Supplement

$X \curvearrowleft T$ Assume X : manifold

$$H^T_*(X^T) \xrightarrow{i_*} H^T_*(X) \underset{\text{P.D.}}{\cong} H_T^{\dim X - *}(X) \qquad i_*^{-1} = \sum_\alpha \frac{1}{e(N_{X_\alpha/X})} i_\alpha^*$$

$X^T = \coprod X_\alpha$ (connected component)

$i_\alpha : X_\alpha \hookrightarrow X$

$e(N_{X_\alpha/X})$: equivariant Euler class of the normal bundle

$$TX|_{X_\alpha} / TX_\alpha \cong N_{X_\alpha/X}$$

T-equivariant bdle

If each X_α is a point $\{x_\alpha\}$, $N_{X_\alpha/X} = T_{x_\alpha}X$ ↶ representation of T

$e(T_{x_\alpha}X) = \prod$ weight (regarded as a polynomial on $\mathfrak{t} = \mathrm{Lie}\, T$)

cf. Examples from yesterday

$$\frac{1}{x_2 - x_1} \text{ for } \mathbb{C}P^1 \qquad \frac{1}{x} \text{ for } \mathbb{C}$$

§2. Instanton moduli spaces

K: cpt Lie group

$\mathcal{M}_0(k, K)$ = framed moduli space of K-instantons on S^4 with charge $= k$

$S^4 = \mathbb{R}^4 \cup \{\infty\}$, $k \in \mathbb{Z}_{\geq 0}$

framed $\leftarrow$ trivialization at ∞

$\cap$

$\mathcal{U}(k, K)$ = its partial compactification $= \bigsqcup_{l \leq k} \mathcal{M}_0(l, K) \times S^{k-l}\mathbb{R}^4$

= moduli of ideal instantons (Uhlenbeck space)

$S^{k-l}\mathbb{R}^4$: singularity

$S^{l}\mathbb{R}^4 = (\mathbb{R}^4)^{l}/\mathfrak{S}_{l}$ l-th symmetric product

$\underline{K = SU(r)}$ ADHM description — space of certain linear matrices

ancestor of quiver varieties

Lectures on Hilb. scheme Chapter 2.

As an Application of ADHM: $\mathcal{M}(k, r)$: (symplectic) resolution of $\mathcal{U}(k, SU(r))$

$r = 1$ $\quad S^k\mathbb{R}^4 = S^k\mathbb{C}^2 \longleftarrow \mathrm{Hilb}^k(\mathbb{C}^2)$

$S^k\mathbb{R}^4 =$ "$\mathcal{U}(k, U(1))$"

$\mathrm{Hilb}^k(\mathbb{C}^2) = \{ I \subset \mathbb{C}[x,y] \text{ ideal} \mid \dim \mathbb{C}[x,y]/I = k \}$

<u>Properties</u> ∘ $\mathcal{U}(k,K) \circlearrowleft \mathbb{T} = T\times(\mathbb{C}^\times)^2$ $\qquad T \subset G = K_{\mathbb{C}}$ max. torus

$\mathcal{M}(k,r) \circlearrowleft \mathbb{T} = T^{r-1}\times(\mathbb{C}^\times)^2$

↳ acts on $S^4 = \mathbb{R}^4 \cup \{\infty\} = \mathbb{C}^2 \cup \{\infty\}$

∘ $\pi : \mathcal{M}(k,r) \to \mathcal{U}(k, SU(r))$ proper map

$\mathrm{Hilb}^k\mathbb{C}^2 \to S^k\mathbb{C}^2$

↑ smooth (mfd) PD holds

<u>Remark</u>

$\mathcal{M}(k,r)$ does <u>not</u> exist for $K \neq SU(r)$.

