PERVERSE SHEAVES ON INSTANTON MODULI SPACES

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HIRAKU NAKAJIMA

Contents

Introduction

Let G be an almost simple simply-connected algebraic group over $\mathbb{C}$ with the Lie algebra $\mathfrak{g}$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. We assume G is of type ADE, as there arise technical issues for type $BCFG$. (We will remark them at relevant places.) At some points, particularly in this introduction, we want to include the case $G = \mathrm{GL}(r)$. We will not make clear distinction between the case $G = \mathrm{SL}(r)$ and $\mathrm{GL}(r)$.

Let G_c denote a maximal compact subgroup of G. Our main player is

$$\mathcal{U}_G^d = \text{the Uhlenbeck partial compactification}$$

of the moduli spaces of framed G_c-instantons on S^4 with instanton number d. The framing means a trivialization of the fiber of the G_c-bundle at $\infty \in S^4$. Framed instantons on S^4 are also called instantons on $\mathbb{R}^4$, as they extend across ∞ if their curvature is in $L^2(\mathbb{R}^4)$. We follow this convention. These spaces were first considered in a differential geometric context by Uhlenbeck, Donaldson and others, for more general 4-manifolds and usually without framing.

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The Uhlenbeck compactification has been used to define differential topological invariants of 4-manifolds, i.e., Donaldson invariants, as integral of cohomology classes over moduli spaces of instantons: Moduli spaces are noncompact, therefore the integral may diverge. Thus compactification is necessary to make the integral well-defined. (See e.g., [DK90].)

Our point of view here is different. We consider Uhlenbeck partial compactification of instanton moduli spaces on $\mathbb{R}^4$ as objects in *geometric representation theory*. We study their intersection cohomology groups and perverse sheaves in view of representation theory of the affine Lie algebra of $\mathfrak{g}$ or the closely related $\mathcal{W}$-algebra.[1] We will be concerned only with a very special 4-manifold, i.e., $\mathbb{R}^4$ (or $\mathbb{C}^2$ as we will use an algebro-geometric framework). On the other hand, we will study instantons for any group G, while $G_c = \mathrm{SU}(2)$ is usually enough for topological applications.

We will study *equivariant* intersection cohomology groups of Uhlenbeck partial compactifications

$$IH^*_{G\times\mathbb{C}^\times\times\mathbb{C}^\times}(\mathcal{U}_G^d),$$

where G acts by the change of the framing and $\mathbb{C}^\times \times \mathbb{C}^\times$ acts on $\mathbb{R}^4 = \mathbb{C}^2$. The G-action has been studied in topological context: it is important to understand singularities of instanton moduli spaces around *reducible* instantons. However the $\mathbb{C}^\times \times \mathbb{C}^\times$-action is specific for $\mathbb{R}^4$, or more generally 4-manifolds with group actions, but not arbitrary 4-manifolds.

More specifically, we will explain the author's joint work with Braverman and Finkelberg [BFN14] with an emphasis on its geometric part in this lecture series. The stable envelop introduced by Maulik-Okounkov [MO12] and its reformulation in [Nak13] via Braden's hyperbolic restriction functors are key technical tools. They also appear in other situations in geometric representation theory. Therefore those will be explained in a general framework.

Prerequisite.

- I understand that the theory of perverse sheaves is introduced in de Cataldo's lectures. I will also use materials in [CG97, §8.6], in particular the isomorphism between convolution algebras and Ext algebras.

[1] If G is not of type ADE, we need to replace the affine Lie algebra of $\mathfrak{g}$ by its Langlands dual $\mathfrak{g}^\vee_{\mathrm{aff}}$. It is a twisted affine Lie algebra, and should not be confused with the untwisted affine Lie algebra of the Langlands dual of $\mathfrak{g}$.

- We assume readers are familiar with [Nak99], at least for Chapters 2, 3, 6. (Chapter 6 presents Hilbert-Chow morphisms as examples of semi-small morphisms. They should be treated in de Cataldo's lectures.) Results explained there will be briefly recalled, but proofs are omitted.
- We will use equivariant cohomology and Borel-Moore homology groups. A brief introduction can be found in [Nak14]. We also use derived categories of equivariant sheaves. See [BL94] (and/or [Lus95, §1] for summary).
- We will not review the theory of $\mathcal{W}$-algebras, such as [FBZ04, Ch. 15]. It is not strictly necessary, but better to have some basic knowledge in order to appreciate the final result.

History. Let us explain history of study of instanton moduli spaces in geometric representation theory. We will present the actual content of this lecture series on the way.

Historically relation between instanton moduli spaces and representation theory of affine Lie algebras was first found by the author in the context of quiver varieties [Nak94]. The relation is different from one we shall study in this paper.[2] Quiver varieties are (partial compactifications of) instanton moduli spaces on $\mathbb{R}^4/\Gamma$ with gauge group $G = \mathrm{GL}(r)$. Here Γ is a finite subgroup of $\mathrm{SL}(2)$, and the corresponding affine Lie algebra is not for G: it corresponds to Γ via the McKay correspondence. The argument in [Nak94] works only for $\Gamma \neq \{1\}$. The case $\Gamma = \{1\}$ corresponds to the Heisenberg algebra, that is the affine Lie algebra for $\mathfrak{gl}(1)$. The result for $\Gamma = \{1\}$ was obtained later by Grojnowski and the author independently [Gro96, Nak97] for $r = 1$, Baranovsky [Bar00] for general r. This result will be recalled in §2, basically for the purpose to explain why they were *not enough* to draw a full picture.

In the context of quiver varieties, a $\mathbb{C}^\times$-action naturally appears from an action on $\mathbb{R}^4/\Gamma$. The equivariant K-theory of quiver varieties are related to representation theory of quantum toroidal algebras, where $\mathbb{C}^\times$ appears as a quantum parameter q. (More precisely the representation ring of $\mathbb{C}^\times$ is the Laurent polynomial ring $\mathbb{Z}[q, q^{-1}]$.) See [Nak01a]. The corresponding result for equivariant homology/affine Yangian version, which is closer to results explained in this lecture series was obtained by Varagnolo [Var00]. But these works covered only the case $\Gamma \neq \{1\}$.

[2] Results explained in this lecture series and the previous one for $\Gamma \neq \{1\}$ are roughly level-rank dual to each other, but we still lack a precise understanding. It is a good direction to pursue in future.

It is basically because the construction relies on a particular presentation of quantum toroidal algebras and affine Yangian, which is available only when $\Gamma \neq \{1\}$.

In physics side, Nekrasov [Nek03] introduced 'partition functions' which are roughtly considered as generating functions of equivariant Donaldson invariants on $\mathbb{R}^4$ with respect to the $\mathbb{C}^\times \times \mathbb{C}^\times$-action. The ordinary Donaldson invariants do not make sense for $\mathbb{R}^4$ (or S^4) as there is no interesting topology on $\mathbb{R}^4$. But equivariant Donaldson invariants are nontrivial, and contain interesting information. Nekrasov partition functions have applications to ordinary Donaldson invariants, see e.g., [GNY08, GNY11].

Another conjectural connection between affine Lie algebras and cohomology groups of instanton moduli spaces on $\mathbb{R}^4$, called the geometric Satake correspondence for affine Kac-Moody groups, was found by [BF10].[3] In this connection, the $\mathbb{C}^\times \times \mathbb{C}^\times$-action is discarded, but the intersection cohomology is considered. The main conjecture says the graded dimension of the intersection cohomology groups can be computed in terms of the q-analog of weight multiplicities of integrable representations of the affine Lie algebra of $\mathfrak{g}$.

In the frame work of quiver varieties, one can replace the Uhlenbeck partial compactification $\mathcal{U}_G^d$ by the Gieseker partial compactification. It is a symplectic resolution of singularities of $\mathcal{U}_G^d$. Then one can work on ordinary homology/K-theory of the resolution. But this is possible only for $G = \mathrm{SL}(r)$, and the intersection cohomology group of $\mathcal{U}_G^d$ is an appropriate object.

Despite its relevance for the study of Nekrasov partition function, the equivariant K-theory and homology group for the case $\Gamma = \{1\}$ were not understood for several years. They were more difficult than the case $\Gamma \neq \{1\}$, because of a technical issue mentioned above. In the context of the geometric Satake correspondence, the role of $\mathbb{C}^\times \times \mathbb{C}^\times$-action was not clear.

In 2009, Alday-Gaiotto-Tachikawa [AGT10] has connected Nekrasov partition for $G = \mathrm{SL}(2)$ with the representation theory of the Virasoro algebra via a hypothetical 6-dimensional quantum field theory. This AGT correspondence is hard to justify in a mathematically rigorous way, but yet gives a very good view point. In particular, it predicts that the equivariant intersection cohomology of Uhlenbeck space is a representation of the Virasoro algebra for $G = \mathrm{SL}(2)$, and of the $\mathcal{W}$-algebra associated with $\mathfrak{g}$ in general.

[3] The first version of the preprint was posted to arXiv in Nov. 2007. It was two years before [AGT10] was posted.

On the other hand, in mathematics side, equivariant K and homology groups for the case $\Gamma = \{1\}$ had been understood gradually around the same time. Before AGT,[4] Feigin-Tsymbaliuk and Schiffmann-Vasserot [FT11, SV13b] studied the equivariant K-theory for the $G = \mathrm{GL}(1)$-case. Then research was continued under the influence of the AGT correspondence, and the equivariant homology for the $G = \mathrm{GL}(r)$-case was studied by Schiffmann-Vasserot, Maulik-Okounkov [SV13a, MO12] for $G = \mathrm{GL}(r)$ and the homology group.

