Towards Geometric Satake Correspondence for Kac-Moody Lie Algebras

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企画特別講演

Part I. Coulomb branch

References

N — 1503.03676

Braverman – Finkelberg – N — 1601.03586

Braverman – Dhillon – Finkelberg – Raskin – Travkin — 2201.09475

$(\mathbb{G}, \mathbb{M}) \overset{\text{physics}}{\rightsquigarrow}$ 3d $\mathcal{N}=4$ SUSY gauge theory $\rightsquigarrow$ Higgs branch $\mathcal{M}_H$

$\rightsquigarrow$ Coulomb branch $\mathcal{M}_C$ $= \mathcal{M}_C(\mathbb{G}, \mathbb{M})$

$\mathbb{G}$: cpx reductive grp (cpt Lie grp)

$\mathbb{M}$: symplectic representation (quaternionic rep.)

hyperKähler manifolds possibly with singularities

- $\mathcal{M}_H = \mathbb{M} /\!/\!/ \mathbb{G}$ (hyperKähler quotient) symplectic quotient e.g. quiver variety

- $\mathcal{M}_C$: more difficult to define mathematically

Define $\mathcal{M}_C$ as a cpx symplectic variety.

[BFN] Assume $\mathbf{M} = \mathbf{N} \oplus \mathbf{N}^*$ cotangent type

cpx repr. of $\mathbb{G}$

$\mathrm{Gr}_{\mathbb{G}}$: affine Grassmannian $\mathcal{K} = \mathbb{C}((z))$, $\mathcal{O} = \mathbb{C}[[z]]$

$= \mathbb{G}(\mathcal{K}) / \mathbb{G}(\mathcal{O})$ (cf. G/B = flag variety)

$\mathcal{T} = \mathbb{G}(\mathcal{K}) \times^{\mathbb{G}(\mathcal{O})} \mathbf{N}(\mathcal{O})$ ∞-rank vector b'dle over ∞-dim'l homog. space

$\pi \downarrow$ $[g(z), s(z)]$

$\mathbf{N}(\mathcal{K}) \ni g(z)s(z)$

$\mathbb{G}(\mathcal{K}) \curvearrowright \mathcal{T} \times_{\mathbf{N}(\mathcal{K})} \mathcal{T}$ fiber product

$H_*^{\mathbb{G}(\mathcal{K})}(\mathcal{T} \times_{\mathbf{N}(\mathcal{K})} \mathcal{T})$ Borel-Moore homology

equiv.

has convolution product. $*$

$$\mathcal{T} \times_{\mathbf{N}(\mathcal{K})} \mathcal{T} \times_{\mathbf{N}(\mathcal{K})} \mathcal{T} \quad \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \quad \mathcal{T} \times_{\mathbf{N}(\mathcal{K})} \mathcal{T} \qquad p_{12}, p_{23}, p_{13} \quad \text{etc}$$

$$c * c' \overset{\text{def.}}{=} p_{13*}(p_{12}^{*} c \cap p_{23}^{*} c')$$

(**Rem** $\exists$ technical issue

replace $H_*^{G(\mathcal{K})}(\mathcal{T} \times_{\mathbf{N}(\mathcal{K})} \mathcal{T})$ by $H_*^{G(\mathcal{O})}(\mathcal{R})$

$\mathcal{R} = \pi^{-1}(\mathbf{N}(\mathcal{O}))$)

Th [BFN]

$\left(H_*^{G(\mathcal{O})}(\mathcal{R}), \text{ convolution product} \right)$ is a <u>commutative</u> ring.

Def. $\mathcal{M}_C \overset{\text{def.}}{=} \operatorname{Spec}(\uparrow)$ Coulomb branch

✲ deformation quantization $\mathbb{C}^\times_{\text{loop}} \curvearrowright \mathcal{O} = \mathbb{C}[[z]] \quad z \mapsto tz$

$\mathcal{K}, G(\mathcal{K})$ etc

$H_*^{G(\mathcal{O}) \rtimes \mathbb{C}^\times_{\text{loop}}}(\mathcal{R})$: non commutative

$\{a, b\} = [\tilde{a}, \tilde{b}]/\hbar \big|_{\hbar=0} \rightsquigarrow$ symplectic form on $\mathcal{M}_C^{\text{reg.}}$