∘ $\mathcal{U}(k,K)^T = S^k\mathbb{R}^4 \left(= \underset{\text{pt}}{\underset{\shortparallel}{\mathcal{M}^0(0,K)}} \times S^k\mathbb{R}^4 \right)$

$$\mathcal{M}(k,r)^T = \coprod_{k_1+\cdots+k_r=k} \mathrm{Hilb}^{k_1}(\mathbb{C}^2)\times\cdots\times\mathrm{Hilb}^{k_r}(\mathbb{C}^2)$$

$$\begin{array}{|c|c|c|c|}
\hline
y^2 & & & \\
\hline
y & xy & & \\
\hline
1 & x & x^2 & x^3 \\
\hline
\end{array}$$

e.g. (3 2 1 2)

$$(\mathrm{Hilb}^k\mathbb{C}^2)^{(\mathbb{C}^\times)^2}$$

$$\shortparallel$$

$$\{ \underset{\text{monomial ideal}}{I \subset \mathbb{C}[x,y]} \mid \dim \mathbb{C}[x,y]/I = k \} \cong \{ \lambda \vdash k \}$$

partition $\qquad \lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$

$$\mathcal{M}(k,r)^{\mathbb{T}} = \{ \vec{\lambda} = (\lambda^1, \lambda^2, \ldots, \lambda^r) \mid \sum_\alpha |\lambda^\alpha| = k \}$$

$\mathcal{V}$ = tautological vector bundle over $\mathcal{M}(k,r)$ (has no analog for $\mathcal{U}(k,K)$ $K \neq SU(r)$)

$r=1$ $\quad \mathrm{Hilb}^k \mathbb{C}^2 = \{ I \mid \dim \mathbb{C}[x,y]/I = k \}$

$$\mathcal{V}|_I \overset{\text{def.}}{=} \mathbb{C}[x,y]/I \qquad \text{rank} = k$$

more generally $\mathcal{V}$ over $\mathcal{M}(k,r)$ rank $= k$

T-equivariant vector bundle

$$\mathcal{M}(k,r) \supset \mathrm{Hilb}^{k_1} \times \mathrm{Hilb}^{k_2} \times \cdots \times \mathrm{Hilb}^{k_r}$$

$$\mathcal{V}| \cong \mathcal{V}_{\mathrm{Hilb}^{k_1}} \oplus \cdots \oplus \mathcal{V}_{\mathrm{Hilb}^{k_r}}$$

$c_i(\mathcal{V})$ $\quad$ ith equivariant Chern class $\in H^{2i}_T(\mathcal{M}(k,r))$

$i = 0, 1, \ldots, \operatorname{rank} \mathcal{V} = k$

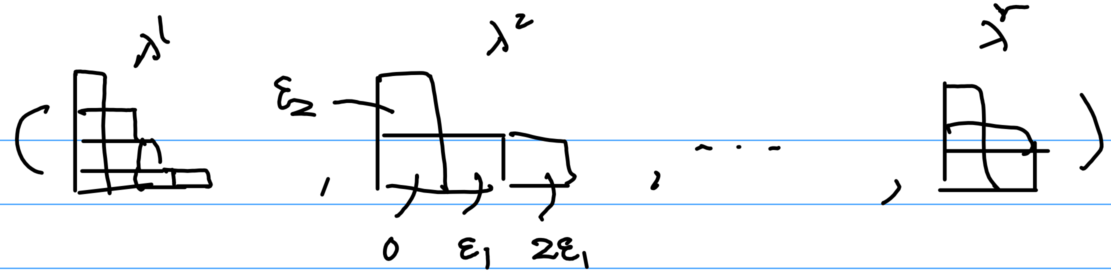
$\vec{\lambda} = (\lambda^1, \ldots, \lambda^r) \leftrightarrow$ fixed pt in $\mathcal{M}(k,r)$ (T-) $\qquad$ inclusion $i_{\vec{\lambda}} : \{\vec{\lambda}\} \hookrightarrow \mathcal{M}(k,r)$

$$i^*_{\vec{\lambda}} c_i(\mathcal{V}) \in H^{2i}_T(\{\vec{\lambda}\}) = \text{degree } 2i \text{ part of } \mathbb{C}[\mathrm{Lie}\, T]$$

ith elementary symmetric function in weights of $\mathcal{V}|_{\vec{\lambda}}$

T-representation

weights of $V|_{\vec{\lambda}}$:



$a_2 +$

$$H^*_{\mathbb{T}}(pt) = \mathbb{C}[\mathrm{Lie}\,\mathbb{T}] = \mathbb{C}[\mathrm{Lie}\,T^{r-1} \times (\mathbb{C}^\times)^2]$$