In particular, the approach taken in [MO12] is considerably different from previous ones. It does not use a particular presentation of an algebra. Rather it constructs the algebra action from the R-matrix, which naturally arises on equivariant homology group. This sounds close to a familiar RTT construction of Yangians and quantum groups, but it is more general: the R-matrix is constructed in a purely geometric way, and has infinite size in general. Also for $\Gamma \neq \{1\}$, it defines a coproduct on the affine Yangian. It was explicitly constructed for the usual Yangian for a finite dimensional complex simple Lie algebra long time ago by Drinfeld [Dri85], but the case of the affine Yangian was new. [5]

In [Nak13], the author reformulated the stable envelop, a geometric device to produce the R-matrix in [MO12], in a sheaf theoretic language, in particular using Braden's hyperbolic restriction functors [Bra03]. This reformulation is necessary in order to generalize the construction of [MO12] for $G = \mathrm{GL}(r)$ to other G. It is because the original formulation of the stable envelop required a symplectic resolution. We have a symplectic resolution of the Uhlenbeck partial compactification for $G = \mathrm{GL}(r)$, as a quiver variety, but not for general G.

Then the author together with Braverman, Finkelberg [BFN14] studies the equivariant intersection cohomology of the Uhlenbeck partial compactification, and constructs the $\mathcal{W}$-algebra action on it. Here the geometric Satake correspondence for the affine Lie algebra of $\mathfrak{g}$ gives a philosophical background: the reformulated stable envelop is used to realize the restriction to the affine Lie algebra of a Levi subalgebra of $\mathfrak{g}$. It nicely fits with Feigin-Frenkel description of the $\mathcal{W}$-algebra [FBZ04, Ch. 15].

Convention.

(1) A partition λ is a nonincreasing sequence $\lambda_1 \geq \lambda_2 \geq \cdots$ of nonnegative integers with $\lambda_N = 0$ for sufficiently large N. We

[4] Preprints of those papers were posted to arXiv, slightly before [AGT10] was appeared on arXiv.

[5] It motivated the author to define the coproduct in terms of standard generators in his joint work in progress with Guay.

set $|\lambda| = \sum \lambda_i$, $l(\lambda) = \#\{i \mid \lambda_i \neq 0\}$. We also write $\lambda = (1^{\alpha_1} 2^{\alpha_2} \cdots)$ with $\alpha_k = \#\{i \mid \lambda_i = k\}$.

(2) For a variety X, let $D^b(X)$ denote the bounded derived category of complexes of constructible $\mathbb{C}$-sheaves on X. Let $\mathrm{IC}(X_0, \mathcal{L})$ denote the intersection cohomology complex associated with a local system $\mathcal{L}$ over a Zariski open subvariety X_0 in the smooth locus of X. We denote it also by $\mathrm{IC}(X)$ if $\mathcal{L}$ is trivial. When X is smooth and irreducible, $\mathcal{C}_X$ denotes the constant sheaf on X shifted by $\dim X$. If X is a disjoint union of irreducible smooth varieties X_α, we understand $\mathcal{C}_X$ as the direct sum of $\mathcal{C}_{X_\alpha}$.

(3) We make a preferred degree shift for Borel-Moore homology groups, and denote them by $H_{[*]}(X)$, where $H_{[*]}(X) = H_{*+\dim X}(X)$ for a smooth variety X. More generally, if L is a closed subvariety in a smooth variety X, we consider $H_{[*]}(L) = H_{*+\dim X}(L)$.

Post-requisite. Further questions and open problems are listed in [BFN14, 1(xi)]. In order to do research in those directions, the followings would be necessary besides what are explained in this lecture series.

- The AGT correspondence predicts a duality between $4d$ $\mathcal{N} = 2$ SUSY quantum field theories and $2d$ conformal field theories. In order to understand it in mathematically rigorous way, one certainly needs to know the theory of vertex algebras (e.g., [FBZ04]). In fact, we still lack a fundamental understanding why equivariant intersection cohomology groups of instanton moduli spaces have structures of vertex algebras. We do want to have an *intrinsic* explanation without any computation, like checking Heisenberg commutation relations.
- The AGT correspondence was originally formulated in terms of Nekrasov partition functions. Their mathematical background is given for example in [NY04].
- In view of the geometric Satake correspondence for affine Kac-Moody groups [BF10], the equivariant intersection cohomology group $IH^*_{G\times\mathbb{C}^\times\times\mathbb{C}^\times}$ for the Uhlenbeck partial compactification of instanton moduli spaces of $\mathbb{R}^4/\mathbb{Z}_\ell$ should be understood in terms of representations of the affine Lie algebra of $\mathfrak{g}$ and the corresponding generalized $\mathcal{W}$-algebra. We believe that necessary technical tools are more or less established in [BFN14], but it still needed to be worked out in detail. Anyhow, one certainly needs knowledge of $\mathcal{W}$-algebras in order to study their generalization.

Acknowledgement. I am grateful to A. Braverman and M. Finkelberg for the collaboration [BFN14], on which the lecture series is based. I still vividly remember when Sasha said that we have symplectic resolution only in rare cases, hence we need to study singular spaces. It was one of motivations for me to start instanton moduli spaces for general groups, eventually it led me to the collaboration.

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1. Uhlenbeck partial compactification – in brief

I suppose that Uhlenbeck partial compactification of an instanton moduli space on $\mathbb{R}^4$ is introduced in Braverman's lectures. We do not use its detailed properties much.

We will use its stratification:

$$\mathcal{U}_G^d = \bigsqcup_{0\le d'\le d} \operatorname{Bun}_G^{d'} \times S^{d-d'}(\mathbb{C}^2), \tag{1.1}$$

where $\operatorname{Bun}_G^{d'}$ is the moduli space of framed holomorphic (or equivalently algebraic) G-bundles $\mathcal{F}$ over $\mathbb{P}^2$ of instanton number d', where the framing is the trivialization φ of the restriction of $\mathcal{F}$ at the line ℓ_∞ at infinity.

By an analytic result due to Bando [Ban93], we can replace instanton moduli spaces on $\mathbb{R}^4$ by Bun_G^d. We use algebro-geometric approaches to instanton moduli spaces hereafter.

Here the instanton number is the integration over $\mathbb{P}^2$ of the characteristic class of G-bundles corresponding to the invariant inner product on $\mathfrak{g}$, normalized as $(\theta,\theta)=2$ for the highest root θ.[6]

Let us refine the stratification in the symmetric product $S^{d-d'}(\mathbb{C}^2)$ part:

$$\mathcal{U}_G^d = \bigsqcup_{d=|\lambda|+d'} \operatorname{Bun}_G^{d'} \times S_\lambda(\mathbb{C}^2), \tag{1.2}$$

where $S_\lambda(\mathbb{C}^2)$ consists of configurations of points in $\mathbb{C}^2$ whose multiplicities are given by the partition λ.

[6]When an embedding $\mathrm{SL}(2)\to G$ corresponding to a coroot $\alpha^\vee$ is gives, we can induction a G-bundle $\mathcal{F}$ from a $\mathrm{SL}(2)$-bundle $\mathcal{F}_{\mathrm{SL}(2)}$ Then we have $d(\mathcal{F}) = d(\mathcal{F}_{\mathrm{SL}(2)})\times(\alpha^\vee,\alpha^\vee)/2$. Instanton numbers are preserved if G is type ADE, but not in general.

We also use that Bun_G^d is a smooth locus of $\mathcal{U}_G^d$, and its tangent space at $(\mathcal{F}, \varphi)$ is

$$H^1(\mathbb{P}^2, \mathfrak{g}_{\mathcal{F}}(-\ell_\infty)),$$

where $\mathfrak{g}_{\mathcal{F}}$ is the associated vector bundle $\mathcal{F} \times_G \mathfrak{g}$.

We will also use the factorization morphism

$$\pi^d \colon \mathcal{U}_G^d \to S^d\mathbb{C}^1.$$

The definition is given in [BFG06, §6.4]. It depends on a choice of the projection $\mathbb{C}^2 \to \mathbb{C}^1$. We do not recall the definition, but its crucial properties are

(1) On the factor $S^{d-d'}(\mathbb{C}^2)$, it is given by the projection $\mathbb{C}^2 \to \mathbb{C}^1$.
(2) Consider $C_1 + C_2 \in S^d\mathbb{C}^1$ such that $C_1 \in S^{d_1}\mathbb{C}^1$, $C_2 \in S^{d_2}\mathbb{C}^1$ are disjoint. Then $(\pi^d)^{-1}(C_1 + C_2)$ is isomorphic to $(\pi^{d_1})^{-1}(C_1) \times (\pi^{d_2})^{-1}(C_2)$.

Intuitively π^d is given as follows. By (1), it is enough to consider the case of *genuine* framed G-bundles $(\mathcal{F}, \varphi)$. Let $x \in \mathbb{C}^1$ and consider the line $\mathbb{P}^1_x = \{z_1 = xz_0\}$ in $\mathbb{P}^2$. If $x = \infty$, we can regard $\mathbb{P}^1_x$ as the line ℓ_∞ at infinity. If we restrict the G-bundle $\mathcal{F}$ to ℓ_∞, it is trivial. Since the triviality is an open condition, the restriction $\mathcal{F}|_{\mathbb{P}^1_x}$ is also trivial except for finitely many $x \in \mathbb{C}^1$, say $x_1, x_2 \dots, x_k$. Then $\pi^d(\mathcal{F}, \varphi)$ is the sum $x_1 + x_2 + \cdots + x_k$ if we assign the multiplicity appropriately.

2. Heisenberg algebra action

This section is an introduction to the actual content of lectures. We consider instanton moduli spaces when the gauge group G is $\mathrm{SL}(r)$. We will explain results about (intersection) cohomology groups of instanton moduli spaces known before the AGT correspondence was found. Then it will be clear what were lacking, and readers are motivated to learn more recent works.

2(i). **Gieseker partial compactification.** When the gauge group G is $\mathrm{SL}(r)$, we denote the corresponding Uhlenbeck partial compactification $\mathcal{U}_G^d$ by $\mathcal{U}_r^d$.