☆ Hamiltonian torus action

$$\pi_0(\mathcal{R}) = \pi_0(\mathrm{Gr}_{G}) \cong \pi_1(G) \rightsquigarrow H_*^{G(\mathcal{O})}(\mathcal{R}) : \pi_1(G)\text{-graded}$$

$$\rightsquigarrow \pi_1(G)^\wedge \curvearrowright \mathcal{M}_C$$

<u>Part II</u> Geometric Satake for Kac-Moody

Reference BFN 1604.03625 ← conjecture

N 1810.04293 ← proof for affine type A

quiver gauge theory

$Q = (Q_0, Q_1)$ quiver $\cancel{\circ\!\circlearrowright}$

(cf. N-Weekes non-symm. Coulomb branch

$\mathfrak{g} \equiv \mathfrak{g}_Q$: Kac-Moody Lie alg (symmetric)

$$\text{Cartan matrix} = 2(\delta_{ij}) - \left(\begin{array}{c}\text{adj matrix}\\ \text{of } Q\end{array}\right) - \left(\,〃\, \right)^t$$

$$V = \bigoplus V_i \qquad W = \bigoplus_{i \in Q_0} W_i$$

Q_0-graded cpx fd. vector spaces

$G = \prod GL(V_i)$, $N = \bigoplus_{h\in Q_1} \mathrm{Hom}(V_{o(h)}, V_{i(h)}) \oplus \bigoplus_{i\in Q_0} \mathrm{Hom}(W_i, V_i)$

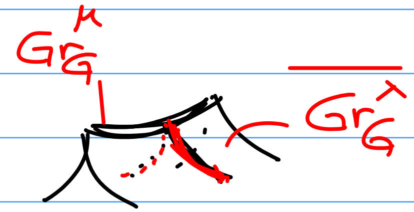
rep. of G by conjugation

$M = N \oplus N^*$ $\qquad \mathcal{M}_H$ = quiver variety

$\lambda \overset{\text{def.}}{=} \sum \dim W_i \cdot \varpi_i$, $\quad \mu \overset{\text{def.}}{=} \lambda - \sum \dim V_i \cdot \alpha_i$

fund coweight of $\mathfrak{g}_Q$ $\qquad$ simple coroot of $\mathfrak{g}_Q$

$\mathcal{M}_C \equiv \mathcal{M}_C(\lambda, \mu)$: Coulomb branch



<u>Th</u> [BFN, NW]

Suppose Q: finite type (ADE, BCFG)

G = adjoint grp of type Q

$\Longrightarrow$ $\mathcal{M}_C$ = generalized slice to Gr_G^μ in $\overline{Gr_G^\lambda}$

$\mu \leq \lambda$

$Gr_G \supset Gr_G^\lambda$

$\parallel$

$G(\mathcal{O})$-orbit through z^λ

Hamiltonian group action $\quad T = \pi_1(G)^\wedge$

$$= \prod_i \underbrace{\pi_1(GL(V_i))}_{\cong\, \mathbb{Z}}{}^\wedge$$

identified with max torus of G

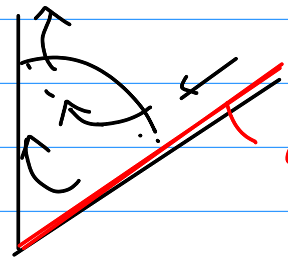
$\rightsquigarrow$ geometric Satake $\quad$ (intersection) cohomology of $\mathcal{M}_C$ (attracting set)

$\quad\quad$ realize

$V_\mu(\lambda)$ $\quad\quad$ $V(\lambda)$: irr. h.w. integrable rep. of $\mathfrak{g}^\vee$ with h.w. $= \lambda$

e.g. $\quad G = PGL_2 \quad , \lambda = 2\varpi_1 \quad , \mu = 0$

$\mathcal{M}_C = \mathbb{C}^2/\pm 1$



attracting set

$\cup$

$V_{\mu = 0}(\lambda = 2\varpi_1)$ $\quad\quad$ $\mathbb{C}^3 \circlearrowleft \mathfrak{sl}_2$

☆ Gr_G does <u>not</u> make sense for G = Kac-Moody group

$\uparrow$
∞-dim'l
unless Q: finite type

$\mathcal{M}_C$ does make sense for any Q.

$\rightsquigarrow$ geometric Satake for Kac-Moody

<u>Conj</u> (<u>Th</u> for affine type A)
or finite type

(0) $\mathcal{M}_C(\lambda,\mu)^T = pt$ or ϕ

Assume $= pt$

Take $\chi = \prod \det : G \longrightarrow \mathbb{C}^\times$

$\rightsquigarrow \underset{\underset{\chi}{\parallel}}{\pi_1(\chi)}^\wedge : \mathbb{C}^\times \longrightarrow T$

$$\mathcal{A}_\chi(\lambda,\mu) \overset{\text{def.}}{=} \{ x \in \mathcal{M}_C(\lambda,\mu) \mid \lim_{t\to 0} \chi(t)\cdot x \}$$

(1) $\uparrow$ lagrangian subvar. in $\mathcal{M}_C$

(2) $$\bigoplus_\mu H_{top}(\mathcal{A}_\chi(\lambda,\mu)) \overset{?}{\cong} V(\lambda)$$

$\uparrow$ has a structure of representation of $\mathfrak{g}^\vee$