PBW / canonical bases for quantum affine algebras (review)

HIRAKU NAKAJIMA (KYOTO)

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Quiver and Representations
Osaka City Univ.

$Q = (I, \Omega)$: quiver $\quad k = \mathbb{F}_{p^r}$: finite field

$\mathcal{H}$ = (twisted) Ringel-Hall algebra

$\mathbb{Q}(q)$-algebra $\quad q = \sqrt{\# k}$

$\{ u_c \mid c \text{ ; iso. class of rep. of } Q \}$: base

$$u_a \cdot u_b \overset{\text{def}}{=} \sqrt{\# k}^{\langle \dim a, \dim b \rangle} \sum_c g_{ab}^c u_c$$

where $g_{ab}^c = \# \left\{ D \;\middle|\; \begin{array}{l} D \subset C \\ D \cong B,\ C/D \cong A \end{array} \right\}$

$A \cdots a$
$B \cdots b$ $\quad$ representative
$C \cdots c$

$\underline{\mathcal{C}} = \langle u_i \rangle \subset \mathcal{H}$ $\quad i$: simple repr. corr. to $i \in I$

<u>composition</u> <u>subalgebra</u>

<u>Th</u>. (Ringel, Lusztig)

$$\underset{\substack{\cup \\ u_i}}{\mathcal{C}} \cong \underset{\substack{\cup \\ E_i}}{U_q^+} \qquad (q = \sqrt{\# k})$$

$u_i \longleftrightarrow E_i$

(upper half of the quantum env. algebra

$$v_c = \sqrt{\# k}^{\,-\dim C + \dim \mathrm{End}_Q(C)} u_c.$$

(normalised)

<u>Lusztig</u> $\qquad V = \bigoplus_i V_i$

$$\mathbb{E}_\Omega(V) = \bigoplus_{h \in \Omega} \mathrm{Hom}(V_{o(h)}, V_{i(h)})$$

$$G_V = \prod_{i \in I} GL(V_i) \curvearrowright \mathbb{E}_\Omega(V)$$

G_V-equiv. perverse sheaves on $\mathbb{E}_\Omega(V)$

$\rightsquigarrow$ canonical base of U_q^+

= Kashiwara's global crystal base

Q : finite type ($\overset{\Leftrightarrow}{\text{def.}}$ underlying graph = Dynkin diagram of type ADE)

- finitely many indec. $\longleftrightarrow \Delta^+$: positive roots

- $\mathcal{C} \cong \mathcal{H}$

- $\{\text{canonical base}\} \cong \{ IC(\overset{\overset{b_c}{\parallel}}{\mathcal{O}_c}) \}$

$\mathcal{O}_c$ ⋯ G_V-orbit corr. to c

$IC(\mathcal{O}_c)$ ⋯ intersection cohomology complex

$b_c \in U_{\mathbb{Z}}^+$

$$\begin{cases} b_c = v_c + \sum_{c<d} a_{cd} v_d & a_{cd} \in \zeta^{-1}\mathbb{Z}[\zeta^{-1}] \\ b_c = \overline{b_c} \end{cases}$$

These properties give us an $\overset{\text{recursive}}{\overbrace{\underline{\text{algorithm}}}}$ to compute $\{b_c\}$ from $\{v_c\}$

$\mathcal{U}_c$ is (an example of) PBW base

w_0 = longest element of Weyl group

$= s_{i_N} s_{i_{N-1}} \cdots s_{i_1}$: reduced expr.

$N = \# \Delta^+$

$\beta_N = \alpha_{i_N} < \beta_{N-1} = s_{i_N}(\alpha_{i_{N-1}}) < \cdots$

$\cdots < \beta_1 = s_{i_N} \cdots s_{i_2}(\alpha_{i_1})$

(total ordering on Δ^+)

$c \cdots\cdots \underset{\{1,\dots,N\}}{\underset{\wr\wr}{\Delta^+}} \to \mathbb{N}$ (multiplicity func.)