For $\mathrm{SL}(r)$, we can consider a modification $\widetilde{\mathcal{U}}_r^d$ of $\mathcal{U}_r^d$, called the *Gieseker partial compactification*. It is a moduli space of framed torsion free sheaves (E, φ) on $\mathbb{P}^2$, where the framing φ is a trivialization of the restriction of E to the line at infinity ℓ_∞. It is known that $\widetilde{\mathcal{U}}_r^d$ is a smooth (holomorphic) symplectic manifold. It is also known that there is a projective morphism $\pi\colon \widetilde{\mathcal{U}}_r^d \to \mathcal{U}_r^d$, which is a resolution of singularities.

When $r = 1$, the group $\mathrm{SL}(1)$ is trivial. But the Giesker space is nontrivial: $\widetilde{\mathcal{U}}_1^d$ is the Hilbert scheme $(\mathbb{C}^2)^{[d]}$ of d points on the plane $\mathbb{C}^2$, and the Uhlenbeck partial compactification[7] $\mathcal{U}_1^d$ is the d^{th} symmetric product $S^d(\mathbb{C}^2)$ of $\mathbb{C}^2$. The former parametrizes ideals I in the polynomial ring $\mathbb{C}[x,y]$ of two variables with colength d, i.e., $\dim \mathbb{C}[x,y]/I = d$. The latter is the quotient of the Cartesian product $(\mathbb{C}^2)^d$ by the symmetric group S_d of d letters. It parametrized d unordered points in $\mathbb{C}^2$, possibly with multiplicities. We will use the summation notation like $p_1 + p_2 + \cdots + p_d$ or $d \cdot p$ to express a point in $S^d(\mathbb{C}^2)$.

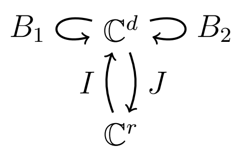

FIGURE 1. Quiver varieties of Jordan type

For general r, these spaces can be understood as quiver varieties associated with Jordan quiver. It is not our intension to explain the theory of quiver varieties, but here is the definition in this case: Take two complex vector spaces of dimension d, r respectively. Consider linear maps B_1, B_2, I, J as in Figure 1. We impose the equation

$$\mu(B_1, B_2, I, J) \overset{\text{def.}}{=} [B_1, B_2] + IJ = 0$$

Then we take two types of quotient of $\mu^{-1}(0)$ by $\mathrm{GL}(d)$. The first one corresponds to $\mathcal{U}_r^d$, and is the affine algebro-geometric quotient $\mu^{-1}(0)/\!\!/\mathrm{GL}(d)$. It is defined as the spectrum of $\mathbb{C}[\mu^{-1}(0)]^{\mathrm{GL}(d)}$, the ring of $\mathrm{GL}(d)$-invariant polynomials on $\mu^{-1}(0)$. Set-theoretically it is the space of closed $\mathrm{GL}(d)$-orbits in $\mu^{-1}(0)$. The second quotient corresponds to $\widetilde{\mathcal{U}}_r^d$, and is the geometric invariant theory quotient with respect to the polarization given by the determinant of $\mathrm{GL}(d)$. It is Proj of $\bigoplus_{n\ge 0} \mathbb{C}[\mu^{-1}(0)]^{\mathrm{GL}(d),\det^n}$, the ring of $\mathrm{GL}(d)$-semi-invariant polynomials. Set-theoretically it is the quotient of *stable* points in $\mu^{-1}(0)$ by $\mathrm{GL}(d)$. Here (B_1, B_2, I, J) is stable if there is no proper subspace T of $\mathbb{C}^d$ which is invariant under B_1, B_2 and is containing the image of I.

From this description, we can check the stratification (1.1). If (B_1, B_2, I, J) has a closed $\mathrm{GL}(d)$-orbit, it is *semisimple*, i.e., a 'submodole' (in appropriate sense) has a complementary submodule. Thus (B_1, B_2, I, J) decomposes into a direct sum of *simple* modules, which do not have nontrivial submodules. There is exactly one simple summand with

[7]Since $\mathrm{Bun}^d_{\mathrm{GL}(1)} = \emptyset$ unless $d = 0$, this is a confusing name.

nontrival I, J, and all others have $I = J = 0$. The former gives a point in $\mathrm{Bun}_G^{d'}$. The latter is a pair of commuting matrices $[B_1, B_2] = 0$, and the simplicity means that the size of matrices is 1. Therefore the simultaneous eigenvalues give a point in $\mathbb{C}^2$.

Let us briefly recall how those linear maps determine points in $\widetilde{\mathcal{U}}_r^d$ and $\mathcal{U}_r^d$. The detail was given in [Nak99, Ch. 2]. Given (B_1, B_2, I, J), we consider the following complex

$$\mathcal{O}^{\oplus d}(-1) \xrightarrow{a} \begin{matrix} \mathcal{O}^{\oplus d} \\ \oplus \\ \mathcal{O}^{\oplus d} \\ \oplus \\ \mathcal{O}^{\oplus r} \end{matrix} \xrightarrow{b} \mathcal{O}^{\oplus d}(1),$$

where

$$a = \begin{pmatrix} z_0 B_1 - z_1 \\ z_0 B_2 - z_2 \\ z_0 J \end{pmatrix}, \qquad b = \begin{pmatrix} -(z_0 B_2 - z_2) & z_0 B_1 - z_1 & z_0 I \end{pmatrix}.$$

Here $[z_0 : z_1 : z_2]$ is the homogeneous coordinate system of $\mathbb{P}^2$ such that $\ell_\infty = \{z_0 = 0\}$. The equation $\mu = 0$ guarantees that this is a complex, i.e., $ba = 0$. One sees easily that a is injective on each fiber over $[z_0 : z_1 : z_2]$ except for finitely many. The stability condition implies that b is surjective on each fiber. It implies that $E \stackrel{\text{def.}}{=} \operatorname{Ker} b / \operatorname{Im} a$ is a torsion free sheaf of rank r with $c_2 = d$. Considering the restriction to $z_0 = 0$, one sees that E has a canonical trivialization φ there. Thus we obtain a framed sheaf (E, φ) on $\mathbb{P}^2$.

Proposition 2.1 ([Nak99, Exercise 5.15])**.** *$\pi\colon \widetilde{\mathcal{U}}_r^d \to \mathcal{U}_r^d$ is semi-small with respect to the stratification (1.2). Moreover the fiber $\pi^{-1}(x)$ is irreducible*[8] *at any point $x \in \mathcal{U}_r^d$.*

Recall that a (surjective) projective morphism $\pi\colon M \to X$ from a nonsingular variety M is *semi-small* if X has a stratification $X = \bigsqcup X_\alpha$ such that $\pi|_{\pi^{-1}(X_\alpha)}$ is a topological fibration, and $\dim \pi^{-1}(x_\alpha) \le \frac{1}{2} \operatorname{codim} X_\alpha$ for $x_\alpha \in X_\alpha$.

This semi-smallness result is proved for general symplectic resolutions by Kaledin [Kal06].

[Nak99, Exercise 5.15] asks the dimension of the central fiber $\pi^{-1}(d \cdot 0)$. Let us explain why the estimate for the central fiber is enough. Let us take $x \in \mathcal{U}_r^d$ and write it as $(F, \varphi, \sum \lambda_i x_i)$, where $(E, \varphi) \in \mathrm{Bun}_{\mathrm{SL}(r)}^{d'}$,

[8] The (solution of) exercise only shows there is only one irreducible component of $\pi^{-1}(x_\alpha)$ with dimension $\frac{1}{2} \operatorname{codim} X_\alpha$. The irreducibility was proved by Baranovsky and Ellingsrud-Lehn.

$x_i \neq x_j$. The morphism π is assigning $(E^{\vee\vee}, \varphi, \operatorname{Supp}(E^{\vee\vee}/E))$ to a framed torsion free sheaf $(E, \varphi) \in \widetilde{\mathcal{U}}_r^d$. See [Nak99, Exercise 3.53]. Then $\pi^{-1}(x)$ parametrizes quotients of $E^{\vee\vee}$ with given multiplicities λ_i at x_i. Then it is clear that $\pi^{-1}(x)$ is isomorphic to the product of quotients of $\mathcal{O}^{\oplus r}$ with multiplicities λ_i at 0, i.e., $\prod_i \pi^{-1}(\lambda_i \cdot 0)$. If one knows each $\pi^{-1}(\lambda_i \cdot 0)$ has dimension $r\lambda_i - 1$, we have $\dim \pi^{-1}(x) = \sum (r\lambda_i - 1) = \frac{1}{2} \operatorname{codim} \operatorname{Bun}^{d'}_{\mathrm{SL}(r)} \times S_\lambda(\mathbb{C}^2)$. Thus it is enough to check that $\pi^{-1}(d \cdot 0) = rd - 1$.

We have an action of a group G on $\widetilde{\mathcal{U}}_r^d$, $\mathcal{U}_r^d$ by the change of framing. We also have an action of $\mathrm{GL}(2)$ on $\widetilde{\mathcal{U}}_r^d$, $\mathcal{U}_r^d$ induced from the $\mathrm{GL}(2)$ action on $\mathbb{C}^2$. In this paper, we only consider the action of the subgroup $\mathbb{C}^\times \times \mathbb{C}^\times$ in $\mathrm{GL}(2)$. Let us introduce the following notation: $\mathbb{G} = G \times \mathbb{C}^\times \times \mathbb{C}^\times$. Taking a maximal torus T of G, we also introduce $\mathbb{T} = T \times \mathbb{C}^\times \times \mathbb{C}^\times$.

We will study the equivariant cohomology groups of Gieseker spaces

$$H^{[*]}_{\mathbb{G}}(\widetilde{\mathcal{U}}_r^d), \quad H^{[*]}_{\mathbb{G},c}(\widetilde{\mathcal{U}}_r^d)$$

with arbitrary and compact support respectively. We will also consider $\mathbb{T}$-equivariant cohomology groups.

We take the following convention on the degree: The degree $2dr$, which is the complex dimension of $\widetilde{\mathcal{U}}_r^d$, in the usual convention is our degree 0. The same applies to $H^{[*]}_{\mathbb{C}^\times \times \mathbb{C}^\times}(\mathbb{C}^2)$. The degree 2 in the usual convention is our degree 0.

