The Hikita conjecture for quiver varieties and Coulomb branches

On the proof in the OpenAI preprint (2026)

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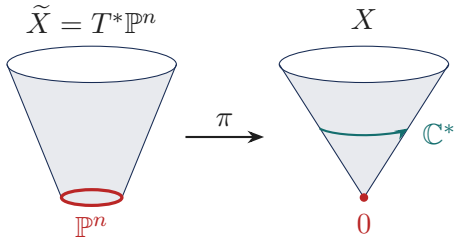
November 10, 2026

These slides were prepared by Claude (Anthropic) from the speaker's handwritten notes.

1. **The Hikita conjecture**
2. Examples of symplectic dual pairs
3. Higgs and Coulomb branches
4. The proof for quiver gauge theories

A first example

- $\widetilde{X} = T^*\mathbb{P}^n \xrightarrow{\pi} X$, collapsing $\mathbb{P}^n$ to a point.
- Indeed, $X = \overline{\mathbb{O}}_{\min}(\mathfrak{gl}_{n+1})$, the minimal nilpotent orbit closure.
- X is a conical symplectic singularity ($\mathbb{C}^*$-action contracting X to 0), and $\widetilde{X}$ is a symplectic resolution.



- $X^\vee = \{\, xy = z^{n+1} \,\} \subset \mathbb{C}^3$, the simple singularity of type A_n:

$$X^\vee = \mathbb{C}^2/(\mathbb{Z}/(n+1)), \qquad (z_1, z_2) \mapsto (\gamma z_1, \gamma^{-1} z_2), \quad \gamma^{n+1} = 1,$$
$$x = z_1^{n+1}, \quad y = z_2^{n+1}, \quad z = z_1 z_2.$$

Cohomology vs. scheme-theoretic fixed points

- Cohomology ring of $\widetilde{X}$ (H: hyperplane bundle):

$$H^*(\widetilde{X}) = H^*(\mathbb{P}^n) = \mathbb{C}[z]/(z^{n+1}), \qquad z = c_1(H) = \mathrm{PD}[\mathbb{P}^{n-1}].$$

- Fixed points of $\mathbb{C}^* \curvearrowright X^\vee$, $(x, y, z) \mapsto (tx, t^{-1}y, z)$: $x = 0,\ y = 0 \Rightarrow z = 0$, a single point.
- But we take the fixed points in the scheme-theoretic sense:

$$\begin{aligned}\mathbb{C}[(X^\vee)^{\mathbb{C}^*}] &= \mathbb{C}[X^\vee]/\big(\text{ideal generated by functions of nonzero weight}\big) \\ &= \mathbb{C}[x, y, z]/\big(xy = z^{n+1},\ x = 0,\ y = 0\big) \ \cong\ \mathbb{C}[z]/(z^{n+1}).\end{aligned}$$

Conclusion

$$H^*(\widetilde{X}) \ \cong\ \mathbb{C}[(X^\vee)^{\mathbb{C}^*}] \quad \text{as rings.}$$

The Hikita conjecture

Conjecture (Hikita 2017)

For a symplectic dual pair $(X, X^\vee)$ of conical symplectic affine varieties,

$$H^*(\widetilde{X}) \;\cong\; \mathbb{C}[(X^\vee)^T],$$

where $\widetilde{X}$ is a symplectic resolution of X, and T is (a maximal torus of) the group of Hamiltonian automorphisms of $X^\vee$.

- This is one instance of symplectic duality (3d mirror symmetry), a mysterious relation between X and $X^\vee$ formulated by Braden–Licata–Proudfoot–Webster [BLPW].

Remark

Expectation: $(X, X^\vee)$ is a symplectic dual pair $\iff$ $(X^\vee, X)$ is a symplectic dual pair.

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Nilpotent orbits in $\mathfrak{gl}_n$

[Kraft–Procesi]

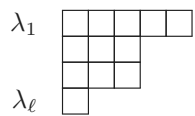
- Conjugacy classes of $n \times n$ nilpotent matrices are classified by Jordan blocks, i.e. by partitions λ of n:

$$\mathbb{O}(\lambda) \ \ni \ J_{\lambda_1} \oplus J_{\lambda_2} \oplus \cdots \oplus J_{\lambda_\ell}.$$

- They are ordered by the closure relation

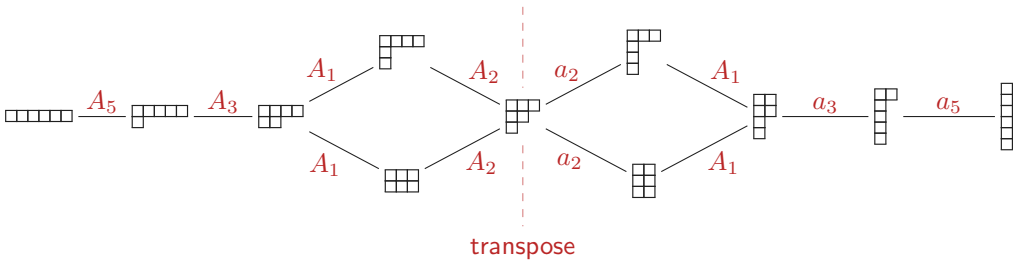
$$\lambda \geq \mu \iff \overline{\mathbb{O}(\lambda)} \supset \mathbb{O}(\mu).$$

$$J_k = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix} \quad (k \times k)$$



Example: $\mathfrak{gl}_6$

[Kraft–Procesi 1982, p. 597]



- A_n = singularity of $xy = z^{n+1}$
- a_n = singularity of $\overline{\mathbb{O}}_{\min}(\mathfrak{gl}_{n+1})$

A_n and a_n are symplectic dual (first example); $A_1 = a_1$.

