CHERKIS BOW VARIETIES
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HIRAKU NAKAJIMA

Cherkis bow varieties were originally introduced by Cherkis as the ADHM type description of moduli spaces of $U(n)$-instantons on multi-Taub-NUT spaces. In a joint work with Takayama [NT17], the author gave an alternative definition as symplectic reduction of certain affine algebraic varieties, which are products of $\operatorname{Hom}(V_1, V_2) \times \operatorname{Hom}(V_2, V_1)$ and $\operatorname{GL}(V) \times$ (slice to a nilpotent orbit). The former is called a 'two-way' part, and the latter a 'triangle' part. Therefore we first study properties of the two-way part and triangle part respectively in §1 and in §2 before giving the definition of Cherkis bow varieties in §3.

1. Two-way part

1(i). **Definition.** Let V_1, V_2 be finite dimensional complex vector spaces of dimensions $\mathbf{v}_1$, $\mathbf{v}_2$. We consider linear maps

$$V_1 \underset{D}{\overset{C}{\rightleftarrows}} V_2.$$

We consider the space of all such linear maps, that is $\mathbf{M}_{\mathrm{tw}} := \operatorname{Hom}(V_1, V_2) \times \operatorname{Hom}(V_2, V_1)$ as a symplectic manifold, where the symplectic form is given by the pairing between the natural pairing between $\operatorname{Hom}(V_1, V_2)$ and $\operatorname{Hom}(V_2, V_1)$. When we want to specify vector spaces, we denote $\mathbf{M}_{\mathrm{tw}}$ by $\mathbf{M}_{\mathrm{tw}}(V_1, V_2)$.

We have a natural action of $\operatorname{GL}(V_1) \times \operatorname{GL}(V_2)$ on $\mathbf{M}_{\mathrm{tw}}$ given by

$$(C, D) \mapsto (g_2 C g_1^{-1}, g_1 D g_2^{-1}) \quad (g_1, g_2) \in \operatorname{GL}(V_1) \times \operatorname{GL}(V_2).$$

It preserves the symplectic form, and in fact, is hamiltonian. The moment map (normalized so that it vanishes at $(C, D) = (0, 0)$) is

$$(-DC, CD).$$

1(ii). **Adjoint orbits and cotangent bundle of Grassmannian.** Let us consider the symplectic reduction of $\mathbf{M}_{\mathrm{tw}}$ by the action of $\operatorname{GL}(V_2)$. It means that we first take the level set of the moment map with respect to $\operatorname{GL}(V_2)$ at a central element in $\mathfrak{gl}(V_2)$, and then take the quotient by $\operatorname{GL}(V_2)$:

$$\mathcal{M}_\zeta := \mathbf{M}_{\mathrm{tw}} /\!\!/\!\!/_\zeta \operatorname{GL}(V_2) = \{(C, D) \mid CD = \zeta \operatorname{id}_{V_2}\} / \operatorname{GL}(V_2). \tag{1.1}$$

Here ζ is a complex number, and the notation $/\!/\!/_\zeta$ indicates that we take the level set $CD = \zeta$. We omit id_{V_2} hereafter. When we want to emphasize vector spaces, we denote $\mathcal{M}_\zeta$ by $\mathcal{M}_\zeta(V_1, V_2)$.

First let us assume $\zeta \neq 0$. Then the equation $CD = \zeta$ implies that C is surjective and D is injective. In particular, we have $\mathbf{v}_1 \geq \mathbf{v}_2$. Let $\mu^{-1}(\zeta) = \{(C, D) \mid CD = \zeta\}$. Then

- the action of $\mathrm{GL}(V_2)$ on $\mu^{-1}(\zeta)$ is free.

If $(g_2 C, D g_2^{-1}) = (C, D)$, the surjectivity of C or injectivity of D implies that $g_2 = \mathrm{id}_{V_2}$. Moreover

- the graph $\{(x, gx) \in \mu^{-1}(\zeta) \times \mu^{-1}(\zeta) \mid g \in \mathrm{GL}(V_2)\}$ is closed in $\mu^{-1}(\zeta) \times \mu^{-1}(\zeta)$.

Suppose that both sequences C_n, $g_n C_n$ converge to C_∞, C'_∞ which are both surjective. If $\|g_n\|$ diverges, we consider the normalized sequence $g_n/\|g_n\|$ in $\mathrm{End}(V_2)$ to take a convergent subsequence. If we denote the limit by ξ_∞, we have $\xi_\infty C_\infty = \lim_{n\to\infty} g_n C_n/\|g_n\| = C'_\infty \lim_{n\to\infty} 1/\|g_n\| = 0$. Since C_∞ is surjective, we have $\xi_\infty = 0$, but it contradicts with $\|\xi_\infty\| = 1$. Therefore $\|g_n\|$ stays bounded, hence we may assume g_n is convergent. We apply a similar consideration for $g_n^{-1}(g_n C_n)$ and $(g_n C_n)$ to conclude that g_n^{-1} is also convergent. Therefore $C'_\infty = g_\infty C_\infty$ with $g_\infty = \lim_{n\to\infty} g_n \in \mathrm{GL}(V_2)$. We can also derive the closedness from the injectivity of D.

Therefore the quotient (1.1) for $\zeta \neq 0$ is a (smooth) complex manifold, and $\{(C, D) \mid CD = \zeta\, \mathrm{id}_{V_2}\}$ is the total space of a principal $\mathrm{GL}(V_2)$-bundle over the quotient space. See e.g., [Var84, §2.9].

Proposition 1.2. *Suppose $\zeta \neq 0$. Then $\mathcal{M}_\zeta$ is bijective to the space of linear maps ξ which is conjugate* $\mathrm{diag}(\underbrace{0, \dots, 0}_{\mathbf{v}_1 - \mathbf{v}_2 \text{ times}}, \underbrace{\zeta, \dots, \zeta}_{\mathbf{v}_2 \text{ times}})$ *by the map*

$$(C, D) \mod \mathrm{GL}(V_2) \mapsto \xi := DC \in \mathrm{End}(V_1).$$

Proof. We have

$$\xi^2 = DCDC = \zeta DC = \zeta\xi, \quad \mathrm{tr}(\xi) = \mathrm{tr}(CD) = \zeta \mathbf{v}_2.$$

Therefore ξ is conjugate to the above diagonal matrix.

Conversely if ξ is conjugate to the above diagonal matrix, we define V_2 as the ζ-eigenspace, D as the inclusion $V_2 \hookrightarrow V_1$, C as the projection $V_1 \twoheadrightarrow V_2$. Taking a base of V_2, we can represent C, D as matrices. □

This proof also shows that the total space of the principal bundle $\{(C, D) \mid CD = \zeta\, \mathrm{id}_{V_2}\}$ is given by a base of the ζ-eigenspace.

Next consider the case $\zeta = 0$. As in the proof of the above proposition, we have

$$\xi^2 = 0, \quad \mathrm{tr}(\xi) = 0$$

for $\xi = DC \in \mathrm{End}(V_1)$. We also have $\mathrm{rank}\,\xi \le \mathbf{v}_2$ by its definition. These conditions do *not* determine the Jordan normal form of ξ. The corresponding partition is of the form $(2^k 1^{\mathbf{v}_1 - 2k})$ (i.e., $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ appears k times and the rest is (0)), but the multiplicity k can be arbitrary number between 0 and $\min(\mathbf{v}_2, \mathbf{v}_1/2)$.

When $k = \mathbf{v}_2$, C is surjective and D is injective. Then (C, D) are determined by ξ uniquely up to the $\mathrm{GL}(V_2)$-action. On the other hand, (C, D) are *not* determined when $k < \mathbf{v}_2$, say in the extreme case $k = 0$, we could have $C = 0$ or $D = 0$, or even the mixture case with $DC = 0$.

Therefore a naive quotient space does *not* behave well. We use the geometric invariant theory (GIT) to introduce a nice quotient space. It is assumed that the reader is familiar with GIT. See [Nak99, Ch. 3] for a brief introduction, or [Muk03] for more thorough treatment. An interesting feature of GIT is that the quotient space depends on a *choice* of a character $\chi\colon \mathrm{GL}(V_2) \to \mathbb{C}^\times$, or more precisely, a character up to its positive multiple. We have three choices for the general linear group, either det, $\det^{-1}$, or 1. We write the corresponding quotient by $\mathcal{M}^\chi_\zeta$. For $\zeta = 0$ as we have assumed, we simply denote it by $\mathcal{M}^\chi$.

When $\zeta \neq 0$, quotients $\mathcal{M}^\chi_\zeta$ are independent of χ, and also isomorphic to the naive quotient considered above.

Let us choose $\chi = \det^{-1}$. Then we restrict the action to *semistable* points with respect to $\det^{-1}$. If we apply the numerical criterion in GIT to this example, it concretely means that D is injective. (See e.g. [Muk03, §8.1].) Note that if we restrict the $\mathrm{GL}(V_2)$-action on the open subset of $\{CD = 0\}$, it becomes free and its graph is closed: in the above argument for the freeness and closedness for $\zeta \neq 0$, we only use either the surjectivity of C or the injectivity of D.

