constructions?

Why dimensions are changed from 4 to 2?

As a mathematician, I only have an unsatisfactory explanation:

We have examples. (My earlier works have origin in Ringel, Lusztig's works of realization of quantum groups)

Physicists are brave and use results which have no mathematical foundation:

It is a consequence of an existence of a hypothetical

$$\underset{\substack{\| \\ 4+2}}{6}$$ dimensional quantum field theory.

(labelled by an ADE Dynkin diagram)

6d QFT assigns

$$\left(\begin{array}{lcl} \text{6d (Riemannian) manifold} & \longrightarrow & \text{number (partition function)} \\ \quad\quad\quad\quad\quad\quad\quad\quad \Big\downarrow \text{bdry} & & \\ \text{5d (Riemannian) manifold} & \longrightarrow & \text{quantum Hilbert space} \end{array}\right.$$

We fix a 4-manifold X^4 and

take $X^4 \times \Sigma^2$ and $X^4 \times S^1$ as 6 and 5 manifolds.

Then 6d theory can be viewed as 2d QFT:

$$\begin{array}{lcl} \text{2d manifold } \Sigma & \longrightarrow & \text{number} \\ \text{1d manifold } S^1 & \longrightarrow & \text{quantum Hilbert space.} \end{array}$$

By physics jargon, this construction is

- topological in 4d
- conformal in 2d

expected to be "cohomology" of instanton moduli space on X^4

gauge grp = ADE

We take

$$X^4 = \underset{\mathbb{R}^4}{\mathbb{C}^2} \text{ with } \underset{(\mathbb{C}^\times \times \mathbb{C}^\times)}{S^1 \times S^1\text{-action}} \qquad \underset{\substack{\cap \\ \mathbb{C}^2}}{(z_1, z_2)} \mapsto (t_1 z_1, t_2 z_2)$$

We expect : quantum Hilbert space = cohomology of KADE-instanton moduli space on $\mathbb{C}^2$.

How $S^1 \times S^1$-action enters the story ?

We use equivariant cohomology groups.

$\longrightarrow$ VOA defined over a polynomial ring

$$H^*_{S^1 \times S^1}(pt) = H^*(B(S^1 \times S^1)) \cong \mathbb{C}[\varepsilon_1, \varepsilon_2]$$

(e.g. level is a rational function in $\varepsilon_1, \varepsilon_2$)

§1. equivariant (co)homology

G : complex reductive group e.g. $GL(n)$, $T^n = (\mathbb{C}^\times)^n$

$X \curvearrowleft G$ reasonable action $X \subset \mathbb{C}^N \xleftarrow{\text{linear}} G$

reasonable topological space complex algebraic variety

$\rightsquigarrow$ equivariant cohomology $H^*_G(X)$ $* = 0, 1, 2, \cdots$

" Borel–Moore homology $H^G_*(X)$ $* = \dim X$, $\dim X - 1$, $\dim X - 2$, $\vdots$

modules over $H^*_G(pt) \longleftarrow$ ~~nontrivial~~ polynomial ring

$EG \xrightarrow{G} BG$

Borel construction classifying space for G

e.g. $G = \mathbb{C}^\times$ $\mathbb{C}^N \setminus \{0\} \circlearrowleft \mathbb{C}^\times$, $N \gg 0$

$$\begin{array}{ccc} \mathbb{C}^N \setminus \{0\} & \leftarrow & \mathbb{C}^\times \\ \downarrow & & \\ \mathbb{CP}^{N-1} & & \end{array}$$

$E\mathbb{C}^\times = \lim_{N\to\infty} \mathbb{C}^N \setminus \{0\}$ $B\mathbb{C}^\times = \lim \mathbb{CP}^{N-1}$

$$H^*_G(X) = H^*(X \times_G EG)$$

$$H^*(\mathbb{C}P^{N-1}) = \mathbb{C}[x]/_{x^N=0}$$

$$\xrightarrow[N\to\infty]{} \mathbb{C}[x] = H^*_{\mathbb{C}^\times}(pt)$$

<u>Fact</u> $H^*_G(pt) = \mathbb{C}[\mathfrak{t}]^W$

W = Weyl group

$\mathfrak{t}$ = Lie T $\quad T \subset G$ max. torus

<u>Fact</u> $H^*_G(X) = H^*_T(X)^W$

<u>Theorem</u> (localization)