We denote the equivariant variables by $\vec{a} = (a_1, \dots, a_r)$ with $a_1 + \cdots + a_r = 0$ for T, and ε_1, ε_2 for $\mathbb{C}^\times \times \mathbb{C}^\times$. Therefore

$$H^*_T(\mathrm{pt}) \cong \mathbb{C}[\operatorname{Lie} T] \cong \mathbb{C}[\vec{a}], \qquad H^*_{\mathbb{C}^\times \times \mathbb{C}^\times}(\mathrm{pt}) \cong \mathbb{C}[\varepsilon_1, \varepsilon_2].$$

We have the intersection pairing

$$H^{[*]}_{\mathbb{G}}(\widetilde{\mathcal{U}}_r^d) \otimes H^{[*]}_{\mathbb{G},c}(\widetilde{\mathcal{U}}_r^d) \to H^*_{\mathbb{G}}(\mathrm{pt}); c \otimes c' \mapsto (-1)^{dr} \int_{\widetilde{\mathcal{U}}_r^d} c \cup c'.$$

This is of degree 0. The sign $(-1)^{dr}$ is introduced to save $(-1)^\star$ in the later formula. The factor dr should be understood as the half of the dimension of $\widetilde{\mathcal{U}}_r^d$. Similarly the intersection form on $H^{[*]}_{\mathbb{C}^\times \times \mathbb{C}^\times}(\mathbb{C}^2)$ has the sign factor $(-1)^1 = (-1)^{\dim \mathbb{C}^2/2}$.

Exercise 1. (a) We define the factorization morphism π^d for $G = \mathrm{SL}(r)$ in terms of (B_1, B_2, I, J). Let $\pi^d([B_1, B_2, I, J]) \in S^d\mathbb{C}$ be the spectrum of B_1 counted with multiplicities. Check that π^d satisfies the properties (1),(2) above.

(b) Check that $\mathcal{F}|_{\mathbb{P}^1_x}$ is trivial if $B_1 - x$ is invertible.

More generally one can define the projection as the spectrum of $a_1B_1 + a_2B_2$ for $(a_1, a_2) \in \mathbb{C}^2 \setminus \{0\}$, but it is enough to check this case after a rotation by the GL(2)-action.

2(ii). **Heisenberg algebra via correspondences.** Let $n > 0$. Let us consider

$$P_n \overset{\text{def.}}{=} \left\{(E_1, \varphi_1, E_2, \varphi_2, x) \in \widetilde{\mathcal{U}}_r^{d+n} \times \widetilde{\mathcal{U}}_r^{d} \times \mathbb{C}^2 \middle| E_1 \subset E_2, \operatorname{Supp}(E_2/E_1) = \{x\}\right\}.$$

Here the condition $\operatorname{Supp}(E_2/E_1) = \{x\}$ means the quotient sheaf E_2/E_1 is 0 outside x. In the left hand side the index d is omitted: we understand either P_n is the disjoint union for various d, or d is implicit from the situation. It is known that P_n is a lagrangian subvariety in $\widetilde{\mathcal{U}}_r^{d+n} \times \widetilde{\mathcal{U}}_r^{d} \times \mathbb{C}^2$.

We have two projections $q_1 \colon P_n \to \widetilde{\mathcal{U}}_r^{d+n}$, $q_2 \colon P_n \to \widetilde{\mathcal{U}}_r^{d} \times \mathbb{C}^2$, which are proper. The convolution product gives an operator

$$P^\Delta_{-n}(\alpha) \colon H^{[*]}_{\mathbb{G}}(\widetilde{\mathcal{U}}_r^{d}) \to H^{[*+\deg\alpha]}_{\mathbb{G}}(\widetilde{\mathcal{U}}_r^{d+n}); c \mapsto q_{1*}\left(q_2^*(c\otimes\alpha) \cap [P_n]\right)$$

for $\alpha \in H^{[*]}_{\mathbb{C}^\times\times\mathbb{C}^\times}(\mathbb{C}^2)$. The meaning of the notation 'Δ' will be explained later. We also consider the adjoint operator

$$P^\Delta_{n}(\alpha) = (P^\Delta_{-n}(\alpha))^* \colon H^{[*]}_{\mathbb{G},c}(\widetilde{\mathcal{U}}_r^{d+n}) \to H^{[*+\deg\alpha]}_{\mathbb{G},c}(\widetilde{\mathcal{U}}_r^{d}).$$

A class $\beta \in H^{[*]}_{\mathbb{C}^\times\times\mathbb{C}^\times,c}(\mathbb{C}^2)$ with compact support gives operators $P^\Delta_{-n}(\beta) \colon H^{[*]}_{\mathbb{G},c}(\widetilde{\mathcal{U}}_r^{d}) \to H^{[*+\deg\beta]}_{\mathbb{G},c}(\widetilde{\mathcal{U}}_r^{d+n})$, and $P^\Delta_{n}(\beta) = (P^\Delta_{-n}(\beta))^* \colon H^{[*]}_{\mathbb{G}}(\widetilde{\mathcal{U}}_r^{d+n}) \to H^{[*+\deg\beta]}_{\mathbb{G}}(\widetilde{\mathcal{U}}_r^{d})$.

Theorem 2.2 ([Gro96, Nak97] for $r = 1$, [Bar00] for $r \geq 2$)**.** *As operators on $\bigoplus_d H^{[*]}_{\mathbb{G}}(\widetilde{\mathcal{U}}_r^{d})$ or $\bigoplus_d H^{[*]}_{\mathbb{G},c}(\widetilde{\mathcal{U}}_r^{d})$, we have the Heisenberg commutator relations*

$$\left[P^\Delta_m(\alpha), P^\Delta_n(\beta)\right] = rm\delta_{m,-n}\langle\alpha,\beta\rangle \operatorname{id}. \tag{2.3}$$

Here α, β are equivariant cohomology classes on $\mathbb{C}^2$ with arbitrary or compact support. When the right hand side is nonzero, m and n have different sign, hence one of α, β is compact support and the other is arbitrary support. Then $\langle\alpha,\beta\rangle$ is well-defined.

Historical Comment 2.4. As mentioned in Introduction, the author [Nak94] found relation between representation theory of affine Lie algebras and moduli spaces of instantons on $\mathbb{C}^2/\Gamma$, where the affine Lie algebra is given by Γ by the McKay correspondence. It was motivated by works by Ringel [Rin90] and Lusztig [Lus90], constructing upper triangular subalgebras of quantum enveloping algebras by representations of quivers.

The above theorem can be regarded as the case $\Gamma = \{1\}$, but the Heisenberg algebra is not a Kac-Moody Lie algebra, and hence it was not covered in [Nak94], and dealt with later [Gro96, Nak97, Bar00]. Note that a Kac-Moody Lie algebra only has finitely many generators and relations, while the Heisenberg algebra has infinitely many.

A particular presentation of an algebra should *not* be fundamental, so it was desirable to have more intrinsic construction of those representations. More precisely, a definition of an algebra by convolution products is natural, but we would like to understand why we get a particular algebra, namely the affine Lie algebra in our case. We do not have a satisfactory explanation yet. The same applies to Ringel, Lusztig's constructions.

When $r = 1$, it is known that the generating function of dimension of $H^{[*]}_{\mathbb{G}}((\mathbb{C}^2)^{[d]})$ (over $H^{[*]}_{\mathbb{G}}(\mathrm{pt})$) for $d \geq 0$ is

$$\sum_{d=0}^{\infty} \dim H^{[*]}_{\mathbb{G}}((\mathbb{C}^2)^{[d]}) q^d = \prod_{d=1}^{\infty} \frac{1}{1-q^d}.$$

(See [Nak99, Chap. 5].) This is also equal to the character of the Fock space[9] of the Heisenberg algebra. Therefore the Heisenberg algebra action produces all cohomology classes from the vacuum vector $|\mathrm{vac}\rangle = 1_{(\mathbb{C}^2)^{[0]}} \in H^{[*]}_{\mathbb{G}}((\mathbb{C}^2)^{[0]})$.

On the other hand we have

$$\sum_{d=0}^{\infty} \dim H^{[*]}_{\mathbb{G}}(\widetilde{\mathcal{U}}^d_r) q^d = \prod_{d=1}^{\infty} \frac{1}{(1-q^d)^r}. \tag{2.5}$$

for general r. Therefore the Heisenberg algebra is smaller than the actual symmetry of the cohomology groups.

Let us explain how to see the formula (2.5). Consider the torus T action on $\widetilde{\mathcal{U}}^d_r$. A framed torsion free sheaf (E, φ) is fixed by T if and only if it is a direct sum $(E_1, \varphi_1) \oplus \cdots \oplus (E_r, \varphi_r)$ of rank 1 framed torsion free sheaves. Rank 1 framed torsion free sheaves are nothing but ideal sheaves on $\mathbb{C}^2$, hence points in Hilbert schemes. Thus

$$(\widetilde{\mathcal{U}}^d_r)^T = \bigsqcup_{d_1+\cdots+d_r=d} (\mathbb{C}^2)^{[d_1]} \times \cdots \times (\mathbb{C}^2)^{[d_r]}. \tag{2.6}$$

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[9] The Fock space is the polynomial ring of infinitely many variables $x_1, x_2, \dots$. The operators $P^\Delta_n(\alpha)$ act by either multiplication of x_n or differentiation with respect to x_n with appropriate constant multiplication. It has the highest weight vector (or the vacuum vector) 1, which is killed by $P^\Delta_n(\alpha)$ with $n > 0$. The Fock space is spanned by vectors given by operators $P^\Delta_n(\alpha)$ successively to the highest weight vector.