Proposition 1.3. *For $\chi = \det^{-1}$ we have a bijection*

$$\mathcal{M}^\chi \cong \left\{ (S, \xi) \,\middle|\, \begin{array}{l} S \subset V_1 \text{ : } \mathbf{v}_2\text{-dimensional subspace} \\ \xi \in \mathrm{End}(V_1) \text{ such that } \xi(S) = 0,\ \mathrm{Im}\,\xi \subset S \end{array} \right\}$$

given by $S := \mathrm{Im}\, D$, $\xi := DC$.

Proof. If we define S, ξ as above, then $\xi(S) = 0$, $\mathrm{Im}(\xi) \subset S$ are satisfied.

Conversely S, ξ as in the right hand side are given, we define $D\colon V_2 \to V_1$ as an inclusion by choosing an isomorphism between S and V_2. Then we can uniquely solve the equation $DC = \xi$ for C. □

The right hand side is the cotangent bundle of Grassmannian of $\mathbf{v}_2$-dimensional subspaces in V_1, as ξ is an element of $\mathrm{Hom}(V_1/S, S)$, which is the dual space to

$\operatorname{Hom}(S, V_1/S)$, the tangent space of the Grassmannian at S. The above bijection is an isomorphism of algebraic varieties. The detail is left as an exercise for the reader.

Similarly we could consider the GIT quotient with respect to det. We restrict the action to the open subset consisting those C is surjective. Then S_2 can be regarded as a $\mathbf{v}_2$-dimensional quotient of V_1. Thus we get the cotangent bundle of the *dual* Grassmannian.

The remaining possibility is to take the trivial character of $\mathrm{GL}(V_2)$. Then all points are semistable. But we take a coarser equivalence relation than a naive quotient. We say $(C, D) \sim (C', D')$ when the closure of $\mathrm{GL}(V_2)$-orbits intersect: $\overline{\mathrm{GL}(V_2)(C, D)} \cap \overline{\mathrm{GL}(V_2)(C', D')} \neq \emptyset$. In each orbit closure, there is the unique smallest one, which is a closed orbit. Therefore the quotient space by the equivalence relation $\sim$ can be also regarded as the space of all *closed* $\mathrm{GL}(V_2)$-orbits. Since χ is trivial, we may write this quotient (called *categorical quotient*) as $\mathcal{M}^0$. We often use the abbreviate notation $\mathcal{M}$ instead.

Proposition 1.4. *We have a bijection*

$$\mathcal{M} \cong \left\{ \xi \in \operatorname{End}(V_1) \,\middle|\, \xi^2 = 0, \operatorname{rank}\xi \leq \mathbf{v}_2 \right\}$$

given by $\xi := DC$.

Proof. We have already observed that $\xi := DC$ defines a well-defined map from the left hand side to the right hand side.

We first claim that V_2 has a decomposition $\operatorname{Im} C \oplus \operatorname{Ker} D$. Let us decompose V_2 as $\operatorname{Im} C \oplus V_2'$ by taking a complementary subspace V_2'. We write $D = \begin{bmatrix} D_1 & D_2 \end{bmatrix}$ accordingly. We take $g_2(t) = \begin{bmatrix} 1 & 0 \\ 0 & t^{-1} \end{bmatrix}$. Then $g_2(t)C = C$, $Dg_2(t)^{-1} = \begin{bmatrix} D_1 & tD_2 \end{bmatrix}$. Since (C, D) has a closed $\mathrm{GL}(V_2)$-orbit, we may assume $D_2 = 0$, i.e., $D|_{V_2'} = 0$. If $\operatorname{Ker} D \supsetneq V_2'$, we use a one parameter subgroup in a similar way to change C so that $\operatorname{Im} C$ is contained in a complementary subspace to $\operatorname{Ker} D$. But $\dim \operatorname{Im} C$ should be unchanged as we change points in the same $\mathrm{GL}(V_2)$-orbit. Therefore we have $V_2 = \operatorname{Im} C \oplus \operatorname{Ker} D$. One can also check that the corresponding $\mathrm{GL}(V_2)$-orbit is closed in this case as before.

We have $\dim \operatorname{Im} C = \operatorname{rank}\xi$. Hence the decomposition is determined by ξ up to the $\mathrm{GL}(V_2)$-action. We further normalize D to

$$\begin{array}{r} \\ \scriptstyle \dim \operatorname{Im} C \\ \scriptstyle \mathbf{v}_1 - \dim \operatorname{Im} C \end{array} \begin{array}{c} \begin{array}{cc} \scriptstyle \dim \operatorname{Im} C & \scriptstyle \mathbf{v}_2 - \dim \operatorname{Im} C \end{array} \\ \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \end{array}.$$

Then C is

$$\begin{bmatrix} 0 & \xi \\ 0 & 0 \end{bmatrix},$$

when ξ is regarded as a homomorphism $V_1/\operatorname{Im} C \to \operatorname{Im} C$. In particular, C is determined by ξ. Hence the map is injective.

For a given ξ with $\operatorname{rank}\xi \le \mathbf{v}_2$, we can take C, D as above. Hence the map is also surjective. □

Remark 1.5. Since $\xi^2 = 0$, we have $\operatorname{rank}\xi \le \mathbf{v}_1/2$. Therefore the condition $\operatorname{rank}\xi \le \mathbf{v}_2$ is superfluous when $\mathbf{v}_1/2 \le \mathbf{v}_2$.

Let $\mathcal{M}^{\mathrm{reg}}(V_1, V_2)$ denote

$$\{(C, D) \mid CD = 0, C \text{ is surjective}, D \text{ is injective}\} / \operatorname{GL}(V_2).$$

Then the above proof show that

- $\mathcal{M}^{\mathrm{reg}}(V_1, V_2) \cong \{\xi \in \operatorname{End}(V_1) \mid \xi^2 = 0, \operatorname{rank}\xi = \mathbf{v}_2\}$,
- we have the inclusion $\mathcal{M}^{\mathrm{reg}}(V_1, V_2) \hookrightarrow \mathcal{M}(V_1, V_2')$ if $V_2 \subset V_2'$.

The image depends only on the dimension of V_2 up to $\operatorname{GL}(V_2')$-action. We also have

$$\bigsqcup_{\mathbf{v}_2 = 0,\dots,\min(\mathbf{v}_2', \mathbf{v}_1/2)} \mathcal{M}^{\mathrm{reg}}(V_1, V_2) = \mathcal{M}(V_1, V_2').$$

Note also that $\mathcal{M}^{\mathrm{reg}}(V_1, V_2)$ is the conjugacy class of the nilpotent matrix corresponding to the partition $(2^{\mathbf{v}_2} 1^{\mathbf{v}_1 - 2\mathbf{v}_2})$.

1(iii). **Gluing two-way parts.** Let us denote $\mathbf{M}_{\mathrm{tw}}$ by $\mathbf{M}_{\mathrm{tw}}(V_1, V_2)$ to indicate vector spaces. We prepare another two-way part $\mathbf{M}_{\mathrm{tw}}(V_2, V_3)$ with common V_2:

$$V_0 \underset{D_0}{\overset{C_0}{\rightleftarrows}} V_1 \underset{D_1}{\overset{C_1}{\rightleftarrows}} V_2 \rightleftarrows \cdots \underset{D_{\ell-1}}{\overset{C_{\ell-1}}{\rightleftarrows}} V_\ell.$$

Then we consider the symplectic reduction

$$\begin{aligned} \mathcal{M}_{(\zeta_1, \zeta_2, \cdots, \zeta_\ell)} := \mathbf{M}_{\mathrm{tw}}(V_0, V_1) \times \mathbf{M}_{\mathrm{tw}}(V_1, V_2) \times \cdots \times \mathbf{M}_{\mathrm{tw}}(V_{\ell-1}, V_\ell) \\ /\!\!/\!\!/_{(\zeta_1, \zeta_2, \cdots, \zeta_\ell)} \operatorname{GL}(V_1) \times \cdots \times \operatorname{GL}(V_\ell), \end{aligned} \tag{1.6}$$

which is the quotient of the space $\{(C_0, D_0, C_1, D_1, \dots, C_{\ell-1}, D_{\ell-1}) \mid C_0 D_0 = D_1 C_1 + \zeta_1, \dots, C_{\ell-1} D_{\ell-1} = \zeta_\ell\}$ by $\operatorname{GL}(V_1) \times \cdots \times \operatorname{GL}(V_\ell)$. Note that we do not take the quotient for $\operatorname{GL}(V_1)$.

As in the previous subsection, this is a categorical quotient corresponding to the trivial character χ of $\operatorname{GL}(V_1) \times \cdots \times \operatorname{GL}(V_\ell)$. For a nontrivial character χ, we denote the GIT quotient by $\mathcal{M}^\chi_{(\zeta_1, \zeta_2, \cdots, \zeta_\ell)}$ and by $\mathcal{M}^\chi$ for $\zeta_1 = \zeta_2 = \cdots = \zeta_\ell = 0$.