$a_1, \dots, a_r$ $\quad$ $\varepsilon_1, \varepsilon_2$

$a_1 + \dots + a_r = 0$

<u>Rem</u> $\mathfrak{M}(k, r)$: quiver variety associated with $\circ\!\circlearrowleft$ i.e. $\circlearrowright \mathbb{C}^k \circlearrowleft$

$$\begin{array}{c} \mathbb{C}^k \\ \downarrow\uparrow \\ \mathbb{C}^r \end{array}$$

§3. Construction of Heisenberg algebra representation

$$\mathrm{Hilb}^k \equiv \mathrm{Hilb}^k \mathbb{C}^2$$

$S = \mathbb{C}^\times \times \mathbb{C}^\times \quad (\mathrm{Hilb}^k)^S = \{ x \vdash k \}$ $\qquad$ $\mathrm{Lie}\, S \ni \varepsilon_1, \varepsilon_2$ std coordinate

$$H^*_S(\mathrm{Hilb}^k) \otimes_{H^*_S(pt)} \underbrace{\mathrm{Frac}\, H^*_S(pt)}_{= \mathbb{C}(\varepsilon_1, \varepsilon_2)} \cong \bigoplus_{\lambda \vdash k} \mathrm{Frac}\, H^*_S(pt) \cdot [\lambda]$$

$\bigoplus_k H^*_S(\mathrm{Hilb}^k) \otimes_{H^*_S(pt)} \mathrm{Frac}\, H^*_S(pt)$ has the size of Fock representation of Heisenberg algebra.

One can construct Heis. repr. on $\bigoplus_k H^*_S(\mathrm{Hilb}^k)$ as follows.

$$\mathrm{Hilb}^{k+n} \times \mathrm{Hilb}^k \times \mathbb{C}^2 \supset P_n := \{ (I_1, I_2, x) \mid I_1 \subset I_2, \ \mathrm{supp}(I_2/I_1) = \{x\} \}$$

closed subvariety

$$\begin{tikzcd} & P_n \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathrm{Hilb}^{k+n} & & \mathrm{Hilb}^k \times \mathbb{C}^2 \end{tikzcd}$$

proper $\Rightarrow$ p_1

$H^*_S(\mathrm{Hilb}^k) \underset{PD}{\cong} H^S_{\dim \mathrm{Hilb}^k - *}(\mathrm{Hilb}^k) \Longrightarrow$ both pull-back and push forward are defined

$\underset{0}{\overset{\shortparallel}{p}} \in \mathbb{C}^2$ fixed pt

$\alpha \in H^S_*(\{p\})$ $\qquad p_{1*} p_2^*(\cdot \times \alpha) =: \overset{\text{creation}}{P_{-n}(\alpha)} : H^S_*(\mathrm{Hilb}^k) \longrightarrow H^S_*(\mathrm{Hilb}^{k+n})$

$H^S_*(\mathrm{Hilb}^k)$ has $(\ ,\ )$ by $(-1)^{\dim \mathrm{Hilb}^k/2} \int_{\mathrm{Hilb}^k} \cdot \cup \cdot$

$P_n(\beta) =$ adjoint of $P_{-n}(\beta) : H^S_*(\mathrm{Hilb}^{k+n}) \longrightarrow H^S_*(\mathrm{Hilb}^k)$

$\beta \in H^S_*(\{p\})$ $\qquad$ annihilation

Th [N 96, Grojnowski]

$$[P_m(\alpha), P_n(\beta)] = m\,\delta_{m+n,0} \overbrace{\left(\int_{\mathbb{C}^2} \alpha \cup \beta\right)}^{=\varepsilon_1\varepsilon_2} \mathrm{id}$$

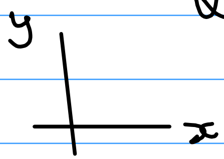
Take $\alpha = [p]$ $\qquad \int_{\mathbb{C}^2} \alpha \cup \beta$ $\qquad$ (cf. $p \in \mathbb{C} \hookrightarrow \mathbb{C}_\varepsilon$
$\beta = $ " $\qquad i_{p*}\alpha = \varepsilon_1\varepsilon_2 \underbrace{[\mathbb{C}^2]}_{=\text{P.D.}\,1}$ $\qquad i_*[p] = \varepsilon[\mathbb{C}]$

$$\int_{\mathbb{C}^2} \alpha \cup \beta = \int_{\mathbb{C}^2} \varepsilon_1 \varepsilon_2 [p] = \varepsilon_1 \varepsilon_2$$