To compute the dimension of $H^{[*]}_{\mathbb{G}}(\widetilde{\mathcal{U}}^d_r)$ over $H^*_{\mathbb{G}}(\mathrm{pt}) = \mathbb{C}[\mathrm{Lie}\,\mathbb{G}]^{\mathbb{G}}$, we restrict equivariant cohomology groups to generic points, that is to consider tensor products with the fractional field $\mathbb{C}(\mathrm{Lie}\,\mathbb{T})$ of $H^*_{\mathbb{T}}(\mathrm{pt}) = \mathbb{C}[\mathrm{Lie}\,\mathbb{T}]$. Then the localization theorem for equivariant cohomology groups gives an isomorphism between $H^{[*]}_{\mathbb{G}}(\widetilde{\mathcal{U}}^d_r)$ and $H^{[*]}_{\mathbb{G}}((\widetilde{\mathcal{U}}^d_r)^T)$ over $\mathbb{C}(\mathrm{Lie}\,\mathbb{T})$. Therefore the above observation gives the formula (2.5).

In view of (2.5), we have the action of r copies of Heisenberg algebra on $\bigoplus_d H^{[*]}_{\mathbb{G}}((\widetilde{\mathcal{U}}^d_r)^T)$ and hence on $\bigoplus_d H^{[*]}_{\mathbb{G}}(\widetilde{\mathcal{U}}^d_r) \otimes_{H^*_{\mathbb{G}}(\mathrm{pt})} \mathbb{C}(\mathrm{Lie}\,\mathbb{T})$ by the localization theorem. It is isomorphic to the tensor product of r copies of the Fock module, so all cohomology classes are produced by the action. This is a good starting point to understand $\bigoplus_d H^{[*]}_{\mathbb{G}}(\widetilde{\mathcal{U}}^d_r)$. However this action cannot be defined over *non localized* equivariant cohomology groups. In fact, $P^\Delta_n(\alpha)$ is the 'diagonal' Heisenberg in the product, and other non diagonal generators have no description like convolution by P_n.

The correct algebra acting on $\bigoplus_d H^{[*]}_{\mathbb{G}}(\widetilde{\mathcal{U}}^d_r)$ is the W-algebra $\mathcal{W}(\mathfrak{gl}(r))$ associated with $\mathfrak{gl}(r)$. It is the tensor product of the W-algebra $\mathcal{W}(\mathfrak{sl}(r))$ and the Heisenberg algebra (as the vertex algebra). Its Verma module has the same size as the tensor product of r copies of the Fock module.

This result is due to Schiffmann-Vasserot [SV13a] and Maulik-Okounkov [MO12] independently.

Exercise 2. [Nak99, Remark 8.19] Define operators $P^\Delta_{\pm 1}(\alpha)$ acting on $\bigoplus_n H^*(S^n X)$ for a (compact) manifold X in a similar way, and check the commutation relation (2.3) with $r = 1$.

2(iii). **Intersection cohomology group.** Recall that the decomposition theorem has a nice form for a semi-small resolution $\pi\colon M \to X$:

$$\pi_*(\mathcal{C}_M) = \bigoplus_{\alpha,\chi} \mathrm{IC}(X_\alpha, \chi) \otimes H_{[0]}(\pi^{-1}(x_\alpha))_\chi, \tag{2.7}$$

where we have used the following notation:

- $\mathcal{C}_M$ denotes the shifted constant sheaf $\mathbb{C}_M[\dim M]$.
- $\mathrm{IC}(X_\alpha, \chi)$ denotes the intersection cohomology complex associated with a simple local system χ on X_α.
- $H_{[0]}(\pi^{-1}(x_\alpha))$ is the homology group of the shifted degree 0, which is the usual degree $\operatorname{codim} X_\alpha$. When x_α moves in X_α, it forms a local system. $H_{[0]}(\pi^{-1}(x_\alpha))_\chi$ denotes its χ-isotropic component.

Exercise 3. Let $\mathrm{Gr}(d, r)$ be the Grassmannian of d-dimensional subspaces in $\mathbb{C}^r$, where $0 \le d \le r$. Let $M = T^*\mathrm{Gr}(d, r)$. Determine $X = \mathrm{Spec}(\mathbb{C}[M])$. Study fibers of the affinization morphism

$\pi\colon M \to X$ and show that π is semi-small. Compute graded dimensions of IH^* of strata, using the well-known computation of Betti numbers of $T^* \operatorname{Gr}(d, R)$.

Consider the Gieseker-Uhlenbeck morphism $\pi\colon \widetilde{\mathcal{U}}_r^d \to \mathcal{U}_r^d$. By Proposition 2.1, any fiber $\pi^{-1}(x_\alpha)$ is irreducible. Therefore all the local systems are trivial, and

$$\pi_*(\mathcal{C}_{\widetilde{\mathcal{U}}_r^d}) = \bigoplus_{d=|\lambda|+d'} \operatorname{IC}(\operatorname{Bun}^{d'}_{\mathrm{SL}(r)} \times S_\lambda(\mathbb{C}^2)) \otimes \mathbb{C}[\pi^{-1}(x_\lambda^{d'})],$$

where $x_\lambda^{d'}$ denotes a point in the stratum $\operatorname{Bun}^{d'}_{\mathrm{SL}(r)} \times S_\lambda(\mathbb{C}^2)$, and $[\pi^{-1}(x_\lambda^{d'})]$ denotes the fundamental class of $\pi^{-1}(x_\lambda^{d'})$, regarded as an element of $H_{[0]}(\pi^{-1}(x_\lambda^{d'}))$.

The main summand is $\operatorname{IC}(\operatorname{Bun}^d_{\mathrm{SL}(r)})$, and other smaller summands could be understood recursively as follows. Let us write the partition λ as $(1^{\alpha_1} 2^{\alpha_2} \dots)$, when 1 appears α_1 times, 2 appears α_2 times, and so on. We set $l(\lambda) = \alpha_1 + \alpha_2 + \cdots$ and $\operatorname{Stab}(\lambda) = S_{\alpha_1} \times S_{\alpha_2} \times \cdots$. Note that we have total $l(\lambda)$ distinct points in $S_\lambda(\mathbb{C}^2)$. The group $\operatorname{Stab}(\lambda)$ is the group of symmetries of a configuration in $S_\lambda(\mathbb{C}^2)$. We have a finite morphism

$$\xi\colon \mathcal{U}^{d'}_{\mathrm{SL}(r)} \times (\mathbb{C}^2)^{l(\lambda)} / \operatorname{Stab}(\lambda) \to \overline{\operatorname{Bun}^{d'}_{\mathrm{SL}(r)} \times S_\lambda(\mathbb{C}^2)}$$

extending the identity on $\operatorname{Bun}^{d'}_{\mathrm{SL}(r)} \times S_\lambda(\mathbb{C}^2)$. Then $\operatorname{IC}(\operatorname{Bun}^{d'}_{\mathrm{SL}(r)} \times S_\lambda(\mathbb{C}^2))$ is the direct image of IC of the domain. We have the Künneth decomposition for the domain, and the factor $(\mathbb{C}^2)^{l(\lambda)} / \operatorname{Stab}(\lambda)$ is a quotient of a smooth space by a finite group. Therefore the IC of the second factor is the (shifted) constant sheaf. We thus have

$$\operatorname{IC}(\operatorname{Bun}^{d'}_{\mathrm{SL}(r)} \times S_\lambda(\mathbb{C}^2)) \cong \xi_* \left(\operatorname{IC}(\operatorname{Bun}^{d'}_{\mathrm{SL}(r)}) \boxtimes \mathcal{C}_{(\mathbb{C}^2)^{l(\lambda)} / \operatorname{Stab}(\lambda)} \right).$$

Thus

$$H^{[*]}_{\mathbb{G}}(\widetilde{\mathcal{U}}_r^d) = \bigoplus_{d=|\lambda|+d'} IH^{[*]}_{\mathbb{G}}(\operatorname{Bun}^{d'}_{\mathrm{SL}(r)}) \otimes H^{[*]}_{\mathbb{G}}((\mathbb{C}^2)^{l(\lambda)} / \operatorname{Stab}(\lambda)) \otimes \mathbb{C}[\pi^{-1}(x_\lambda^{d'})].$$

This decomposition nicely fits with the Heisenberg algebra action. Note that the second and third factors are both 1-dimensional. Thus we have 1-dimensional space for each partition λ. If we take the sum over d, it has the size of the Fock module, and it is indeed the submodule generated by the vacuum vector $|\mathrm{vac}\rangle = 1_{\widetilde{\mathcal{U}}_r^0} \in H^{[*]}(\widetilde{\mathcal{U}}_r^d)$. This statement can be proved by the analysis of the convolution algebra in

[CG97, Chap. 8], but it is intuitively clear as the 1-dimensional space corresponding to λ is the span of $P^{\Delta}_{-1}(1)^{\alpha_1} P^{\Delta}_{-2}(1)^{\alpha_2} \cdots |\mathrm{vac}\rangle$.

The Heisenberg algebra acts trivially on the remaining factor

$$\bigoplus_d IH^{[*]}_{\mathbb{G}}(\mathrm{Bun}^d_{\mathrm{SL}(r)}).$$

The goal of this paper is to see that it is a module of $\mathcal{W}(\mathfrak{sl}(r))$, and the same is true for ADE groups G, not only for $\mathrm{SL}(r)$.

Exercise 4. Show the above assertion that the Heisenberg algebra acts trivially on the first factor $\bigoplus_d IH^{[*]}_{\mathbb{G}}(\mathrm{Bun}^d_{\mathrm{SL}(r)})$.

3. Stable envelops

The purpose of this lecture is to explain the stable envelop introduced in [MO12]. It will nicely explain a relation between $H^{[*]}_{\mathbb{T}}(\widetilde{\mathcal{U}}^d_r)$ and $H^{[*]}_{\mathbb{T}}((\widetilde{\mathcal{U}}^d_r)^T)$. This is what we need to clarify, as we have explained in the previous lecture. The stable envelop also arises in many other situations in geometric representation theory. Therefore we explain it in a wider context, as in the original paper [MO12].