X^T = T-fixed point set in X $\quad X^T \overset{i}{\hookrightarrow} X$ inclusion

$$\left| \begin{array}{l} Y = X \setminus X^T \\ Y^T = \emptyset \\ H^*_T(Y) \otimes_{H^*_T(pt)} \mathrm{Frac} = 0 \end{array} \right.$$

$$\left.\begin{array}{l} H^*_T(X) \xrightarrow{i^*} H^*_T(X^T) \\ H^T_*(X^T) \xrightarrow[i_*]{} H^T_*(X) \end{array}\right\}$$ become <u>isomorphism</u> after $\bullet \otimes_{H^*_T(pt)} \mathrm{Frac}\, H^*_T(pt)$.

e.g. $T = \mathbb{C}^\times$ $\quad \bullet \otimes_{\mathbb{C}[x]} \mathbb{C}(x)$

<u>Example</u> $X = \mathbb{CP}^1 \curvearrowleft T = T^2$ $(t_1, t_2) \circ [z_1 : z_2] = [t_1 z_1 ; t_2 z_2]$

fixed pts $[1:0]$, $[0:1]$

$$H_T^*(X) = \mathbb{C}[h, x_1, x_2] \big/ (h + x_1)(h + x_2) = 0$$

$\ni f(h, x_1, x_2)$ modulo ——

deformation of $h^2 = 0$ for $H^*(\mathbb{CP}^1) = \mathbb{C}[h]/h^2 = 0$

$$H_T^*(pt) = \mathbb{C}[x_1, x_2]$$

$$\xrightarrow{i^*} H_T^*(X^T) = \mathbb{C}[x_1, x_2] \oplus \mathbb{C}[x_1, x_2]$$

$$\rotatebox{90}{$\in$}$$

$$f|_{h = -x_1} \oplus f|_{h = -x_2}$$

$$\quad\; \| \qquad\qquad\qquad \|$$

$$\leftarrow \cdots\cdots \mapsto \quad g_1(x_1, x_2) \qquad g_2(x_1, x_2)$$

$$\frac{1}{x_2 - x_1}\left((x_2 + h) g_1 - (x_1 + h) g_2\right)$$

inverse is <u>not</u> defined along $x_1 = x_2$

Borel-Moore homology • fundamental class of a smooth mfd X (cpx algebraic variety) irreducible

is defined even when X is noncompact.

• $\underset{\text{proper}}{f : X \longrightarrow Y} \rightsquigarrow H_*(X) \xrightarrow[f_*]{} H_*(Y)$

Ex. $X = \mathbb{C} \curvearrowleft T = \mathbb{C}^\times$ $\quad H_2^T(\mathbb{C}) \ni [X]$ fundamental class

$$H_*^T(X) \cong \underset{\substack{\parallel \\ \mathbb{C}[x]}}{H_T^*(pt)} \cap [X] = \bigoplus_{n=0}^{\infty} \mathbb{C}\underbrace{x^n \cap [X]}_{\text{in degree } 2-n}$$

$$H_*^T(\underset{\substack{\parallel \\ \{p\}}}{X^T}) \cong H_T^*(pt) \cap [\{p\}]$$

$\uparrow$ degree $0, -2, -4, \cdots$

$$i_* : \underset{\substack{\cup \\ [\{p\}]}}{H_*^T(X^T)} \underset{\longmapsto}{\longrightarrow} \underset{x \cap [X]}{H_*^T(X)}$$

inverse $[X] \longmapsto \frac{1}{x}[\{p\}]$

not defined at $x = 0$