At the rightmost part, $C_{\ell-1}$ is surjective and $D_{\ell-1}$ is injective when $\zeta_\ell \ne 0$ as in §1(ii). Next consider the equation $C_{\ell-2} D_{\ell-2} = D_{\ell-1} C_{\ell-1} + \zeta_{\ell-1}$. We claim that $C_{\ell-2}$ is surjective and $D_{\ell-2}$ is injective when $\zeta_\ell \ne 0$, $\zeta_{\ell-1} + \zeta_\ell \ne 0$. Indeed note first that

$V_{\ell-1} = \operatorname{Im} D_{\ell-1} \oplus \operatorname{Ker} C_{\ell-1}$, as $\zeta_\ell \neq 0$. When $\zeta_{\ell-1} + \zeta_\ell \neq 0$, this decomposition is preserved under $C_{\ell-2}D_{\ell-2}$ and it is $(\zeta_{\ell-1} + \zeta_\ell)$ on $\operatorname{Im} D_{\ell-1}$ and $\zeta_{\ell-1}$ on $\operatorname{Ker} C_{\ell-1}$ as

$$C_{\ell-2}D_{\ell-2}D_{\ell-1} = D_{\ell-1}C_{\ell-1}D_{\ell-1} + \zeta_{\ell-1}D_{\ell-1} = (\zeta_{\ell-1} + \zeta_\ell)D_{\ell-1},$$
$$C_{\ell-2}D_{\ell-2}|_{\operatorname{Ker} C_{\ell-1}} = (D_{\ell-1}C_{\ell-1} + \zeta_{\ell-1})|_{\operatorname{Ker} C_{\ell-1}} = \zeta_{\ell-1} \operatorname{id}|_{\operatorname{Ker} C_{\ell-1}}.$$

We also see that $C_{\ell-2}D_{\ell-2}$ is an isomorphism, hence $C_{\ell-2}$ is surjective and $D_{\ell-2}$ is injective. We next have $V_{\ell-2} = \operatorname{Im} D_{\ell-2} \oplus \operatorname{Ker} C_{\ell-2} = \operatorname{Im} D_{\ell-2}D_{\ell-1} \oplus D_{\ell-2}(\operatorname{Ker} C_{\ell-1}) \oplus \operatorname{Ker} C_{\ell-2}$, and it is preserved under $C_{\ell-3}D_{\ell-3}$ if $\zeta_{\ell-2} + \zeta_{\ell-1} + \zeta_\ell \neq 0$. Continuing this argument, we get

Proposition 1.7. *Suppose* 0, ζ_1, $\zeta_1 + \zeta_2$, ..., $\zeta_1 + \zeta_2 + \cdots + \zeta_\ell$ *are pairwise distinct. Then* $\mathcal{M}_{(\zeta_1,\dots,\zeta_\ell)}$ *is bijective to the space of the conjugacy class of*

$$\operatorname{diag}(\underbrace{0,\dots,0}_{\mathbf{v}_0 - \mathbf{v}_1 \text{ times}}, \underbrace{\zeta_1,\dots,\zeta_1}_{\mathbf{v}_2 - \mathbf{v}_1 \text{ times}}, \cdots, \underbrace{\zeta_1 + \cdots + \zeta_\ell, \dots, \zeta_1 + \cdots + \zeta_\ell}_{\mathbf{v}_\ell \text{ times}})$$

by the map

$$(C_i, D_i)_{i=0}^{\ell-1} \mod \operatorname{GL}(V_1) \times \cdots \times \operatorname{GL}(V_\ell) \mapsto \xi := D_0C_0 \in \operatorname{End}(V_0).$$

This is a generalization of Proposition 1.2.

As a generalization of Proposition 1.3 we have

Proposition 1.8. *Let* χ *be the product of* $\det^{-1}$ *on* $\operatorname{GL}(V_i)$ *for* $i = 1, \dots, \ell$. *We have a bijection*

$$\mathcal{M}^\chi \cong \left\{ \begin{matrix} (V_0 = S_0 \supset S_1 \supset \cdots \supset S_\ell \supset S_{\ell+1} = 0, \xi) \\ \in \mathcal{F} \times \operatorname{End}(V_0) \end{matrix} \,\middle|\, \xi(S_i) \subset S_{i+1} \text{ for } i = 0, \dots, \ell \right\},$$

where $\mathcal{F}$ *is the partial flag variety of subspaces* $S_\bullet$ *with* $\dim S_i = \mathbf{v}_i$. *It is given by* $S_i := \operatorname{Im} D_0D_1 \dots D_{i-1}$, $\xi := D_0C_0$.

The right hand side is the cotangent bundle of the partial flag variety $\mathcal{F}$.

Next consider the case corresponding to Proposition 1.4. We assume

$$\mathbf{v}_0 - \mathbf{v}_1 \geq \mathbf{v}_1 - \mathbf{v}_2 \geq \cdots \geq \mathbf{v}_{\ell-1} - \mathbf{v}_\ell \geq \mathbf{v}_\ell. \tag{1.9}$$

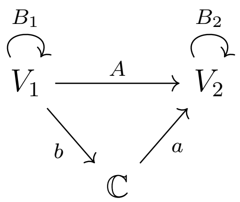
We consider this as a partition of $\mathbf{v}_0$. Let μ be its transpose. For example, for $\mathbf{v}_i = \mathbf{v}_0 - i$, $\mu = (\mathbf{v}_0)$. For $\mathbf{v}_1 = 0$, $\mu = (1^{\mathbf{v}_0})$. Note that any partition of $\mathbf{v}_0$ can be realized in this way.

Proposition 1.10 (Kraft-Procesi [KP79])**.** *Under the assumption* (1.9), $\mathcal{M}$ *is isomorphic to the closure* $\overline{\mathcal{O}_\mu}$ *of the nilpotent conjugacy class whose Jordan normal form corresponds to the partition* μ. *The isomorphism is given by* $\xi := D_0C_0$.

2. Triangle

2(i). **Definition.** Let V_1, V_2 be finite dimensional complex vector spaces of dimensions $\mathbf{v}_1$, $\mathbf{v}_2$. We consider the set $\mathbf{M}_{\mathrm{tri}}$ of linear maps A, B_1, B_2, a, b satisfying one algebraic equation and two open conditions (S1,2) as follows:

(2.1)



$$B_2A - AB_1 + ab = 0,$$

(S1) $B_1(S_1) \subset S_1, S_1 \subset \operatorname{Ker} A \cap \operatorname{Ker} b \Rightarrow S_1 = 0,$

(S2) $B_2(T_2) \subset T_2, T_2 \supset \operatorname{Im} A + \operatorname{Im} a \Rightarrow T_2 = V_2.$

When we emphasize vector spaces, we denote $\mathbf{M}_{\mathrm{tri}}$ by $\mathbf{M}_{\mathrm{tri}}(V_1, V_2)$.

The meaning of the conditions (S1,2) might be difficult to understand at this stage, though they guarantee smoothness of $\mathbf{M}_{\mathrm{tri}}$ as we will show below. In fact, different stability conditions were used, earlier than [NT17], in the context of handsaw/chainsaw quiver varieties. See §§2(vi), 2(vii) below.

We add a linear map $\eta\colon V_2 \to V_1$ and consider a potential

$$W = \operatorname{tr}\left(\eta(B_2A - AB_1 + ab)\right).$$

Then the defining equation $B_2A - AB_1 + ab = 0$ is understood as $\partial W/\partial\eta = 0$. On the other hand, consider other derivatives of W:

$$\frac{\partial W}{\partial A} = \eta B_2 - B_1\eta, \quad \frac{\partial W}{\partial B_1} = -\eta A, \quad \frac{\partial W}{\partial B_2} = A\eta, \quad \frac{\partial W}{\partial a} = b\eta, \quad \frac{\partial W}{\partial b} = \eta a.$$

Suppose that all these derivatives vanish. If (S1) is satisfied, $S_1 = \operatorname{Im}\eta$ must be 0. Therefore $\eta = 0$. Similarly (S2) implies that $T_2 = \operatorname{Ker}\eta$ must be V_2, and hence $\eta = 0$. In particular, this shows

Proposition 2.2. $\mathbf{M}_{\mathrm{tri}}$ *is a nonsingular variety of dimension* $\mathbf{v}_1^2 + \mathbf{v}_2^2 + \mathbf{v}_1 + \mathbf{v}_2$.

We have a natural action of $\mathrm{GL}(V_1) \times \mathrm{GL}(V_2)$ on $\mathbf{M}_{\mathrm{tri}}$ given by

(2.3)
$$(A, B_1, B_2, a, b) \mapsto (g_2Ag_1^{-1}, g_1B_1g_1^{-1}, g_2B_2g_2^{-1}, g_2a, bg_1^{-1}) \quad (g_1, g_2) \in \mathrm{GL}(V_1)\times\mathrm{GL}(V_2).$$

2(ii). **Slice.** Let us consider the case $V_1 = 0$ as an example. Then the condition (S2) means that a, B_2a, …span V_2. It is also equivalent to say that $\langle a, B_2a, \dots, B_2^{\mathbf{v}_2-1}a\rangle$ forms a basis of V_2. Under this basis, the linear map B_2 is represented by a matrix of the form

(2.4)
$$\begin{pmatrix} 0 & \cdots & 0 & x_1 \\ 1 & & 0 & x_2 \\ & \ddots & & \vdots \\ 0 & & 1 & x_{\mathbf{v}_2} \end{pmatrix} \qquad \text{for } x_1, x_2, \dots, x_{\mathbf{v}_2} \in \mathbb{C}.$$

We have

Proposition 2.5. *Suppose* $V_1 = 0$. *Then* $\mathbf{M}_{\mathrm{tri}}$ *is isomorphic to* $\mathrm{GL}(V_2) \times \mathcal{S}$, *where* $\mathcal{S}$ *is the space of matrices* X *of the form* (2.4). *The isomorphism is given by the map*

$$(B_2, a) \mapsto u := \begin{bmatrix} a & (-B_2)a & \dots & (-B_2)^{\mathbf{v}_2 - 1} a \end{bmatrix}^{-1}, \quad X := -u B_2 u^{-1}.$$

The inverse map is given by $(u, X) \mapsto B_2 := -u^{-1} X u$, $a := u^{-1} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$.