3(i). **Setting – symplectic resolution.** Let $\pi\colon M \to X$ be a resolution of singularities of an affine algebraic variety X. We assume M is symplectic. We suppose that a torus $\mathbb{T}$ acts on both M, X so that π is $\mathbb{T}$-equivariant. We suppose $\mathbb{T}$-action on X is linear. Let T be a subtorus of $\mathbb{T}$ which preserves the symplectic form of M.

Example 3.1. Our basic example is $M = \widetilde{\mathcal{U}}^d_r$, $X = \mathcal{U}^d_r$ and π is the Gieseker-Uhlenbeck morphism with the same $\mathbb{T}$, T as above. In fact, we can also take a larger torus $T \times \mathbb{C}^\times_{\mathrm{hyp}}$ in $\mathbb{T}$, where $\mathbb{C}^\times_{\mathrm{hyp}} \subset \mathbb{C}^\times \times \mathbb{C}^\times$ is given by $t \mapsto (t, t^{-1})$.

Example 3.2. Another example is $M = T^*(\text{flag variety}) = T^*(G/B)$, $X = (\text{nilpotent variety})$ and π is the Grothendieck-Springer resolution. Here T is a maximal torus of G contained in B, and $\mathbb{T} = T \times \mathbb{C}^\times$, where $\mathbb{C}^\times$ acts on X by scaling on fibers.

We can also consider the same $\pi\colon M \to X$ as above with smaller $\mathbb{T}$, T.

Let M^T be the T-fixed point locus in M. It decomposes $M^T = \bigsqcup F_\alpha$ to connected components, and each F_α is a smooth symplectic submanifold of M. Let $i\colon M^T \to M$ be the inclusion. We have the pull-back homomorphism

$$i^*\colon H^{[*]}_{\mathbb{T}}(M) \to H^{[*+\operatorname{codim} X^T]}_{\mathbb{T}}(M^T) = \bigoplus_\alpha H^{[*+\operatorname{codim} F_\alpha]}_{\mathbb{T}}(F_\alpha).$$

Here we take the degree convention as before. Our degree 0 is the usual degree $\dim_{\mathbb{C}} M$ for $H^{[*]}_{\mathbb{T}}(M)$, and $\dim_{\mathbb{C}} F_\alpha$ for $H^{[*]}_{\mathbb{T}}(F_\alpha)$. Since i^* preserves the usual degree, it shifts our degree by $\operatorname{codim} F_\alpha$. Each F_α has its own codimension, but we denote the direct sum as $H^{[*+\operatorname{codim} X^T]}_{\mathbb{T}}(M^T)$ for brevity.

The stable envelop we are going to construct goes in the opposite direction $H^{[*]}_{\mathbb{T}}(M^T) \to H^{[*]}_{\mathbb{T}}(M)$ and preserves (our) degree.

In the above example $M = \widetilde{\mathcal{U}}^d_r$, the T and $T \times \mathbb{C}^\times_{\text{hyp}}$-fixed point loci are

$$(\widetilde{\mathcal{U}}^d_r)^T = \bigsqcup_{d_1+\cdots+d_r=d} \widetilde{\mathcal{U}}^{d_1}_1 \times \cdots \times \widetilde{\mathcal{U}}^{d_r}_1,$$

$$(\widetilde{\mathcal{U}}^d_r)^{T\times\mathbb{C}^\times_{\text{hyp}}} = \bigsqcup_{|\lambda_1|+\cdots+|\lambda_r|} \{I_{\lambda_1} \oplus \cdots \oplus I_{\lambda_r}\},$$

where λ_i is a partition, and I_{λ_i} is the corresponding monomial ideal sheaf (with the induced framing).

Also

$$T^*(G/B)^T = W,$$

where W is the Weyl group.

3(ii). **Chamber structure.** Let us consider the space $\operatorname{Hom}_{\text{grp}}(\mathbb{C}^\times, T)$ of one parameter subgroups in T, and its real form $\operatorname{Hom}_{\text{grp}}(\mathbb{C}^\times, T) \otimes_{\mathbb{Z}} \mathbb{R}$. A generic one parameter subgroup ρ satisfies $M^{\rho(\mathbb{C}^\times)} = M^T$. But if ρ is special (the most extreme case is ρ is the trivial), the fixed point set $M^{\rho(\mathbb{C}^\times)}$ could be larger. This gives us a 'chamber' structure on $\operatorname{Hom}_{\text{grp}}(\mathbb{C}^\times, T) \otimes_{\mathbb{Z}} \mathbb{R}$, where a chamber is a connected component of the complement of the union of hyperplanes given by ρ such that $M^{\rho(\mathbb{C}^\times)} \neq M^T$.

Exercise 5. (1) In terms of T-weights on tangent spaces T_pM at various fixed points $p \in M^T$, describe the hyperplanes.

(2) Show that the chamber structure for $M = T^*(\text{flag variety})$ is identified with usual Weyl chambers.

(3) Show that the chamber structure for $M = \widetilde{\mathcal{U}}^d_r$ is identified with the usual Weyl chambers for $\mathrm{SL}(r)$.

(4) Compute the chamber structure for $M = \widetilde{\mathcal{U}}^d_r$, but with the larger torus $T \times \mathbb{C}^\times_{\text{hyp}}$.

For a chamber $\mathcal{C}$, we have the *opposite chamber* $-\mathcal{C}$ consisting of one parameter subgroups $t \mapsto \rho(t^{-1})$ for $\rho \in \mathcal{C}$.

The stable envelop depends on a choice of a chamber $\mathcal{C}$.

3(iii). **Attracting set.** Let $\mathcal{C}$ be a chamber and $\rho \in \mathcal{C}$. We define the *attracting set* $\mathcal{A}_X$ by

$$\mathcal{A}_X = \left\{ x \in X \,\middle|\, \exists \lim_{t\to 0} \rho(t)x \right\}.$$

We similarly define the attracting set $\mathcal{A}_M$ in M in the same way. As π is proper, we have $\mathcal{A}_M = \pi^{-1}(\mathcal{A}_X)$. We put the scheme structure on $\mathcal{A}_M$ as $\pi^{-1}(\mathcal{A}_X)$ in this paper.

Example 3.3. Let $X = \mathcal{U}_2^d$. In the quiver description, $\mathcal{A}_X$ consists of closed $\mathrm{GL}(d)$-orbits $\mathrm{GL}(d)(B_1, B_2, I, J)$ such that $Jf(B_1, B_2)I$ is upper triangular for any noncommutative monomial $f \in \mathbb{C}\langle x, y\rangle$. It is the *tensor product variety* introduced in [Nak01b], denoted by $\pi(\mathfrak{Z})$ therein.

As framed sheaves, $\mathcal{A}_M$ consists of (E, φ) which are written as an extension $0 \to E_1 \to E \to E_2 \to 0$ (compatible with the framing) for some $E_1 \in \widetilde{\mathcal{U}}_1^{d_1}$, $E_2 \in \widetilde{\mathcal{U}}_1^{d_2}$ with $d = d_1 + d_2$.

We have the following diagram

$$X^{\rho(\mathbb{C}^\times)} = X^T \underset{i}{\overset{p}{\leftrightarrows}} \mathcal{A}_X \xrightarrow{j} X, \tag{3.4}$$

where i, j are natural inclusion, and p is given by $\mathcal{A}_X \ni x \mapsto \lim_{t\to 0} \rho(t)x$.

Since $M^T = \bigsqcup F_\alpha$, we have the corresponding decomposition of $\mathcal{A}_M = \bigsqcup p^{-1}(F_\alpha)$. Let $\mathrm{Leaf}_\alpha = p^{-1}(F_\alpha)$. By the Bialynicki-Birula theorem ([?], see also [?]), $p\colon \mathrm{Leaf}_\alpha \to F_\alpha$ is a vector bundle. Similarly $\mathrm{Leaf}_\alpha^- \to F_\alpha$ denote the corresponding vector bundle for the opposite chamber $-\mathcal{C}$.

Let us consider the restriction of the tangent bundle TM to a fixed point component F_α. It decomposes into weight spaces with respect to ρ:

$$TM|_{F_\alpha} = \bigoplus T(m), \qquad T(m) = \{v \mid \rho(t)v = t^m v\}. \tag{3.5}$$

Then $\mathrm{Leaf}_\alpha = \bigoplus_{m>0} T(m)$. Note also $T(0) = TF_\alpha$. Since T preserves the symplectic form, $T(m)$ and $T(-m)$ are dual to each other. From these, one can also check that $\mathrm{Leaf}_\alpha \xrightarrow{j\times p} M \times F_\alpha$ is a lagrangian embedding.

Example 3.6. Let $\pi\colon M = T^*\mathbb{P}^1 \to X = \mathbb{C}^2/\pm$. Let $T = \mathbb{C}^\times$ act on X and $\mathbb{C}^2/\pm 1$ so that it is given by $t(z_1, z_2) \bmod \pm = (tz_1, tz_2^{-1}) \bmod \pm$. Then X^T consists of two points $\{0, \infty\}$ in the zero section $\mathbb{P}^1$ of $T^*\mathbb{P}^1$. If we take the 'standard' chamber containing the identity operator, Leaf_0 is the zero section $\mathbb{P}^1$ minus ∞. On the other hand Leaf_∞ is the (strict transform of) the axis $z_2 = 0$. See Figure 2.

For the opposite chamber, Leaf_0 is the axis $z_1 = 0$, and Leaf_∞ is the zero section minus 0.

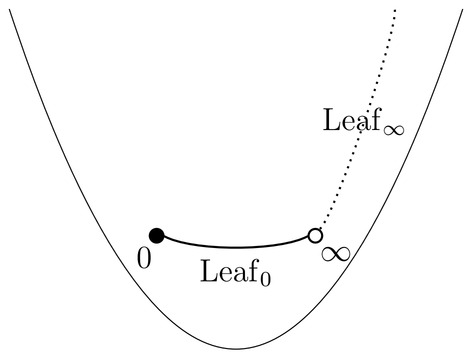

FIGURE 2. Leaves in $T^*\mathbb{P}^1$

Definition 3.7. We define a partial order $\geq$ on the index set $\{\alpha\}$ for the fixed point components so that

$$\overline{\mathrm{Leaf}_\beta} \cap F_\alpha \neq \emptyset \Longrightarrow \alpha \leq \beta.$$

We have $\infty \leq 0$ in Example 3.6.