$E_{\beta_s} := T_{i_N}^{-1} T_{i_{N-1}}^{-1} \cdots T_{i_{s+1}}^{-1}(E_{i_s})$

root vector

T_i : Lusztig's braid group operator

$\mathcal{U}_q \circlearrowleft$ $\quad$ q-analogue of s_i

$L_c = E_{\beta_N}^{(c(N))} \cdots\cdots E_{\beta_1}^{(c(1))}$

$E_{\beta_i}^{(n)} = \dfrac{1}{[n]_q!} E_{\beta_i}^{n}$

◎ depending on reduced expr

Th (Lusztig, Beck-N) ····· purely algebraic proof

$$b_c = L_c + \sum_{c<d}' a_{cd} L_d$$

$a_{cd} \in q^{-1}\mathbb{Z}[q^{-1}]$ ··· positive in geom. case

$<$ ··· lexicographic order

$(\overline{b_c} = b_c)$

reduced expr. of w_0 is <u>adapted</u> to Q

$\Longleftrightarrow$ i_N is sink of Q

i_{N-1} " of $s_{i_N} Q$

$\vdots$

Example

Coxeter element : (used later)

i_0 : sink of Q

i_{-1} : " $s_{i_0} Q$ $\Rightarrow$ ordering on I

$\vdots$

i_{-n+1} $\qquad \# I = n$

$$\underbrace{s_{i_{-n+1}} \cdots s_{i_0}}_{\text{"coxeter element}} Q = Q$$

$$w_0 = (\text{Coxeter element})^{h/2}$$

h: Coxeter #

Rem A reduced expr. may <u>not</u> be adapted to any orientation Ω

Q : affine ($\underset{\text{def.}}{\Leftrightarrow}$ underlying graph is of affine Dynkin type)

Lusztig ⋯⋯ parametrization of canonical base elements $\{b_c\}$ using rep. th. of Q

Beck-Chari-Pressley
Beck-N
⋯⋯ purely algebraic constr. of $\{L_c\}$: PBW base

Lin-Xiao-Zhang
⋯⋯ "geometric" constr. of $\{L'_c\}$

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reduced expr. of w_0

$\underset{\text{affine}}{\Longrightarrow}$ (doubly) infinite sequence

$$\cdots\cdots i_{-1}, i_0, i_1, i_2, \cdots$$

s.t. $s_{i_m} \cdots s_{i_p}$ is reduced for any $m < p$

[BCP, BN] ⋯⋯ $\sum_{i \neq 0} \varpi_i = \rho$ sum of fund. wts

[LXZ] ⋯ Coxeter element

periodic

$$\beta_k = \begin{cases} s_{i_0} s_{i_{-1}} \cdots s_{i_{k+1}} (\alpha_{i_k}) & k \leqq 0 \\ s_{i_1} s_{i_2} \cdots (\alpha_{i_k}) & k > 0 \end{cases}$$

[BCP, BN]

$\{\beta_k\} = \Delta_+^{\text{real}}$

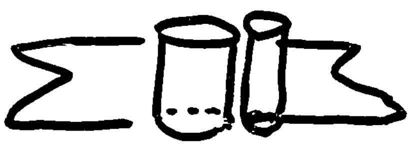
missing imaginary roots $n\delta$

$n \in \mathbb{N} \setminus 0$

[LXZ]

$\{\beta_k\} =$ irregular roots $\subset \Delta_+^{\text{real}}$

missing regular roots



reg.

.... imaginary roots + regular real roots

Def. α : regular root

$\iff$ $c^N \alpha = \alpha$ for some N

NB. $c\delta = \delta$

Problem.

Construct "root vectors" for missing roots

[BCP, BN]

root vector E_α for real roots α

.... as in finite type case

imaginary root vector $\leftarrow$ q-commutator

$i \in I_0 = I \setminus \{0\}$

- $\tilde{\psi}_{i,k} := E_{k\delta-\alpha_i} E_{\alpha_i} - q^{-2} E_{\alpha_i} E_{k\delta-\alpha_i}$

- $\tilde{P}_{i,k} = \dfrac{1}{[k]} \displaystyle\sum_{s=1}^{k} q^{s-k} \tilde{\psi}_{i,s} \tilde{P}_{i,k-s}$