A matrix X of the form (2.4) has the following properties:

- The characteristic polynomial $\det(t - X)$ is $t^{\mathbf{v}_2} - x_{\mathbf{v}_2} t^{\mathbf{v}_2 - 1} - \dots - x_1$.
- X is *regular*, that is the centralizer of X in $\mathrm{End}(V_2)$ is of minimum dimension $\mathbf{v}_2$.

For a given polynomial $f(t) = t^{\mathbf{v}_2} + \dots$ of degree $\mathbf{v}_2$, there are finitely many conjugacy classes of matrices whose characteristic polynomials are $f(t)$, corresponding to various Jordan normal forms with fixed eigenvalues with multiplicities. Among them, there is the unique open conjugacy class, formed by regular matrices. And in the open conjugacy class, there is the unique matrix of the form (2.4). Thus the space of matrices of the form (2.4) gives a section of the projection

$$\{\text{regular matrices}\} \xrightarrow{\text{characteristic polynomial}} \{f(t) = t^{\mathbf{v}_2} + \dots\} \cong \mathbb{C}^{\mathbf{v}_2}.$$

Let X_0 be the nilpotent matrix of the form (2.4), that is one with all entries $*$ in the last column are 0. The conjugacy class of X_0 is a (locally closed) smooth subvariety in $\mathrm{End}(V_2)$ and its tangent space at X is the space of matrices of $[\xi, X_0]$ for $\xi \in \mathrm{End}(V_2)$. The linear space consisting of matrices of the form

$$\begin{pmatrix} 0 & \cdots & 0 & * \\ 0 & & \text{\Large 0} & * \\ \text{\Large 0} & \ddots & & \vdots \\ & & 0 & * \end{pmatrix}$$

is a complementary subspace to the tangent space. Hence the space of matrices of the form (2.4) is a *slice* to the conjugacy class of X_0.

Let us return back to the general triangle $\mathbf{M}_{\mathrm{tri}}$. We have

Proposition 2.6 ([Tak16, §2.4])**.** (1) *Suppose* $\mathbf{v}_1 = \mathbf{v}_2$. *Then* $\mathbf{M}_{\mathrm{tri}}$ *is isomorphic to* $\mathrm{GL}(\mathbf{v}_1) \times \mathfrak{gl}(\mathbf{v}_1) \times \mathbb{C}^{\mathbf{v}_1} \times (\mathbb{C}^{\mathbf{v}_1})^*$ *by*

$$(u, X, I, J) \mapsto (A, B_1, B_2, a, b) := (u, u^{-1} X u, X - IJ, I, Ju).$$

(2) *Suppose* $\mathbf{v}_1 < \mathbf{v}_2$. *Then* $\mathbf{M}_{\mathrm{tri}}$ *is isomorphic to the product* $\mathrm{GL}(V_2) \times \mathcal{S}$, *where* $\mathcal{S}$ *is the space of the size* $\mathbf{v}_2 \times \mathbf{v}_2$ *matrices of forms*

$$X = \left(\begin{array}{c|cccc} \ast & & 0 & & \begin{matrix}\ast\\ \vdots \\ \ast\end{matrix} \\ \hline \ast \ \cdots \ \ast & 0 & \cdots & 0 & \ast \\ & 1 & & 0 & \ast \\ 0 & & \ddots & & \vdots \\ & 0 & & 1 & \ast \end{array}\right). \tag{2.7}$$

Here $\mathbf{v}_2 \times \mathbf{v}_2$ *is divided as* $(\mathbf{v}_1 + (\mathbf{v}_2 - \mathbf{v}_1)) \times (\mathbf{v}_1 + (\mathbf{v}_2 - \mathbf{v}_1))$. *The isomorphism is given by*

$$u := \begin{bmatrix} -A & a & -B_2 a & \cdots & (-B_2)^{\mathbf{v}_2 - \mathbf{v}_1 - 1} a \end{bmatrix}^{-1}, \quad X := -u B_2 u^{-1}.$$

(3) *If* $\mathbf{v}_1 > \mathbf{v}_2$, *we have the same conclusion after exchanging* V_1 *and* V_2.

Proof. We first show that A is of full rank. Let $x \in \operatorname{Ker} A$, $y \in \operatorname{Ker} A^t$. Then

$$0 = \langle (B_2 A - A B_1 + ab)x, y\rangle = bx \cdot a^t y.$$

Since bx and $a^t y$ are scalar, we have either $bx = 0$ or $a^t y = 0$. If $\operatorname{Ker} A \not\subset \operatorname{Ker} b$, we take $x \in \operatorname{Ker} A$ such that $bx \neq 0$. Then $a^t y = 0$ for any $y \in \operatorname{Ker} A^t$, i.e., $\operatorname{Ker} A^t \subset \operatorname{Ker} a^t$. Then we conclude $\operatorname{Ker} A^t = 0$ by (S2), hence A is surjective.

Otherwise we have $\operatorname{Ker} A \subset \operatorname{Ker} b$. Then we have $\operatorname{Ker} A = 0$ by (S1), hence A is injective.

(1) By the above observation, A is an isomorphism. Then the conclusion is clear.

(2) As in the special case $V_1 = 0$, we first observe that (S2) implies that the matrix

$$\begin{bmatrix} -A & a & -B_2 a & \cdots & (-B_2)^{\mathbf{v}_2 - \mathbf{v}_1 - 1} a \end{bmatrix}$$

is invertible. We rewrite B_2 under the corresponding base. Since $B_2 A - A B_1 + ab = 0$ implies $B_2(\operatorname{Im} A) \subset \operatorname{Im} A + \operatorname{Im} a$, we see that the resulted matrix is of above form.

(3) The argument is similar and omitted. □

We can refine the assertion given during the proof as

Proposition 2.8. *If* $\mathbf{v}_1 < \mathbf{v}_2$, $\det \begin{bmatrix} -A & a & -B_2 a & \cdots & (-B_2)^{\mathbf{v}_2 - \mathbf{v}_1 - 1} a \end{bmatrix} \neq 0$ *if and only if both* (S1,2) are satisfied.

Therefore $\mathbf{M}_{\mathrm{tri}}$ is an affine algebraic variety defined from the space of linear maps (A, B_1, B_2, a, b) by imposing $B_2 A - A B_1 + ab = 0$, and inverting $\det \begin{bmatrix} -A & a & -B_2 a & \cdots & (-B_2)^{\mathbf{v}_2 - \mathbf{v}_1 - 1} a \end{bmatrix}$.

Proof. One direction is already proved. Let us suppose $\begin{bmatrix} -A & a & -B_2 a & \cdots & (-B_2)^{\mathbf{v}_2 - \mathbf{v}_1 - 1} a \end{bmatrix}$ is invertible. Then (S2) is satisfied. (S1) is also satisfied as A is injective. □

Let X_0 be the matrix of the form (2.7) such that all $*$ vanish. It is a nilpotent matrix corresponding to a partition $(\mathbf{v}_2 - \mathbf{v}_1, \underbrace{1, \dots, 1}_{\mathbf{v}_1 \text{ times}})$. Then one can directly check that

- the tangent space of $\mathcal{S}$ at X_0 is complementary to the tangent space of the conjugacy class of X_0, which is a locally closed smooth subvariety in $\operatorname{End}(V_2)$.

Thus $\mathcal{S}$ is a slice to the conjugacy class of X_0, as in the case of $V_1 = 0$.

Remark 2.9. An affine subspace in the semisimple Lie algebra $\mathfrak{g}$ with the same property with respect to the adjoint action was introduced by Kostant and Slodowy. It is defined by using an $\mathfrak{sl}(2)$-triple associated with a regular nilpotent element. It is different from the above slice for $\mathfrak{g} = \mathfrak{gl}(V_2)$.

