§4. Higher rank case

Recall

$\mathcal{M}(k, r)$ resolution of $\mathcal{U}(k, SU(r))$

$\circlearrowleft$

$\mathbb{T} = (\mathbb{C}^\times)^2 \times T \qquad T \subset SL(r) = SU(r)_{\mathbb{C}}$

max. torus

Recall

$$\mathcal{M}(k,r)^T = \coprod_{k_1 + k_2 = k} \mathrm{Hilb}^{k_1}(\mathbb{C}^2) \times \mathrm{Hilb}^{k_2}(\mathbb{C}^2) \times \cdots \times \mathrm{Hilb}^{k_r}$$

$$\mathcal{M}(k,r)^{\mathbb{T}} = \{\vec{\lambda} = (\lambda^1, \ldots, \lambda^r) \mid \text{pair of partitions s.t. } \sum_{\alpha=1}^{r} |\lambda^\alpha| = k\}$$

$$\therefore \bigoplus_k H^*_{\mathbb{T}}(\mathcal{M}(k,r)) \otimes_{H^*_{\mathbb{T}}(pt)} \mathrm{Frac}\, H^*_{\mathbb{T}}(pt) \cong \mathrm{Fock} \otimes \mathrm{Fock} \otimes \cdots \otimes \mathrm{Frac}\, H^*_{\mathbb{T}}(pt)$$

$\circlearrowleft P^1_m(\alpha) \qquad \circlearrowleft P^2_n(\alpha)$

Heisenberg for $T^r \subset GL(r)$

Which subalgebra of Heis_t preserve $\bigoplus_k H^*_{\mathbb{T}}(\mathcal{M}(k,r))$?

$$\mathcal{M}(k+n, r) \times \mathcal{M}(k, r) \times \mathbb{C}^2 \supset P_n$$

is still possible to define in the same way.

Th (Baranovsky)

$$[P_m, P_n] = r\, m\, \delta_{m+n,0}\, \varepsilon_1 \varepsilon_2\, \mathrm{id}$$

$\alpha = [pt]$

level r Heisenberg

$$\bigoplus_k H^*_{\mathbb{T}}(\mathcal{M}(k, r)) \cong \underset{P_k(\alpha)}{\text{Fock fr}} \otimes \underline{\qquad\qquad}$$

$$\left(\bigoplus_k IH^*_{\mathbb{T}}(\mathcal{U}(k, SU(r))) \right)$$

equivariant intersection cohomology

We want to construct W-algebra representation on

Two approaches

1°. Schiffmann-Vasserot, Feigin-Tsymbaliuk

Use $c_1(\mathcal{V})$ and $P_{\pm 1}$

(← correspondence is smooth.)

to construct $Y(\hat{gl}_1)$ Yangian for $\hat{gl}_1$.

e.g. $[c_1(\mathcal{V}), P_{\pm 1}] = L_{\pm 1}$ for Virasoro.

Then $W(gl_r) = \mathrm{Heis} \otimes W(sl_r)$ is its quotient.

2°. Maulik-Okounkov, Braverman-Finkelberg-N for G: ADE

Realise Feigin-Frenkel : $\mathfrak{g}$: simple Lie alg. $\supset \mathfrak{t}$: Cartan

α_i : simple root

$$\mathrm{Heis}_{\mathfrak{t}} \supset \bigcap_i \mathrm{Vir}_{\alpha_i} \otimes \mathrm{Heis}_{\alpha_i^\perp} = W(\mathfrak{g})$$

in a geometric way.

2°. a) realise this picture, i.e. α_i etc.

b) construct Vir_{α_i} ($G = SL_2$ case) $\quad \mathrm{Heis} \supset \mathrm{Vir}$

(Feigin-Fuchs)