Let

$$\mathcal{A}_{M,\leq\alpha} = \bigsqcup_{\beta:\beta\leq\alpha} \mathrm{Leaf}_\beta\,.$$

Then $\mathcal{A}_{M,\leq\alpha}$ is a closed subvariety. We define $\mathcal{A}_{M,<\alpha}$ in the same way.

Proposition 3.8. (1) $H^{\mathbb{T}}_{[*]}(\mathcal{A}_{M,\leq\alpha})$ *vanishes in the odd degree.*

(2) *We have an exact sequence*

$$0 \to H^{\mathbb{T}}_{[*]}(\mathcal{A}_{M,<\alpha}) \to H^{\mathbb{T}}_{[*]}(\mathcal{A}_{M,\leq\alpha}) \to H^{\mathbb{T}}_{[*]}(\mathrm{Leaf}_\alpha) \to 0.$$

Proof. Consider the usual long exact sequence

$$H^{\mathbb{T}}_{[*]}(\mathcal{A}_{M,<\alpha}) \to H^{\mathbb{T}}_{[*]}(\mathcal{A}_{M,\leq\alpha}) \to H^{\mathbb{T}}_{[*]}(\mathrm{Leaf}_\alpha) \xrightarrow{\delta} H^{\mathbb{T}}_{[*-1]}(\mathcal{A}_{M,<\alpha}).$$

Recall that Leaf_α is a vector bundle over F_α. It is known that $H^{\mathbb{T}}_{[*]}(F_\alpha)$ vanishes in odd degrees. It follows from [Nak99, Exercise 5.15] for $\widetilde{\mathcal{U}}^d_r$, and is a result of Kaledin [Kal06] in general.

Let us show that $H^{\mathbb{T}}_{[*]}(\mathcal{A}_{M,\leq\alpha})$ vanishes in odd degrees by the descending induction on α. In particular, the assertion for $H^{\mathbb{T}}_{[*]}(\mathcal{A}_{M,<\alpha})$ implies $\delta = 0$, i.e., (2).

If α is larger than the maximal element, $\mathcal{A}_{M,\leq\alpha} = \emptyset$, hence the assertion is true. Suppose that the assertion is true for $H^{\mathbb{T}}_{[*]}(\mathcal{A}_{M,<\alpha})$. Then the above exact sequence and the odd vanishing of $H^{\mathbb{T}}_{[*]}(\mathrm{Leaf}_\alpha)$ implies the odd vanishing of $H^{\mathbb{T}}_{[*]}(\mathcal{A}_{M,\leq\alpha})$. □

Exercise 6. (1) Determine $\mathcal{A}_M$ for $M = T^*(\text{flag variety})$.

(2) Determine $\mathcal{A}_M$ for $M = \widetilde{\mathcal{U}}_r^d$ with respect to $T \times \mathbb{C}^\times_{\text{hyp}}$.

3(iv). **Steinberg type variety.** Recall that Steinberg variety is the fiber product of $T^*(\text{flag variety})$ with itself over nilpotent variety. Its equivariant K-group realizes the affine Hecke algebra (see [CG97, Ch. 7]), and it plays an important role in the geometric representation theory.

Let us recall the definition of the product in our situation. We define $Z = M \times_X M$, the fiber product of M itself over X. Its equivariant Borel-Moore homology group has the convolution product:

$$H^{\mathbb{G}}_{[*]}(Z) \otimes H^{\mathbb{G}}_{[*]}(Z) \ni c \otimes c' \mapsto p_{13*}(p_{12}^* c \cap p_{23}^* c') \in H^{\mathbb{G}}_{[*]}(Z),$$

where $p_{ij}\colon M \times M \times M \to M \times M$ is the projection to the product of i^{th} and j^{th} factors.[10] When $M \to X$ is semi-small, one can check that the multiplication preserves the shifted degree $[*]$.

We introduce a variant, mixing the fixed point set M^T and the whole variety M: Let $Z_{\mathcal{A}}$ be the fiber product of $\mathcal{A}_M$ and M^T over X^T, considered as a closed subvariety in $M \times M^T$:

$$Z_{\mathcal{A}} = \mathcal{A}_M \times_{X^T} M^T \subset M \times M^T,$$

where $\mathcal{A}_M \to X^T$ is the composite of $p\colon \mathcal{A}_M \to M^T$ and $\pi^T\colon M^T \to M_0^T$, or alternatively $\pi|_{\mathcal{A}_M}\colon \mathcal{A}_M \to \mathcal{A}_{M_0}$ and $p\colon \mathcal{A}_{M_0} \to X^T$. Here we denote projections $\mathcal{A}_M \to M^T$ and $\mathcal{A}_X \to X^T$ both by p for brevity. As a subvariety of $M \times M^T$, $Z_{\mathcal{A}}$ consists of pairs (x, x') such that $\lim_{t\to 0} \rho(t)\pi(x) = \pi^T(x')$.

The convolution product as above defines a $(H^{\mathbb{T}}_{[*]}(Z), H^{\mathbb{T}}_{[*]}(Z^T))$-bimodule structure on $H^{\mathbb{T}}_{[*]}(Z_{\mathcal{A}})$. Here Z^T is the T-fixed point set in Z, or equivalently the fiber product of M^T with itself over X^T.

In our application, we use $H^{\mathbb{T}}_{[*]}(Z)$ as follows: we shall construct an operator $H_{\mathbb{T}}^{[*]}(M^T) \to H_{\mathbb{T}}^{[*]}(M)$ by

$$H_{\mathbb{T}}^{[*]}(M^T) \ni c \mapsto p_{1*}(p_2^* c \cap \mathcal{L}) \in H_{\mathbb{T}}^{[*]}(M) \tag{3.9}$$

for a suitably chosen (degree 0) equivariant class $\mathcal{L}$ in $H^{\mathbb{T}}_{[0]}(Z_{\mathcal{A}})$. Note that the projection $Z_{\mathcal{A}} \to M$ is proper, hence the operator in this direction is well-defined. On the other hand, $Z_{\mathcal{A}} \to M^T$ is *not* proper. See §3(vi)(a) below.

Recall $\mathcal{A}_M$ and M^T decompose as $\bigsqcup \operatorname{Leaf}_\beta$, $\bigsqcup F_\alpha$ respectively. Therefore

$$Z_{\mathcal{A}} = \bigsqcup_{\alpha,\beta} \operatorname{Leaf}_\beta \times_{X^T} F_\alpha.$$

[10] We omit explanation of pull-back with supports p_{12}^*, p_{23}^*, etc. See [CG97] for more detail.

We have the projection $p\colon \operatorname{Leaf}_\beta \times_{X^T} F_\alpha \to F_\beta \times_{X^T} F_\alpha$.

Proposition 3.10. *$Z_{\mathcal{A}}$ is a lagrangian subvariety in $M \times M^T$. If Z_1^F, Z_2^F, … denote the irreducible components of $\bigsqcup_{\alpha,\beta} F_\beta \times_{X^T} F_\alpha$, then closures of their inverse images*

$$Z_1 \overset{\text{def.}}{=} \overline{p^{-1}(Z_1^F)}, Z_2 \overset{\text{def.}}{=} \overline{p^{-1}(Z_1^F)}, \dots$$

are the irreducible components of $Z_{\mathcal{A}}$.

Proof. Consider an irreducible component Z_ν^F of $F_\beta \times_{X^T} F_\alpha$. Using the semi-smallness of $\pi^T\colon M^T \to X^T$, one can check that Z_ν^F is half-dimensional. Then the dimension of Z_ν is

$$\frac{1}{2}(\dim F_\beta + \dim F_\alpha) + \frac{1}{2}\operatorname{codim}_X F_\beta = \frac{1}{2}(\dim X + \dim F_\alpha),$$

as the rank of $\operatorname{Leaf}_\beta$ is the half of codimension of F_β. Therefore Z_ν is half-dimensional in $F_\beta \times M$.

We omit the proof that Z_ν is lagrangian. □

3(v). **Polarization.** Consider the normal bundle $N_{F_\alpha/M}$ of F_α in M. It is the direct sum $\operatorname{Leaf}_\alpha \oplus \operatorname{Leaf}_\alpha^-$. Since $\operatorname{Leaf}_\alpha$, $\operatorname{Leaf}_\alpha^-$ are dual to each other with respect to the symplectic form, the equivariant Euler class $e(N_{F_\alpha/M})$ is a square up to sign:

$$e(N_{F_\alpha/M}) = e(\operatorname{Leaf}_\alpha)e(\operatorname{Leaf}_\alpha^-) = (-1)^{\operatorname{codim}_M F_\alpha/2} e(\operatorname{Leaf}_\alpha)^2.$$

Let us consider $H_T^*(\mathrm{pt})$-part of the equivariant classes, i.e., weights of fibers of vector bundles as T-modules. In many situations, we have another preferred choice of a square root of $(-1)^{\operatorname{codim}_M F_\alpha/2} e(N_{F_\alpha/M})|_{H_T^*(\mathrm{pt})}$. For example, suppose $M = T^*Y$ and the T-action on M is coming from a T-action on Y. Then $F_\alpha = T^*N$ for a submanifold $N \subset Y$, we have $T(T^*Y) = TY \oplus T^*Y$, and $N_{F_\alpha/M} = N_{N/Y} \oplus N_{N/Y}^*$, hence $(-1)^{\operatorname{codim}_M F_\alpha/2} e(N_{F_\alpha/M}) = e(N_{N/Y})^2$. This choice $e(N_{N/Y})$ of the square root is more canonical than $e(\operatorname{Leaf}_\alpha)$ as it behaves more uniformly in α. We call such a choice of a square root of $(-1)^{\operatorname{codim}_M F_\alpha/2} e(N_{F_\alpha/M})|_{H_T^*(\mathrm{pt})}$ *polarization*. We will understand a polarization as a choice of $\pm$ for each α, e.g., $e(\operatorname{Leaf}_\alpha)|_{H_T^*(\mathrm{pt})} = \pm e(N_{N/Y})|_{H_T^*(\mathrm{pt})}$ for this example.