2(iii). **Simple singularity as intersection.** Consider a special case $\mathbf{v}_1 = 1$, $\mathbf{v}_2 = n$. Then $X \in \mathcal{S}$ is

$$X = \begin{pmatrix} w & 0 & \cdots & 0 & x_1 \\ y & 0 & \cdots & 0 & x_2 \\ 0 & 1 & & \text{\Large 0} & \vdots \\ \vdots & \text{\Large 0} & \ddots & & \vdots \\ 0 & & & 1 & x_n \end{pmatrix}.$$

Its characteristic polynomial is

$$t^n - (w + x_n)t^{n-1} - (-wx_n + x_{n-1})t^{n-2} + \cdots - (-wx_3 + x_2)t + wx_2 - yx_1.$$

Therefore X is nilpotent if and only if

$$x_n = -w, x_{n-1} = -w^2, \dots, x_2 = -w^{n-1}, yx_1 = -w^n.$$

Thus the intersection of $\mathcal{S}$ with the nilpotent cone is a hypersurface $yx_1 = -w^n$ in $\mathbb{C}^3$, which is the simple singularity of type A_n. It has an isolated singularity at 0, and isomorphic to the quotient singularity $\mathbb{C}^2/(\mathbb{Z}/(n+1))$, where the action is given by $(z_1, z_2) \mapsto (\zeta z_1, \zeta z_2)$ for $\zeta^{n+1} = 1$. The isomorphism is given by (z_1, z_2) mod $\mathbb{Z}/(n+1) \mapsto (x_1 = -z_1^{n+1}, y = z_2^{n+1}, w = z_1 z_2)$.

Remark 2.10. This is the simplest example of Brieskorn-Slodowy's result relating a simple singularity and the intersection of the nilpotent cone and a slice in a complex simple Lie algebra.

2(iv). **Symplectic structure.** Using Proposition 2.6 we define a symplectic form on $\mathbf{M}_{\mathrm{tri}}$ by

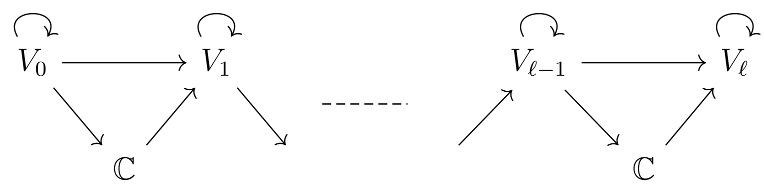
$$\operatorname{tr}\left(dX \wedge duu^{-1} + Xduu^{-1} \wedge duu^{-1}\right) \quad \text{if } \mathbf{v}_1 \neq \mathbf{v}_2,$$
$$\operatorname{tr}\left(dX \wedge duu^{-1} + Xduu^{-1} \wedge duu^{-1} + dI \wedge dJ\right) \quad \text{if } \mathbf{v}_1 = \mathbf{v}_2.$$

The moment map with respect to the $\mathrm{GL}(V_1) \times \mathrm{GL}(V_2)$-action in (2.3) is

$$(-B_1, B_2).$$

2(v). **Gluing triangles.** We consider the following diagram, as a generalization of the triangle:

(2.11)



Linear maps in each triangle satisfy the algebraic equation and (S1,2) in (2.1).

We take the quotient by $\mathrm{GL}(V_1) \times \cdots \times \mathrm{GL}(V_{\ell-1})$. Note that we do not include the left most $\mathrm{GL}(V_0)$ and the right most $\mathrm{GL}(V_\ell)$. Let us denote by $\mathcal{M}$ the space of all linear maps modulo $\mathrm{GL}(V_1) \times \cdots \times \mathrm{GL}(V_{\ell-1})$. It can be regarded as a symplectic reduction:

$$\mathcal{M} \cong \mathbf{M}_{\mathrm{tri}}(V_0, V_1) \times \cdots \times \mathbf{M}_{\mathrm{tri}}(V_{\ell-1}, V_\ell) /\!\!/\!\!/ \mathrm{GL}(V_1) \times \cdots \times \mathrm{GL}(V_{\ell-1}), \tag{2.12}$$

as the moment map equation means that $B_i \in \mathrm{End}(V_i)$ from the left triangle $\mathbf{M}_{\mathrm{tri}}(V_{i-1}, V_i)$ and one from the right triangle $\mathbf{M}_{\mathrm{tri}}(V_i, V_{i+1})$ are equal.

Lemma 2.13. *The actions of* $\mathrm{GL}(V_0) \times \cdots \times \mathrm{GL}(V_{\ell-1})$ *and* $\mathrm{GL}(V_1) \times \cdots \times \mathrm{GL}(V_\ell)$ *are free.*

Proof. Consider the first case. Suppose $(g_0, \dots, g_{\ell-1})$ stabilizes a point in $\mathbf{M}_{\mathrm{tri}}(V_0, V_1) \times \cdots \times \mathbf{M}_{\mathrm{tri}}(V_{\ell-1}, V_\ell)$. At the rightmost triangle, $\mathrm{Im}(g_{\ell-1} - 1)$ satisfies the assumption for (S1). Therefore we must have $\mathrm{Im}(g_{\ell-1} - 1) = 0$, i.e., $g_{\ell-1} = 1$. We continue to apply (S1) for triangles from the right to conclude that all g_i are 1. □

It is a direct consequence of this freeness and general results in GIT (see e.g., [Muk03, Ch. 5]) that the graph of the action is closed. It is also easy to check it directly as in §1(ii). Thus $\mathcal{M}$ is a (nonsingular) affine algebraic variety, and the quotient (2.12) is a principal bundle.

Proposition 2.14. (1) *Assume* $\mathbf{v}_0 = 0$*, and* $0 < \mathbf{v}_1 < \mathbf{v}_2 < \cdots < \mathbf{v}_\ell$*. Then* $\mathcal{M}$ *is isomorphic to* $\mathrm{GL}(V_\ell) \times \mathcal{S}$*, where* $\mathcal{S}$ *is the space of size* $\mathbf{v}_\ell \times \mathbf{v}_\ell$ *matrices of the form*

$$X = \left[\begin{array}{cccc|cccc|cc|c}
0 & \cdots & 0 & * & & & & * & & * & \\
1 & & 0 & * & & 0 & & * & 0 & * & \cdots \\
0 & \ddots & & \vdots & & & & \vdots & & \vdots & \\
& & 1 & * & & & & * & & * & \\ \hline
* & \cdots & * & * & 0 & \cdots & 0 & * & & * & \\
& & & & 1 & & 0 & * & 0 & * & \cdots \\
& 0 & & & 0 & \ddots & & \vdots & & \vdots & \\
& & & & & & 1 & * & & * & \\ \hline
* & \cdots & * & * & * & \cdots & * & * & & & \\
& 0 & & & & 0 & & & \ddots & & \\ \hline
& \vdots & & & & \vdots & & & & &
\end{array}\right],$$

where row and columns are divided into $\mathbf{v}_1$*,* $\mathbf{v}_2 - \mathbf{v}_1$*, ...,* $\mathbf{v}_\ell - \mathbf{v}_{\ell-1}$*.*

(2) *Assume further*

$$\mathbf{v}_\ell - \mathbf{v}_{\ell-1} \geq \mathbf{v}_{\ell-1} - \mathbf{v}_{\ell-2} \geq \cdots \geq \mathbf{v}_2 - \mathbf{v}_1 \geq \mathbf{v}_1.$$

Then $\mathcal{S}$ *is a slice to the conjugacy class of the matrix* X *above with all* $*$ *vanish, that is the nilpotent conjugacy class corresponding to the partition given by the above sequence.*

For example, we have $\mathcal{M} \cong \mathrm{GL}(V_\ell) \times \mathfrak{gl}(V_\ell) = T^*\mathrm{GL}(V_\ell)$ when $\mathbf{v}_i = i$ $(0 \leq i \leq \ell)$.

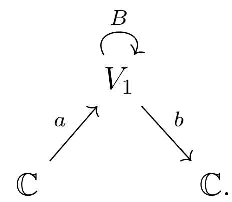
If we do not assume the inequality in (2), the space $\mathcal{S}$ in (1) has dimension larger than or equal to the slice to the conjugacy class of the matrix X.

Remark 2.15. We could consider the symplectic reduction at a different level. It means that we impose $B_i^{\mathrm{left}} = B_i^{\mathrm{right}} + \zeta_i$ for $\zeta_i \in \mathbb{C}$ at V_i, where B_i^{left} (resp. B_i^{right}) is the linear map from the left (resp. right) triangle. But it can be absorbed in the next triangle by changing B_{i+1}^{left} to $B_{i+1}^{\mathrm{left}} + \zeta_i$, as the defining equation $A_i(B_i^{\mathrm{right}} + \zeta_i) - (B_{i+1}^{\mathrm{left}} + \zeta_i)A_i + a_i b_i = 0$, as well as (S1,2), are preserved. Continuing this process, we can move the effect of all ζ_i to the leftmost or rightmost triangle. Hence $\mathcal{M}$ is unchanged as a variety. Only the moment map with respect to the residual $\mathrm{GL}(V_0) \times \mathrm{GL}(V_\ell)$-action is shifted.

This argument fails when we glue triangles on a circle as in §2(vii).

2(vi). **Based maps.** Consider the special case $\ell = 2$ and $V_0 = V_\ell = 0$, that is

(2.16)



Let $n := \dim V_1$.

Theorem 2.17 (Donaldson [Don84])**.** *The space $\mathcal{M}$ in this case is isomorphic to the space of rational maps $f(z)\colon \mathbb{CP}^1 \to \mathbb{CP}^1$ of degree $= n$ with $f(\infty) = 0$ by*

$$(B, a, b) \mod \mathrm{GL}(V_1) \mapsto f(z) = b(z - B)^{-1}a.$$

The latter space is called the space of *based* maps, thanks to the condition $f(\infty) = 0$.