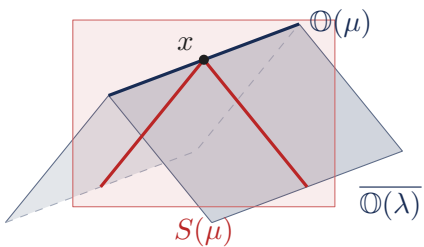
- Edge labels: singularity of $\overline{\mathbb{O}(\lambda)}$ along $\mathbb{O}(\mu)$.
- Transpose = reflection in the dashed line; exchanges $A_n \leftrightarrow a_n$.

Transversal slices

Look at the singularity of $\overline{\mathbb{O}(\lambda)}$ at $x \in \mathbb{O}(\mu)$, for $\lambda \geq \mu$, via a transversal slice $S(\mu)$ through x.

- When $\lambda > \mu$ are adjacent, the singularity type is either A_n or a_n.
- $A_n \leftrightarrow a_n$ are exchanged by

$$(\lambda, \mu) \longleftrightarrow (\mu^t, \lambda^t).$$



More general pairs

One can take more general pairs (λ, μ), not necessarily adjacent:

$$X = \overline{\mathbb{O}(\lambda)} \cap S(\mu) \quad \longleftrightarrow \quad X^\vee = \overline{\mathbb{O}(\mu^t)} \cap S(\lambda^t) \qquad (\text{note } \mu^t \geq \lambda^t).$$

Theorem (Hikita)

The conjecture holds for these pairs $(X, X^\vee)$.

★ There is no apparent geometric meaning of the transpose.

Remark

One can look for a similar symmetry for other complex simple Lie algebras: the transpose fails $\rightsquigarrow$ special orbits, Barbasch–Vogan–Lusztig–Spaltenstein duality.
But the naive statement is false.

Toric hyperkähler manifolds

- An exact sequence of tori $1 \to T \to (\mathbb{C}^*)^N \to T' \to 1$,
  $(\mathbb{C}^*)^N \curvearrowright M = T^*\mathbb{C}^N \ni (z_1, w_1), \dots, (z_N, w_N)$.
- Moment maps:

$$\tilde{\mu}\colon M \to \mathbb{C}^N = \big(\mathrm{Lie}(\mathbb{C}^*)^N\big)^*, \quad \text{given by } (z_i w_i)_{i=1}^N,$$
$$\mu\colon M \xrightarrow{\tilde{\mu}} \big(\mathrm{Lie}(\mathbb{C}^*)^N\big)^* \xrightarrow{\text{natural proj.}} \mathfrak{t}^*.$$

- Quotients:

$$\widetilde{X} = \mu^{-1}(0)^{\zeta\text{-stable}}/T \;=:\; M /\!\!/\!\!/_\zeta\, T, \qquad \zeta\colon T \to \mathbb{C}^* \text{ generic character},$$
$$X = \mu^{-1}(0)^{\text{closed orbits}}/T \;=:\; M /\!\!/\!\!/_0\, T, \qquad \text{categorical quotient}.$$

- Example: $T^*\mathbb{P}^n = T^*\mathbb{C}^{n+1} /\!\!/\!\!/_\zeta\, \mathbb{C}^*$.

Toric hyperkähler manifolds: the dual side

- Note that X and $\widetilde{X}$ have an action of T'.
- Dualize the exact sequence:

$$1 \to (T')^\vee \to (\mathbb{C}^*)^N \to T^\vee \to 1, \qquad X^\vee = M /\!\!/\!\!/ (T')^\vee \curvearrowleft T^\vee.$$

Theorem (Hikita)

The conjecture is true for $(X, X^\vee)$.

- This example explains why resolution and torus fixed points are exchanged:

$$\underbrace{\zeta \in \mathrm{Hom}(T, \mathbb{C}^*)}_{\text{resolution of } X} = \underbrace{\mathrm{Hom}(\mathbb{C}^*, T^\vee)}_{\text{cocharacter in } T^\vee}.$$

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Next examples: Higgs branches

So far

The proofs are computational: compute both sides and check that they match.

- G: complex reductive group, e.g. GL_n.
- M: symplectic representation of G, here $M = \mathbf{N} \oplus \mathbf{N}^*$.