λ: partition

$$S_{i,\lambda} = \det(\tilde{P}_{i,\lambda_k - k + m})_{k,m}$$

Schur func. for $\tilde{P}_i$

$$L_c = L_{c_+} S_{c_0} L_{c_-}$$

$c_+ : \{0, -1, -2, \cdots\} \to \mathbb{N}$ (β_k $k \leq 0$)

$c_- : \{1, 2, 3, \cdots\} \to \mathbb{N}$ (β_k $k > 0$)

$c_0 = (\lambda_1, \cdots \lambda_n)$: n-tuple of partitions

$n = \# I$

$L_{C_\pm}$: as in finite type

$$S_{c_0} = S_{1,\lambda_1} \cdots S_{n,\lambda_n}$$

product of Schur func.

Th [BN]

$$\cdot\, b_c = L_c + \sum_{c<d} a_{cd} L_d \qquad a_{cd} \in q^{-1}\mathbb{Z}[q^{-1}]$$

lex. order on $C_\pm$
(no ordering on C_0)

This gives a parametrization of the canonical base elements

Open Problem

Describe Kashiwara operators $\tilde{e}_i, \tilde{f}_i$ in this parametrization

cf. Finite type
piecewise linear combinatorics
Berenstein - Zelvinsky

Kronecker quiver

$\longrightarrow$ no real regular roots !!!

$0 \rightrightarrows 1$

$k = \overline{k}$: alg. closed

- $k^n \rightrightarrows k^{n+1}$ or $k^{n+1} \rightrightarrows k^n$ $\quad (n \in \mathbb{Z}_{\geq 0})$

$\begin{bmatrix} 1_n \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1_n \end{bmatrix}$ $\qquad [1_n\ 0], [0\ 1_n]$

$\Longrightarrow$ unique indecomposables

preprojective	preinjective	
$n\delta + \alpha_1$	$(n+1)\delta - \alpha_1$	real roots

- $k^n \rightrightarrows k^n$ $\qquad \lambda 1_n + J_n,\ 1_n$ or $1_n,\ J_n$ $\qquad J_n$: Jordan matrix of size n

$(\lambda \in k)$

$\Longrightarrow$ parametrised by $\mathbb{P}^1(k)$

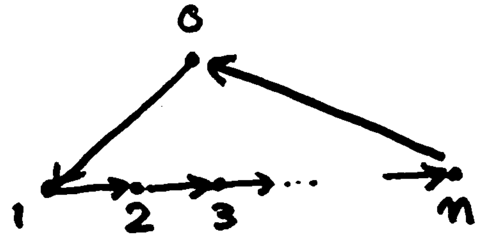
regular

$n\delta$: imaginary root

Th. (McGerty) based on Zhang

$$\widetilde{P}_n = (\# k)^{-n} \sum_{\substack{C:\text{ regular} \\ \text{rep.} \\ \text{of } \dim = (n,n)}} u_C$$

Cyclic quiver



$\mathcal{N}(Q)$ = category of nilpotent representations

indec. rep. $= \{ M(i,p) \mid i \in I,\ p > 0 \}$

fold by mod n

$$\underset{i}{k} \leftarrow \underset{i-1}{k} \leftarrow \cdots \leftarrow \underset{i-p+1}{k}$$

$c = (c(i,p))$

$$M_c = \bigoplus_{i,p} M(i,p)^{\oplus c(i,p)} \quad \text{is } \underline{\text{aperiodic}}$$

$\overset{\text{def.}}{\Longleftrightarrow}$ $\forall p \ \exists i$ s.t. $c(i,p) = 0$

Th (Lusztig)

$\{ IC(\mathcal{O}_c) \mid c : \text{aperiodic} \}$ is the canonical base of U_q^+.

Q. $\{ v_c \mid c : \text{aperiodic} \}$ base of U_q^+ ?

A. No. $v_c \notin \mathcal{C}$ in general

Deng - Du - Xiao

$\exists \{E_c\}$: base of U_q^+ s.t.

- $E_c = v_c + \sum_{d} a_{cd} v_d$

periodic

- $b_c = E_c + \sum_{c<d} c_{cd} E_d$

aperiodic $\qquad c_{cd} \in q^{-1}\mathbb{Z}[q^{-1}]$

$(\overline{b_c} = b_c)$

v_c, E_c is computable in principle.