Proof. Suppose (B, a, b) is given. Note that

$$f(z) = b(z - B)^{-1}a = \frac{1}{\det(z - B)} b(\text{cofactor matrix of } z - B)^t a.$$

The denominator $\det(z - B)$ is of degree n, while the cofactor matrix of $z - B$ is of degree $n - 1$. Therefore $f(\infty) = 0$ is satisfied.

Let us check that $\det(z - B)$ and $b(\text{cofactor matrix of } z - B)^t a$ do not have common zeros. If not, there is a polynomial $q_0 + q_1 z + \cdots + q_{n-1}z^{n-1}$ with $q_{n-1} \neq 0$ such that $f(z)(q_0 + q_1 z + \cdots + q_{n-1}z^{n-1})$ is a polynomial in z. We expand it as

$$b\left(\frac{1}{z} + \frac{B^2}{z} + \frac{B^2}{z^3} + \cdots\right) a(q_0 + q_1 z + \cdots + q_{n-1}z^{n-1}) = \sum_{i,j\geq 0} bB^j a q_i z^{i-j-1}.$$

Let us take coefficients of z^{i-j-1} with $i \leq j$ which are zero by the assumption:

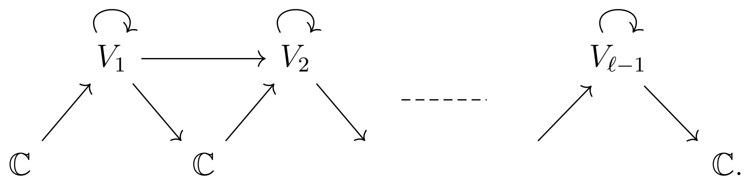
$$\begin{aligned} &q_0 ba + q_1 bBa + \cdots + q_{n-1} bB^{n-1}a = 0, \\ &q_0 bBa + q_1 bB^2 a + \cdots + q_{n-1} bB^n a = 0, \\ &\cdots \\ &q_0 bB^{n-1}a + q_1 bB^n a + \cdots + q_{n-1} bB^{2n-2}a = 0. \end{aligned}$$

Therefore the matrix $(bB^{i+j}a)_{0\leq i,j\leq n-1}$ has rank less than n. But it contradicts with the conditions (S1,2) which say $\langle a, Ba, \dots, B^{n-1}a\rangle$ and $\langle b, bB, \dots, bB^{n-1}\rangle$ are bases of V_1 and V_1^* respectively. Therefore $\det(z - B)$ and $b(\text{cofactor matrix of } z - B)^t a$ do not have common zeros.

Conversely $f(z)$ is given. Let us write $f(z) = p(z)/q(z)$ with polynomials p, q such that $\deg q = n$, $\deg p < n$. We define $V := \mathbb{C}[z]/(q(z))$, $B :=$ multiplication by z,

$a := 1 \mod (q(z))$ and $b(g(z) \mod (q(z))$ as the coefficient z^{-1} in $f(z)g(z)$. This is well-defined as $f(z)q(z) = p(z)$ is a polynomial. The condition (S2) says that a, Ba, ...span V_1. It is obvious from the definition. Let us check (S1). Suppose that $g(z)$ satisfies $b(g(z) \mod (q(z))) = b(zg(z) \mod (q(z))) = \cdots = 0$. It means that $f(z)g(z)$ is a polynomial in z. Therefore $g(z)$ is divisible by $q(z)$, hence is 0 in $\mathbb{C}[z]/(q(z))$. □

Hurtubise generalized this result to the case of a longer chain of triangles [Hur89]. Consider



Theorem 2.18 (Hurtubise [Hur89])**.** *The space $\mathcal{M}$ in this case is isomorphic to the space of based holomorphic maps f from $\mathbb{CP}^1$ to the flag variety $0 \subset S_1 \subset S_2 \subset \cdots \subset S_{\ell-1} \subset \mathbb{C}^\ell$ of degree* $\dim V_1, \dots, \dim V_{\ell-1}$.

Here the based condition means that ∞ is sent to the standard flag $0 \subset \mathbb{C} \subset \mathbb{C}^2 \subset \cdots \subset \mathbb{C}^{\ell-1} \subset \mathbb{C}^\ell$.

A proof is given, for example, in [Nak12, §3].

This variety for which conditions (S1,2) are replaced by different stability conditions was called a *handsaw quiver variety*. See [Nak12] and references therein. As an example, let us consider the simplest case (2.16). Instead of imposing conditions (S1,2) and taking the set-theoretical quotient by $\mathrm{GL}(V_1)$, we do not impose (S1,2), but take the categorical quotient by $\mathrm{GL}(V_1)$. Then we have the morphism

$$[B, a, b] \mapsto \left(\det(z - B), b(\text{cofactor matrix of } z - B)^t a\right).$$

The target is the space of pairs of polynomials of degree n and $n-1$ where the former starts with z^n, which is isomorphic to $\mathbb{C}^{2n}$. Contrary to Theorem 2.17, we do not impose that $\det(z - B)$, $b(\text{cofactor matrix of } z - B)^t a$ have no common zeros. We factor common zeros so that they are $p(z)\prod_{i=1}^k (z - \alpha_i)$, $q(z)\prod_{i=1}^k (z - \alpha_i)$ such that $p(z)$ and $q(z)$ have no common zeros. Then $q(z)/p(z)$ is the target of (B', a', b') by Theorem 2.17 and (B, a, b) can be given as

$$B = B' \oplus \operatorname{diag}(\alpha_1, \alpha_2, \dots, \alpha_k), \quad a = a' \oplus 0, \quad b = \begin{bmatrix} b' & 0 \end{bmatrix}.$$

From this observation, we see that the categorical quotient is

$$\bigsqcup_{k=0}^{n} S^{n-k}\mathbb{C} \times (\text{the space of based maps } \mathbb{CP}^1 \to \mathbb{CP}^1 \text{ of degree } k).$$

This is the *zastava* space of degree n for $\mathbb{P}^1$.

Laumon space should be explained.

Remark 2.19. In (2.11) we did not take the quotient by $\mathrm{GL}(V_0)\times\mathrm{GL}(V_\ell)$. But we can regard it as the space of based maps by adding sequences of triangles such that dimension are decreasing one by one to the right of V_ℓ, and also to the left of V_0. This is a consequence of the observation explained after Proposition 2.14 and the fact that the symplectic reduction of $X\times T^*GL(n)$ by $\mathrm{GL}(n)$ is nothing but X. In particular, $\mathrm{GL}(n)\times\mathcal{S}$ for a slice $\mathcal{S}$ to a nilpotent conjugacy class of $\mathrm{GL}(n)$ is isomorphic to the space of based maps to the flag variety for $\mathbb{C}^{2\ell}$.

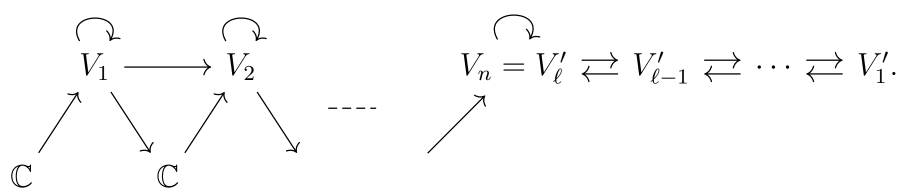
2(vii). **Chainsaw.**

To be written.

3. Cherkis bow varieties of affine type A

3(i). **Intersection of a slice and a nilpotent orbit.** Let us give an example of Cherkis bow varieties, before giving a general definition. Let us consider

(3.1)



Let us denote dimensions of V_i, V_i' by $\mathbf{v}_i$, $\mathbf{v}_i'$ respectively. Note that this can be regarded as the symplectic reduction of the product of the varieties in (2.12) (the left side) and (1.6) (the right side) by $\mathrm{GL}(V_n)=\mathrm{GL}(V_\ell')$ (the middle), where vector spaces for the latter are renamed.

We assume

$$\begin{aligned}\mathbf{v}_n-\mathbf{v}_{n-1}\geq\mathbf{v}_{n-1}-\mathbf{v}_{n-2}\geq\cdots\geq\mathbf{v}_2-\mathbf{v}_1\geq\mathbf{v}_1,\\ \mathbf{v}_\ell'-\mathbf{v}_{\ell-1}'\geq\mathbf{v}_{\ell-1}'-\mathbf{v}_{\ell-2}'\geq\cdots\geq\mathbf{v}_2'-\mathbf{v}_1'\geq\mathbf{v}_1'.\end{aligned}\tag{3.2}$$

We regard both as partitions of $\mathbf{v}_n=\mathbf{v}_\ell'$ and denote them by λ, μ.