One can modify $P_n(\alpha)$, $P_n(\beta)$ so that $[P_m(\alpha), P_n(\beta)] = m\,\delta_{m+n,0}\,\cancel{\varepsilon_1\varepsilon_2}$

e.g. $\alpha = [p]$, $\beta = [\mathbb{C}^2]$

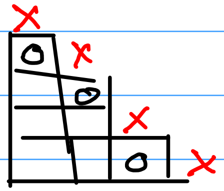
$\alpha = [x\text{-axis}]$, $\beta = [y\text{-axis}]$



$$\int_{\mathbb{C}^2} \alpha \cup \beta = x\text{-axis} \cap y\text{-axis} = 1.$$

<u>Example of computation</u>

$m = 1$, $n = -1$ $\qquad [P_1, P_{-1}] = \mathrm{id}$



× : addable part

o : removable box

P_{-1} : add a box (to get a new Young diagram) $\qquad$ P_1 : remove a box (")

$P_{-1}P_1[\lambda]$: remove and add $\longrightarrow [\mu]$

$P_1 P_{-1}[\lambda]$: add and remove s.t. $|\lambda| = |\mu|$

If $\lambda \neq \mu$ $\Rightarrow$ order of remove/add is irrelevant

bijection between $P_{-1}P_1$ and P_1P_{-1}

$\longrightarrow$ commutator vanishes for $\lambda \neq \mu$

If $\lambda = \mu$ $\Rightarrow$ addable case is always 1 case larger than removable

$\Longrightarrow$ $[P_1, P_{-1}] = 1$

- over $\otimes_{H_S^*(pt)} \mathrm{Frac}\, H_S^*(pt)$, this is an irreducible representation of Heis.

- By more works $\bigoplus_{k} H_S^*(\mathrm{Hilb}^k) \cong \mathrm{Span}\{P_{-m_1}(\alpha)P_{-m_2}(\alpha)\cdots P_{-m_r}(\alpha)|\mathrm{vac}\rangle\}$

defined over $H_S^*(pt) = \mathbb{C}[\varepsilon_1, \varepsilon_2]$ $\quad \alpha = [\mathbb{C}^2]$ $\quad$ fund. class of $\mathrm{Hilb}^0(\mathbb{C}^2) = \{pt\}$

cf. Jack polynomial with parameter $= -\varepsilon_1/\varepsilon_2$

§ Higher rank case

$\mathcal{M}(k,r)$

$\circlearrowleft \ \mathbb{T} = (\mathbb{C}^\times)^2 \times T$

$\mathcal{M}(k,r)^{\mathbb{T}} = \{ \vec{\lambda} \mid \sum |\lambda^\alpha| = k \}$

$$\bigoplus_k H^*_{\mathbb{T}}(\mathcal{M}(k,r)) \otimes_{H^*_{\mathbb{T}}(pt)} \operatorname{Frac} H^*_{\mathbb{T}}(pt) = \bigoplus_{\vec{\lambda}} \operatorname{Frac} \cdot [\vec{\lambda}]$$

$\vec{\lambda} = (\lambda^1, \dots, \lambda^r)$ r-tuple of partitions

$$\cong \overset{\lambda^1}{\text{Fock}} \otimes \overset{\lambda^2}{\text{Fock}} \otimes \cdots \otimes \overset{\lambda^r}{\text{Fock}} \otimes \operatorname{Frac} H^*_{\mathbb{T}}(pt)$$

$\circlearrowleft P^1_m(\alpha)$ $\quad \circlearrowleft P^2_m(\alpha)$ $\quad \cdots \quad \circlearrowleft P^n_m(\alpha)$

representation of $\text{Heis} \otimes \cdots \otimes \text{Heis}$

$= \text{Heis}_{\mathfrak{t}}$ $\qquad \mathfrak{t} = \text{Lie}\, T$

$T =$ max torus of $GL(r)$

This $\text{Heis}_{\mathfrak{t}}$ acts only after the localization $\otimes$ Frac

(recall $S^k\mathbb{C}^2 = \mathcal{U}(k, \underset{\sim}{U(1)})$)