$\Rightarrow b_c$ is computable

NB. The situation is simpler for the whole $\mathcal{H}$

$$b_c = v_c + \sum_{d} e_{cd} v_d$$

as in finite type case

... Ariki. Varagnoro-Vasserot Schiffmann ......

$c \longleftrightarrow$ Young tableau (Fock space) Kashiwara op. can be described !!

Q : affine (not cyclic $A_n^{(1)}$)

⊛ Review of Representation Th. of Q

- irregular roots α

  $\longleftrightarrow$ unique indec. repr M_α

  s.t. $\dim M_\alpha = \alpha$

- regular real roots (finitely many)

  p : period $\quad \underbrace{\alpha, c\alpha, \cdots, c^{p-1}\alpha}_{\text{disjoint}}, \ c^p\alpha = \alpha$

  "tube"

  $\exists_1$ regular simple M_α

$$\begin{tikzcd} & M_\alpha \arrow[dl] & \\ M_{c\alpha} \arrow[r] & \cdots \arrow[r] & M_{c^{p-1}\alpha} \arrow[ul] \end{tikzcd}$$

$$\text{Rep Tube} \cong \text{Rep Nil } A_{(p-1)}^{(1)}$$

- imaginary roots $n\delta$

  For each n, indec. rep M with $\dim M = n\delta$

  are parametrised by

  $\mathbb{P}^1(k) \setminus \{\text{finite pts}\}$

  For each $x \in \mathbb{P}^1(k) \setminus \{\ \}$,

  $\text{Rep tube} \cong \text{Rep Nil } \circlearrowleft$ (Jordan quiver

Lusztig parametrization

$c = (c_+, c_-, c_{nh}, \lambda)$ — λ: partition

$c_\pm : \{0, -1, \cdots\} \to \mathbb{N}$
$\{1, 2, \cdots\} \to \mathbb{N}$ — as before

irr. root multiplicity

$c_{nh} = (c_{nh}^1, \cdots, c_{nh}^s)$ — $s = \#$ of nonhom. tubes

$c_{nh}^k : \{ M(i, g) \mid \begin{smallmatrix} g \geq 1 \\ i = 0, 1, \cdots, p-1 \end{smallmatrix} \} \to \mathbb{N}$

p = period of tube T_k

$$X_c = \{ M_{c_+} \oplus M_{c_-} \oplus M_{nh} \oplus R_1 \oplus \cdots \oplus R_{|\lambda|} \mid R_i : \text{regular}, \dim R_i = \delta \text{ from diff. tube} \}$$

$\lambda \leftrightarrow$ repr. of $S_{|\lambda|}$ (symmetric group)
defines a local system on X_c

$\{\text{canonical base}\} = \{ IC(X_c, \lambda) \mid c, \lambda \}$

[LXZ]

$$L_c = L_{c_+} E_{c_{nh}} S_\lambda L_{c_-}$$

$L_{c_\pm}$: as before

$E_{c_{nh}}$: as in cyclic orientation

S_λ : based on Kronecker quiver

$\exists\ F : \mathrm{Rep}\,K \longrightarrow \mathrm{Rep}\,Q$

fully faithful
exact functor

$$\rightsquigarrow \begin{array}{ccc} \mathcal{C}(K) & \longrightarrow & \mathcal{C}(Q) \\ \Psi & & \Psi \\ S_\lambda & \longmapsto & S_\lambda \end{array}$$

Th ① $\{L_c\}$: $\mathbb{Q}[\xi,\xi^{-1}]$ basis of $U_{\mathbb{Q}}^+$

② $\overline{L_c} = L_c + \sum a_{cd} L_d$

Cor. $\exists 1\ \{b_c'\}$ s.t. $\overline{b_c'} = b_c'$

$$b_c' = L_c + \sum_{c<d} a_{cd} L_d, \quad a_{cd} \in \xi^{-1}\mathbb{Q}[\xi^{-1}]$$

Question

b_c' coincides with canonical base ?