The 0^{th} step of 2°. a) is the following observation:

$$\mathcal{M}(k,r) \curvearrowleft T = T^r = \mathbb{C}^\times \times \cdots \times \underbrace{\underset{i}{\mathbb{C}^\times} \times \underset{i+1}{\mathbb{C}^\times}}_{\cup \atop \Delta\mathbb{C}^\times} \times \cdots \times \mathbb{C}^\times$$

$$\text{Consider} \quad \cup \quad T_i$$

The fixed point set is larger:

$$\mathcal{M}(k,r)^{T_i} = \coprod_{\sum k_\alpha + \overline{k_i} = k} \mathrm{Hilb}^{k_1}(\mathbb{C}^2) \times \cdots \times \mathrm{Hilb}^{k_{i-1}}(\mathbb{C}^2) \times \mathcal{M}(\overline{k_i}, 2) \times \mathrm{Hilb}^{k_{i+2}}(\mathbb{C}^2) \times \cdots$$

$\mathcal{M}(\overline{k_i},2) \hookrightarrow \mathcal{M}(k,r)$

$\circlearrowleft$ fixed $\mathbb{C}^\times \times \mathbb{C}^\times$

$$\mathcal{M}(k,r)^T = \coprod_{\overline{k_i} = k_i + k_{i+1}} \quad {}'' \quad \times \mathrm{Hilb}^{k_i} \times \mathrm{Hilb}^{k_{i+1}} \times \quad {}''$$

By a simple computation, i^* (i: inclusion of fixed pt sets)

does <u>not</u> give a correct subspace.

(even Virasoro $\subset$ Heis. is no realised.)
Feigin–Fuchs

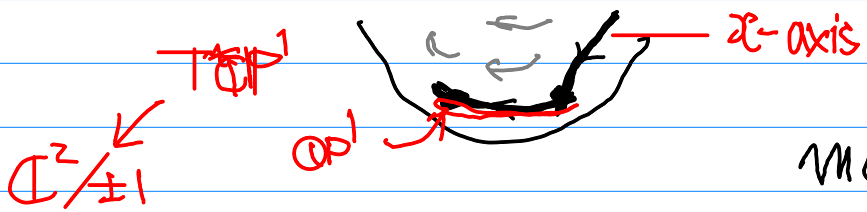
Stable envelope (Maulik–Okounkov)

$\underset{N}{\sim}$ hyperbolic restriction introduced by Braden

Choose $\rho : \mathbb{C}^\times \longrightarrow T$ s.t. $\quad \mathcal{M}(k,r)^{\rho(\mathbb{C}^\times)} = \mathcal{M}(k,r)^T$

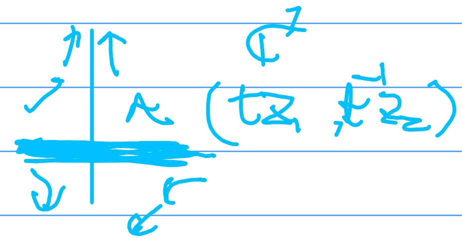
Consider the attracting set

$$\mathcal{M}(k,r)^{\rho \geq 0} := \{ x \in \mathcal{M}(k,r) \mid \lim_{t\to 0} \rho(t) x \quad \text{exists} \}$$



limit

$\mathcal{M}(k,r)^T$



$$H_*^{\mathbb{T}}(\mathcal{M}(k,r)^T) \xrightarrow[\cong]{\text{Stab}} H_*^{\mathbb{T}}(\mathcal{M}(k,r)^{\rho \geq 0}) \longrightarrow H_*^{\mathbb{T}}(\mathcal{M}(k,r))$$

$H_*^{\mathbb{T}, \text{ord}}(\mathcal{M}(k,r))$

Similarly we consider stable envelopes for $\rho_i : \mathbb{C}^\times \longrightarrow T_i$

These realise the transpose of $W(\mathfrak{gl}(r)) \hookrightarrow \mathrm{Heis}_r$

and can be generalised to $G = ADE$.

↳ Need to understand SL_3.