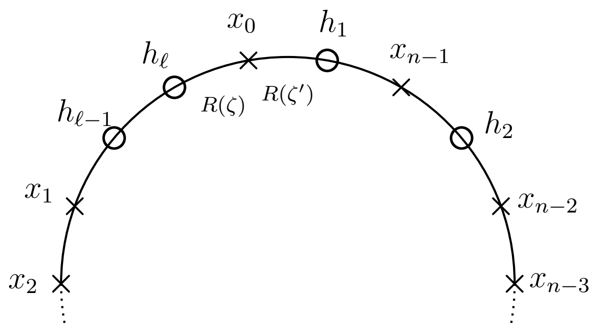
As consequences of Propositions 1.10 and 2.14, we have

Proposition 3.3. *Under* (3.2) *the variety* $\mathcal{M}$ *for* (3.1) (*with the trivial* ζ *and* χ) *is isomorphic to the intersection* $\overline{\mathcal{O}_{\mu^t}}\cap\mathcal{S}_\lambda$, *where* $\overline{\mathcal{O}_{\mu^t}}$ *is the closure of the nilpotent conjugacy class corresponding to* μ^t, *and* $\mathcal{S}_\lambda$ *is the slice for the nilpotent conjugacy class corresponding to* λ *given in Proposition 2.14.*

This is a generalization of the example discussed in §2(iii).

We can consider other variants of quotients discussed in §1(iii). They are simply varieties described in §1(iii) with the additional constraint $\xi\in\mathcal{S}_\lambda$.

3(ii). **Definition.** A *bow diagram* is a configuration of × and ○ put on a circle together with nonnegative integers $R(\zeta)$ assigned for segments ζ of the circle cut out by × or ○ as



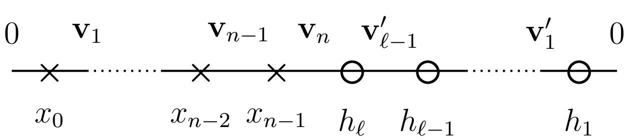
Let n (resp. ℓ) be the number of × (resp. ○). We name × (resp. ○) as x_0, x_1, …, x_{n-1} (resp. h_1, h_2, …, h_ℓ) in the anticlockwise (resp. clockwise) orientation. For x (resp. h) let $\mathrm{o}(x)$, $\mathrm{i}(x)$ (resp. $\mathrm{o}(h)$, $\mathrm{i}(h)$) be the adjacent segments so that $\mathrm{o}(x)$ —×x→ $\mathrm{i}(x)$ (resp. $\mathrm{o}(h)$ —○h→ $\mathrm{i}(h)$) in the anticlockwise orientation.

We define Cherkis bow variety $\mathcal{M}$ associated with a bow diagram by a symplectic reduction of the product of $\mathbf{M}_{\mathrm{tw}}$ and $\mathbf{M}_{\mathrm{tri}}$ by $\prod \mathrm{GL}(R(\zeta))$:

(1) We first put $\mathbf{M}_{\mathrm{tw}}$ (resp. $\mathbf{M}_{\mathrm{tri}}$) at each h (resp. x) such that two vector spaces have dimensions $R(\mathrm{o}(h \text{ (resp. } x)))$, $R(\mathrm{o}(h \text{ (resp. } x)))$, numbers associated with segments adjacent to ○. We put $\mathbf{M}_{\mathrm{tri}}$ so that the linear map A goes in the anticlockwise direction.

(2) A segment ζ is cut out by two ○ or ×. Let us denote them by p and q so that $\mathrm{i}(p) = \mathrm{o}(q)$. (There are four possibilities $\overset{p}{○}$—$\overset{q}{○}$, $\overset{p}{×}$—$\overset{q}{○}$, $\overset{p}{○}$—$\overset{q}{×}$, $\overset{p}{×}$—$\overset{q}{×}$.) Then we have the common vector space $\mathbb{C}^{R(\zeta)}$ for two $\mathbf{M}_{\mathrm{tw}}$ or $\mathbf{M}_{\mathrm{tri}}$ assigned for p and q. We have the action of $\mathrm{GL}(R(\zeta))$ on two $\mathbf{M}_{\mathrm{tw}}$ or $\mathbf{M}_{\mathrm{tri}}$.

The example in the previous subsection is associated with

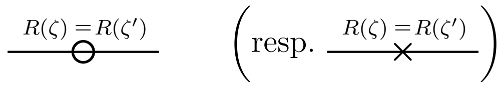
(3.4) 

The circle is replaced by a line, as the number between h_1 and x_0 is 0.

3(iii). **GIT quotients.** We can also consider GIT quotients by $\prod \mathrm{GL}(R(\zeta))$.

To be written.

3(iv). **Cobalanced condition.** We say a bow diagram satisfies the *balanced* (resp. *cobalanced*) *condition* if $R(\zeta) = R(\zeta')$ for any pair of segments ζ, ζ' connected by ○ (resp. ×):

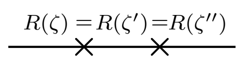


Bow varieties with the balanced condition are identified with Coulomb branches of quiver gauge theories of affine type A, and will be discussed in ***. We consider bow varieties with the cobalanced condition in this subsection.

By Proposition 2.6 the corresponding $\mathbf{M}_{\mathrm{tri}}$ is $T^* \mathrm{GL}(n) \times \mathbb{C}^n \times \mathbb{C}^n$ ($n = R(\zeta)$). We take the quotient with respect to $\mathrm{GL}(n)$ (which can be regarded either as $\mathrm{GL}(R(\zeta))$ or $\mathrm{GL}(R(\zeta'))$). Then we have an isomorphism

$$\mathbf{M}_{\mathrm{tri}} /\!\!/\!\!/ \mathrm{GL}(n) \cong \mathbb{C}^n \times \mathbb{C}^n \cong \mathbf{M}_{\mathrm{tw}}(\mathbb{C}^n, \mathbb{C}),$$

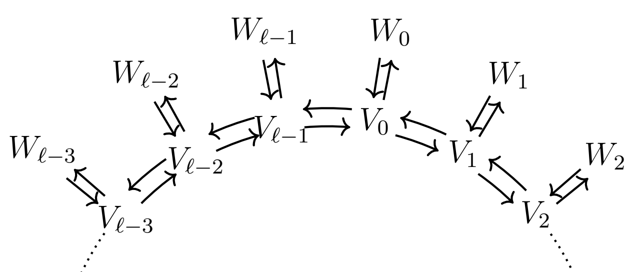
which respects the symplectic form. The residual $\mathrm{GL}(n)$-action on the left hand side is identfied with the natural $\mathrm{GL}(n)$-action on $\mathbf{M}_{\mathrm{tw}}(\mathbb{C}^n, \mathbb{C})$. If two ×'s appear consecutively as



We can take the reduction by one more $\mathrm{GL}(n)$ and get

$$\mathrm{GL}(n) \curvearrowright \mathbf{M}_{\mathrm{tw}}(\mathbb{C}^n, \mathbb{C}^2).$$

We continue this process to show that the bow variety with the cobalanced condition is the symplectic reduction of vector spaces consisting of linear maps



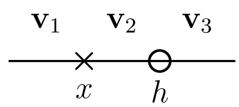
Here $\dim V_i = R(\zeta)$ for ζ between h_i and h_{i+1}, $\dim W_i$ is the number of × between h_i and h_{i+1}.

This is nothing but the definition of a quiver variety of affine type A [Nak94, Nak98].

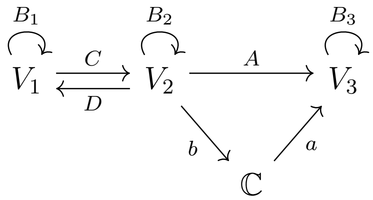
Theorem 3.5 (Cherkis [Che11]. See also [NT17, Th. 2.15])**.** *A bow variety $\mathcal{M}$ satisfying the cobalanced condition is isomorphic to a quiver variety of affine type A.*

Bow varieties satisfying the balanced condition will be studied in §4.

3(v). **Hanany-Witten transition.** Consider the following part of a bow diagram

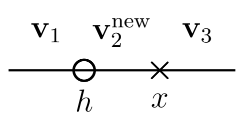


The corresponding linear maps are

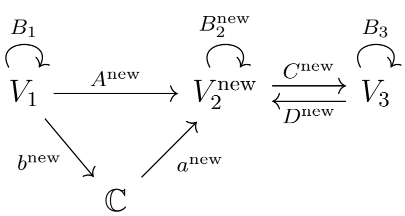


$$CD + B_2 = \nu^{\mathbb{C}}, \quad DC + B_1 = \nu^{\mathbb{C}},$$
$$B_3 A - A B_2 + ab = 0.$$

Let $\mathcal{M}$ be the quotient by $\mathrm{GL}(V_2)$. (The quotient is a principal bundle thanks to Lemma 2.13.) On the other hand $\mathcal{M}^{\mathrm{new}}$ be the corresponding part associated with



such that $\mathbf{v}_2 + \mathbf{v}_2^{\mathrm{new}} = \mathbf{v}_1 + \mathbf{v}_3 + 1$. The corresponding linear maps are



$$C^{\mathrm{new}} D^{\mathrm{new}} + B_3 = \nu^{\mathbb{C}}, \quad D^{\mathrm{new}} C^{\mathrm{new}} + B_1 = \nu^{\mathbb{C}},$$
$$B_2^{\mathrm{new}} A^{\mathrm{new}} - A^{\mathrm{new}} B_1 + a^{\mathrm{new}} b^{\mathrm{new}} = 0.$$

Proposition 3.6 (Nakajima-Takayama [NT17, Prop. 7.1])**.** *There is an explicitly defined* $\mathrm{GL}(V_1) \times \mathrm{GL}(V_3)$*-isomorphism of symplectic manifolds* $\mathcal{M} \cong \mathcal{M}^{\mathrm{new}}$*. It respects* B_1*,* B_3*.*

The vector space V_2^{new} is defined as $\operatorname{Coker} \alpha$, where α is

$$V_2 \xrightarrow{\alpha = \begin{bmatrix} D \\ A \\ b \end{bmatrix}} V_1 \oplus V_3 \oplus \mathbb{C}.$$

See the original paper for the definition of other linear maps.