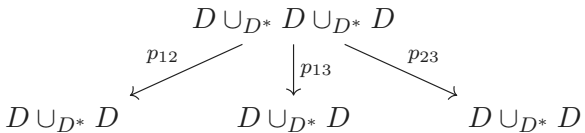
$$\widetilde{X} = M /\!\!/\!\!/_\zeta G, \quad \zeta \colon G \to \mathbb{C}^* \text{ character}; \qquad X = M /\!\!/\!\!/_0 G.$$

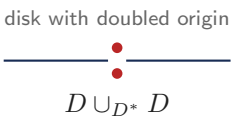
- In some cases (e.g. M comes from a framed quiver), $\widetilde{X}$ is a symplectic resolution of X.
- Then $\widetilde{X}$ is a quiver variety.

Coulomb branches

[Braverman–Finkelberg–Nakajima]

- $M = T^*\mathbf{N}$, $G \curvearrowright \mathbf{N}$. Consider $H_*(\mathrm{Maps})$, where $\mathrm{Maps} := \mathrm{Maps}\big(D \cup_{D^*} D, [\mathbf{N}/G]\big)$.





- $p_{13*}\big(p_{12}^*(\ ) \cap p_{23}^*(\ )\big)$ gives the convolution product $*$.

Coulomb branch

$$X^\vee := \mathrm{Spec}\big(H_*(\mathrm{Maps}), *\big).$$

Coulomb branches vs. the Hikita conjecture

The construction has a similar flavor to the Hikita conjecture:

Hikita conjecture

$$(X^\vee)^T = \mathrm{Spec}\big(H^*(\widetilde{X}),\ \cup\big)$$

cohomology, cup product

Coulomb branch

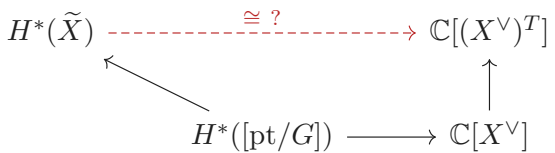
$$X^\vee = \mathrm{Spec}\big(H_*(\mathrm{Maps}),\ *\big)$$

homology, convolution product

- But the cup product and the convolution product are different.
- One additional ingredient: $T = \pi_1(G)^\vee$ acts on $X^\vee$ by the construction (explanation omitted).

The refined Hikita conjecture

Both rings receive a homomorphism from $H^*([\mathrm{pt}/G])$:



$$\widetilde{X} = \mu^{-1}(0)^{\zeta\text{-ss}}/G.$$

Refined Hikita conjecture

$H^*(\widetilde{X}) \cong \mathbb{C}[(X^\vee)^T]$, compatibly with the homomorphisms from $H^*([\mathrm{pt}/G])$.

- For quiver gauge theories, $H^*([\mathrm{pt}/G]) \to H^*(\widetilde{X})$ and $H^*([\mathrm{pt}/G]) \to \mathbb{C}[(X^\vee)^T]$ are surjective: McGerty–Nevins (Kirwan surjectivity) and Weekes.

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The theorem

Theorem (OpenAI 2026)

Let Q be any finite quiver (loops and multiple arrows allowed), v, w dimension vectors, and ζ a generic stability parameter. Then

$$H^*(\widetilde{X}) \;\cong\; \mathbb{C}[(X^\vee)^T]$$

compatibly with the homomorphisms from $H^*([\mathrm{pt}/G])$, i.e. the refined Hikita conjecture holds.

- Both maps from $H^*([\mathrm{pt}/G])$ are surjective, so the statement means: they have the same kernel.

The comparison map

[Botta–Tamagni 2026]

Theorem (Botta–Tamagni)

There is a natural map $\Phi\colon \mathbb{C}[(X^\vee)^T] \longrightarrow H^*(\widetilde{X})$.

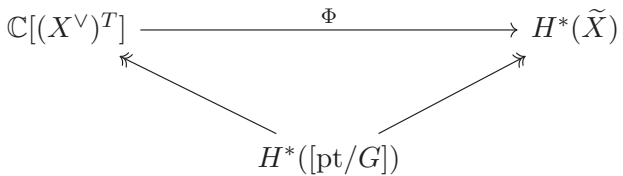
Idea of [BT]:

$$\underset{\text{(whole Coulomb branch)}}{\mathbb{C}[X^\vee]} \quad \curvearrowright \quad H^*\big(\mathrm{QMap}(\mathbb{P}^1, \widetilde{X})\big)$$

$$\cup$$

$$H^*(\widetilde{X}) \qquad \text{degree } 0 \text{ subspace, preserved by } \mathbb{C}[(X^\vee)^T]$$

Reduction to a dimension estimate



- Φ is surjective by [McGerty–Nevins, Weekes].

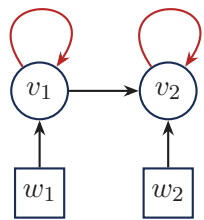
Enough to show

$$\dim \mathbb{C}[(X^\vee)^T] \ \leq \ \dim H^*(\widetilde{X}).$$

Finally: the OpenAI argument

(1) Compute both dimensions $\dim \mathbb{C}[(X^\vee)^T]$ and $\dim H^*(\widetilde{X})$ for quiver gauge theories with an edge loop at every vertex.

(2) Reduction to the edge loop case.



- (2) is standard.
- The most mysterious part is (1), but this part of the proof is correct, as far as I checked.

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