§5. Higher level

work in ~~progress~~ stuck with D. Muthiah

(type $A_{\ell-1}$ singularity)

Instantons on $\mathbb{R}^4/\mathbb{Z}/\ell$ $\qquad \zeta \in \mathbb{Z}/\ell \quad (z_1, z_2) \mapsto (\zeta z_1, \zeta^{-1} z_2)$

<u>Expectation</u>: K - instanton moduli spaces on $\mathbb{R}^4/\mathbb{Z}/\ell$

$\subsetneqq$ Coulomb branches of framed affine quiver gauge theories of type K_{aff}.. $\quad K = SU(2)$

$\qquad\qquad (\mathfrak{sl}_2)_{\mathrm{aff}}$

In particular, we have "moduli space" $\mathcal{M}_C(\lambda, \mu)$

parametrised by $\begin{cases} \lambda : \text{dominant coweight of } K_{\mathrm{aff}}. \\ \mu : \text{coweight s.t. } \mu \leqq \lambda \end{cases}$

<u>Remark</u>. Even $\ell = 1$ $(\lambda = \Lambda_0)$, we have more "moduli spaces"

$\mathcal{U}(k, K) = \mathcal{M}_C(\Lambda_0, \Lambda_0 - k\delta)$.

ℓ = level of λ $\qquad\qquad \mu \leqq \lambda$ is more general.

$\mathcal{M}_C(\lambda,\mu) \curvearrowleft \mathbb{T} = \underbrace{T \times \mathbb{C}^\times}_{T} \times \mathbb{C}^\times = T_{\mathrm{aff}} \times \mathbb{C}^\times$

acts on $\mathbb{C}^2$ $\quad$ maximal torus of K_{aff}

$(x,y) \mapsto (tx, t^{-1}y)$

Conjectural Geometric Satake for affine Lie algebras (BFN)

⓪ $\mathcal{M}_C(\lambda,\mu)^{T_{\mathrm{aff}}} = \mathrm{pt}$ or ϕ

① Choose $\rho_{\mathrm{aff}} : \mathbb{C}^\times \longrightarrow T_{\mathrm{aff}}$ "dominant" coweight

$\Longrightarrow \bigoplus_\mu H_{\mathrm{top}}(\mathcal{M}_C(\lambda,\mu)^{\rho_{\mathrm{aff}} \geq 0})$ (more precisely hyperbolic restriction)

has a structure of $(K_{\mathrm{aff}})^\vee$-representation $V_\mu(\lambda)$

s.t. • μ ---- weight

• compatible with restriction to Levi subalgebras

as a Kac-Moody

not affine of Levi for finite dim.

proved for affine type A [N].

$$\mathcal{V}_\mu(\lambda)\otimes_{\mathbb{C}} H^*_T(pt) \cong \text{“}H^T_*(\mathcal{M}_C(\lambda,\mu)^{P_{\mathrm{aff}}\geq 0})\text{”} \longleftarrow \text{“}H^T_{*,\mathrm{ord}}(\mathcal{M}_C(\lambda,\mu))\text{”}$$

more precisely hyperbolic restriction of IC

more precisely costalk of IC.

injective by localization thm.

<u>Conjecture (Muthiah–N)</u> (based on a similar statement for finite dim'l setting by Ginzburg–Riche)

$$\begin{array}{ccc} H^T_{*,\mathrm{ord}}(\mathcal{M}_C(\lambda,\mu)) & \overset{\exists}{\cong} & (\mathcal{V}(\lambda)\otimes \mathbb{M}(\mu))^{B^\vee_{\mathrm{aff}}} \\ \downarrow & \circlearrowleft & \downarrow \\ H^T_*(\mathcal{M}_C(\lambda,\mu)^{P\leq 0}) & \cong & \mathcal{V}_\mu(\lambda)\otimes_{\mathbb{C}} H^*_T(pt) \end{array}$$

coset VOA

$B^\vee_{\mathrm{aff}}$ = Borel of the $(K_{\mathrm{aff}})^\vee$

$\mathbb{M}(\mu)$ = universal asymptotic Verma module

$(H^*_T(pt), U_\hbar(K^\vee_{\mathrm{aff}}))$-bimodule s.t. $v_\mu \cdot a = (a+\hbar(\mu+\check{\rho})(a))v_\mu$

$\hbar = \varepsilon_1+\varepsilon_2$

$XY-YX = \hbar[X,Y]$

Idea of proof) Check $\downarrow \longrightarrow$ extends across all “root hyperplanes” in $\operatorname{Spec} H^*_T(pt) = \mathfrak{t}\times\mathbb{C}^2$

$\longrightarrow$ reduction to $\mathfrak{sl}_2$ (finite) and Heis.