3(vi). **Application of Hanany-Witten transition.** Let us consider bow varieties of finite type A, i.e., one with $R(\zeta) = 0$ for the segment ζ between h_1 and x_0. Then the bow diagram is written on a line, and we move all $\times$ to the left, and all $\bigcirc$ to the right as in (3.4) by a successive application of Hanany-Witten transitions.

Let us examine how dimensions $\mathbf{v}_1, \dots, \mathbf{v}_n, \mathbf{v}'_{\ell-1}, \dots, \mathbf{v}'_1$ can be read off from the original bow diagram before the application of Hanany-Witten transitions.

By the formula for $\mathbf{v}_2^{\mathrm{new}}$, we have $\mathbf{v}_3 - \mathbf{v}_2^{\mathrm{new}} + 1 = \mathbf{v}_2 - \mathbf{v}_1$. We understand this formula as

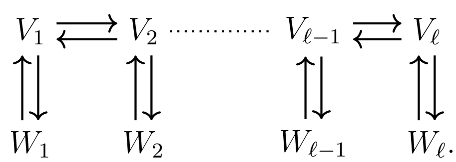
$$R(\mathrm{i}(x)) - R(\mathrm{o}(x)) + \#\{\bigcirc\text{'s left of } x\}$$

is preserved under Hanany-Witten transition. Similarly we understand $\mathbf{v}_1 - \mathbf{v}_2^{\mathrm{new}} = \mathbf{v}_2 - \mathbf{v}_3 - 1$ as

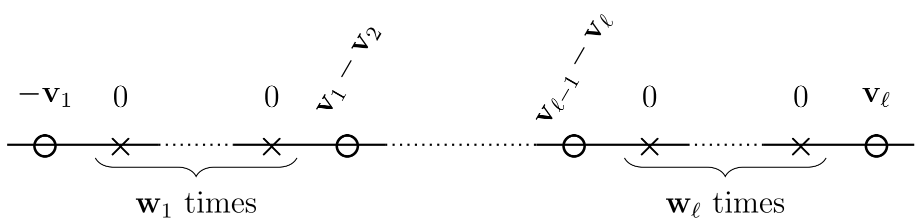
$$R(\mathrm{o}(h)) - R(\mathrm{i}(h)) + \#\{\times\text{'s right of } h\}$$

is preserved.

Consider the quiver variety associated with

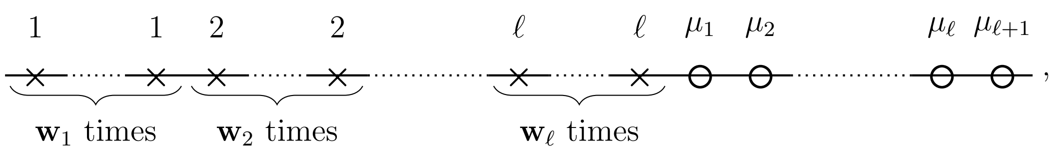
(3.7) 

Let $\mathbf{v}_i = \dim V_i$, $\mathbf{w}_i = \dim W_i$. By Theorem 3.5 the quiver variety can be written as the bow variety associated with the bow diagram satisfying the cobalanced condition:



Here we write $R(\mathrm{i}(x)) - R(\mathrm{o}(x))$ (resp. $R(\mathrm{o}(h)) - R(\mathrm{i}(h))$) above x (resp. h). We have $R(\zeta) = 0$ for the leftmost and rightmost segments ζ, hence all $R(\zeta)$ can be recovered these numbers.

After a successive application of Hanany-Witten transitions, we arrive at the bow diagram



where

$$\mu_{\ell+1} = \mathbf{v}_\ell, \quad \mu_\ell = \mathbf{v}_{\ell-1} - \mathbf{v}_\ell + \mathbf{w}_\ell, \dots,$$
$$\mu_2 = \mathbf{v}_1 - \mathbf{v}_2 + \mathbf{w}_2 + \cdots + \mathbf{w}_\ell, \quad \mu_1 = -\mathbf{v}_1 + \mathbf{w}_1 + \cdots + \mathbf{w}_\ell.$$

The left side defines a partition λ given by $[1^{\mathbf{w}_1} 2^{\mathbf{w}_2} \dots \ell^{\mathbf{w}_\ell}]$. On the other hand, the right side defines a partition, i.e. $\mu_1 \ge \mu_2 \ge \cdots \ge \mu_{\ell+1}$ if and only if

$$\mathbf{w}_1 + \mathbf{v}_2 \ge 2\mathbf{v}_1, \quad \mathbf{w}_2 + \mathbf{v}_1 + \mathbf{v}_3 \ge 2\mathbf{v}_2, \quad \dots, \quad \mathbf{w}_\ell + \mathbf{v}_{\ell-1} \ge 2\mathbf{v}_\ell. \tag{3.8}$$

By Proposition 3.3

Theorem 3.9. *When (3.8) is satisfied, the quiver variety associated with (3.7) is isomorphic to* $\overline{\mathcal{O}_{\mu^t}} \cap \mathcal{S}_\lambda$.

This result was originally proved by a different argument in [Nak94, Th. 8.4]. The above proof is given in [NT17, Th. 7.10]. This new proof has an advantage: the same proof works also for variants of the quotient discussed in §1(iii). Such a generalization for a GIT quotient was conjectured in [Nak94, Conj. 8.6] and proved by Maffei [Maf05].

3(vii). **Hamiltonian torus action.** Consider the $\mathbb{C}^\times$-action on the bottom $\mathbb{C}$ in the triangle part (2.1). It induces a $\mathbb{C}^\times$-action on $\mathbf{M}_{\mathrm{tri}}$ given by

$$(A, B_1, B_2, a, b) \mapsto (A, B_1, B_2, t^{-1}a, tb) \quad t \in \mathbb{C}^\times.$$

It is identified with the action of the diagonal subgroup $\Delta\mathbb{C}^\times \subset \mathrm{GL}(V_1) \times \mathrm{GL}(V_2)$ in (2.3). Therefore it is hamiltonian, and the moment map is given by $-\operatorname{tr} B_1 + \operatorname{tr} B_2$.

For a bow variety $\mathcal{M}$, we have the induced $(\mathbb{C}^\times)^n$-action from the $\mathbb{C}^\times$-action for each $\times$. Note that it is induced from the $\mathbb{C}^\times$-action on $\mathbb{C}$ in the triangle, not from $\Delta\mathbb{C}^\times \subset \mathrm{GL}(V_1) \times \mathrm{GL}(V_2)$. Therefore it is nontrivial on $\mathcal{M}$, which is a quotient by a product of $\mathrm{GL}(R(\zeta))$. However the action of the diagonal $\mathbb{C}^\times \subset (\mathbb{C}^\times)^n$ is absorbed into the diagonal subgroup in $\prod \mathrm{GL}(R(\zeta))$. Therefore we have the induced $(\mathbb{C}^\times)^n/\mathbb{C}^\times \cong (\mathbb{C}^\times)^{n-1}$-action.

We have an additional $\mathbb{C}^\times$-action given by

$$(A, B_1, B_2, a, b) \mapsto (tA, B_1, B_2, a, tb) \quad t \in \mathbb{C}^\times$$

at the triangle for x_0. Since it is dentified with the action of $\mathbb{C}^\times \subset \mathrm{GL}(V_1)$ on $\mathbf{M}_{\mathrm{tri}}$ for x_0, the moment map is $\operatorname{tr} B_1$.

4. Coulomb branches

Let us consider a bow variety satisfying the balanced condition. Let us take vector spaces V_i, W_i so that $\dim V_i = R(\zeta)$ for ζ between x_i and x_{i+1}, and $\dim W_i$ is equal to the number of $\bigcirc$ between x_i and x_{i+1}. This is the same rule as in §3(iv) after exchanging $\bigcirc$ and $\times$.

We consider the quiver gauge theory of

Theorem 4.1 ([NT17]). *Let $\mathcal{M}$ be a bow variety satisfying the balanced condition. We take V_i, W_i as above. Then $\mathcal{M}$ is isomorphic to the Coulomb branch of the quiver gauge theory of affine type A_{n-1} associated with V_i, W_i $(i = 0, \dots, n-1)$.*

We give a sketch of the proof in this section. See the original paper [NT17] for the full detail.

4(i). Factorization.

To be written.

4(ii). Stratification.

To be written.

5. Geometric Satake for affine type A

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Kavli Institute for the Physics and Mathematics of the Universe (WPI), The University of Tokyo, 5-1-5 Kashiwanoha, Kashiwa, Chiba, 277-8583, Japan
E-mail address: hiraku.nakajima@ipmu.jp