Our main example $\widetilde{\mathcal{U}}_r^d$ is not a cotangent bundle, but is a symplectic reduction of a symplectic vector space, which is a cotangent bundle. Therefore we also have a natural choice of a polarization for $\widetilde{\mathcal{U}}_r^d$.

3(vi). **Definition of the stable envelop.** Note that T acts on M^T trivially. Therefore the equivariant cohomology $H_T^{[*]}(M^T)$ is isomorphic to $H^{[*]}(M^T) \otimes H_T^*(\mathrm{pt})$. The second factor $H_T^*(\mathrm{pt})$ is the polynomial ring $\mathbb{C}[\operatorname{Lie} T]$.

We define the *degree* $\deg_T$ on $H_T^{[*]}(M^T)$ as the degree of the component of $H_T^*(\mathrm{pt}) = \mathbb{C}[\operatorname{Lie} T]$.

Let $\iota_{\beta,\alpha}\colon F_\beta \times F_\alpha \to M \times M^T$ denonte the inclusion.

Theorem 3.11. *Choose and fix a chamber* $\mathcal{C}$ *and a polarization* $\pm$. *There exists a unique homology class* $\mathcal{L} \in H^T_{[0]}(Z_{\mathcal{A}})$ *with the following three properties*:

(1) $\mathcal{L}|_{M\times F_\alpha}$ *is supported on* $\bigcup_{\beta\le\alpha} \overline{\mathrm{Leaf}_\beta} \times_{X^T} F_\alpha$.

(2) *Let* $e(\mathrm{Leaf}_\alpha^-)$ *denote the equivariant Euler class of the bundle* Leaf_α^- *over* F_α. *We have*

$$\iota_{\alpha,\alpha}^* \mathcal{L} = \pm e(\mathrm{Leaf}_\alpha^-) \cap [\Delta_{F_\alpha}].$$

(3) *For* $\beta < \alpha$,

$$\deg_T \iota_{\beta,\alpha}^* \mathcal{L} < \frac{1}{2} \operatorname{codim}_M F_\beta.$$

In fact, in our situation, we have a stronger statement $\iota_{\beta,\alpha}^*\mathcal{L} = 0$.

Proof. We first prove the existence. We use the 1-parameter deformation $\mathcal{M}$, $\mathcal{X}$ appeared in §4(i). Moreover the T-action extends to M^t, X^t.

We define F_α^t, Leaf_α^t for M^t in the same way. Since M^t is affine for $t \neq 0$, Leaf_α^t is a closed subvariety in M^t. Considering it as a correspondence in $M^t \times_{(X^t)^T} (M^t)^T$, we consider its fundamental class $[\mathrm{Leaf}_\alpha^t]$. We now define

$$\mathcal{L} = \sum_\alpha \mathcal{L}_\alpha; \qquad \mathcal{L}_\alpha \overset{\text{def.}}{=} \pm \lim_{t\to 0} [\mathrm{Leaf}_\alpha^t],$$

where $\lim_{t\to 0}$ is the specialization in Borel-Moore homology groups (see [CG97, §2.6.30]).

The conditions (1),(2),(3) are satisfied for $[\mathrm{Leaf}_\alpha^t]$. Moreover $\iota_{\beta,\alpha}^*[\mathrm{Leaf}_\alpha^t] = 0$ for $\beta \neq \alpha$. Taking the limit $t \to 0$, we see that three conditions (1),(2),(3) are satisfied also for $\mathcal{L}$.

Next let us show the uniqueness. We decompose $\mathcal{L} = \sum \mathcal{L}_\alpha$ according to $M^T = \bigsqcup F_\alpha$ as above. Let Z_1, Z_2, … be irreducible components of Z as in Proposition 3.10. Each Z_k is coming from an irreducible component of $F_\beta \times_{X^T} F_\alpha$. So let us indicate it as $Z_k^{(\beta,\alpha)}$. Since $\mathcal{L}$ is the top degree class, we have $\mathcal{L}_\alpha = \sum a_k [Z_k^{(\beta,\alpha)}]$ with some $a_k \in \mathbb{C}$. Moreover by (1), $Z_k^{(\beta,\alpha)}$ with nonzero a_k satisfies $\beta \le \alpha$.

Consider $Z_k^{(\alpha,\alpha)}$. By (2), we must have $Z_k^{(\alpha,\alpha)} = \overline{\mathrm{Leaf}_\alpha}$, $a_k = \pm 1$, where $\overline{\mathrm{Leaf}_\alpha}$ is mapped to $Z_{\mathcal{A}} = M \times_{X^T} M^T$ by (inclusion) $\times\, p$.

Suppose that $\mathcal{L}_\alpha^1$, $\mathcal{L}_\alpha^2$ satisfy conditions (1),(2),(3). Then the above discussion says $\mathcal{L}_\alpha^1 - \mathcal{L}_\alpha^2 = \sum a_k' [Z_k^{(\beta,\alpha)}]$, where $Z_k^{(\beta,\alpha)}$ with nonzero a_k'

satisfies $\beta < \alpha$. Suppose that $\mathcal{L}^1_\alpha \neq \mathcal{L}^2_\alpha$ and take a maximum β_0 among β such that there exists $Z_k^{(\beta,\alpha)}$ with $a'_k \neq 0$.

Consider the restriction $\iota^*_{\beta_0,\alpha}(\mathcal{L}^1_\alpha - \mathcal{L}^2_\alpha)$. By the maximality, only those $Z_k^{(\beta,\alpha)}$'s with $\beta = \beta_0$ contribute and hence

$$\iota^*_{\beta_0,\alpha}(\mathcal{L}^1_\alpha - \mathcal{L}^2_\alpha) = \sum a'_k e(\mathrm{Leaf}^-_{\beta_0}) \cap [Z_k^F],$$

where Z_k^F is an irreducible component of $F_{\beta_0} \times_{X^T} F_\alpha$ corresponding to $Z_k^{(\beta_0,\alpha)}$.

Since $e(\mathrm{Leaf}^-_{\beta_0})|_{H^*_T(\mathrm{pt})}$ is the product of weights λ such that $\langle \rho, \lambda\rangle < 0$ (cf. (3.5)), it has degree exactly equal to codim $F_\beta/2$. But it contradicts with the condition (3). Therefore $\mathcal{L}^1_\alpha = \mathcal{L}^2_\alpha$. □

Since $\mathcal{L}$ lives in the top degree, we have

$$H_{[0]}(Z_{\mathcal{A}}) \cong H^T_{[0]}(Z_{\mathcal{A}}) \cong H^{\mathbb{T}}_{[0]}(Z_{\mathcal{A}}).$$

In particular, $\mathcal{L}$ gives an operator on $\mathbb{T}$-equivariant cohomology by the formula (3.9), even though we construct it in the T-equivariant cohomology group originally. The property (2) still holds in $\mathbb{T}$-equivariant cohomology group by the construction. However the deformation we have used cannot be $\mathbb{T}$-equivariant, so $\iota^*_{\beta,\alpha}\mathcal{L} \neq 0$ in general.

Definition 3.12. The operator defined by the formula (3.9) given by the class $\mathcal{L}$ constructed in Theorem 3.11 is called the *stable envelop*. It is denoted by $\mathrm{Stab}_{\mathcal{C}}$. (The dependence on the polarization is usually suppressed.)

Let us list several properties of stable envelops.

(a). *Adjoint.* By a general property of the convolution product, the adjoint operator

$$\mathrm{Stab}^*_{\mathcal{C}}\colon H^{[*]}_{\mathbb{T},c}(M) \to H^{[*]}_{\mathbb{T},c}(M^T)$$

is given by changing the first and second factors in (3.9):

$$H^{[*]}_{\mathbb{T},c}(M) \ni c \mapsto (-1)^{\operatorname{codim} X^T} p_{2*}(p_1^* c \cap \mathcal{L}) \in H^{[*]}_{\mathbb{T},c}(M^T),$$

where the sign comes from our convention on the intersection form.

(b). *Image of the stable envelop.* From the definition, the stable envelop defines a homomorphism

$$H^{[*]}_{\mathbb{T}}(M^T) \to H^{\mathbb{T}}_{[-*]}(\mathcal{A}_M), \tag{3.13}$$

so that the original $\mathrm{Stab}_{\mathcal{C}}$ is the composition of the above together with the pushforward $H^{\mathbb{T}}_{[-*]}(\mathcal{A}_M) \to H^{\mathbb{T}}_{[-*]}(M)$ of the inclusion $\mathcal{A}_M \to M$ and the Poincaré duality $H^{\mathbb{T}}_{[-*]}(M) \cong H^{[*]}_{\mathbb{T}}(M)$.

Proposition 3.14. (3.13) *is an isomorphism.*

By Proposition 3.8, $H^{\mathbb{T}}_{[*]}(\mathcal{A}_M)$ has a natural filtration $\bigcup_\alpha H^{\mathbb{T}}_{[*]}(\mathcal{A}_{M,\le\alpha})$ whose associated graded is $\bigoplus_\alpha H^{\mathbb{T}}_{[*]}(\mathrm{Leaf}_\alpha)$. From the construction, the stable envelop is compatible with the filtration, where the filtration on $H_{\mathbb{T}}^{[*]}(M^T)$ is one induced from the decomposition to $\bigoplus H_{\mathbb{T}}^{[*]}(F_\alpha)$. Moreover the induced homomorphism $H_{\mathbb{T}}^{[*]}(F_\alpha) \to H^{\mathbb{T}}_{[*]}(\mathrm{Leaf}_\alpha)$ is the pull-back, and hence it is an isomorphism. Therefore the original stable envelop is also an isomorphism.

(c). *Subtorus.*

Historical Comment 3.15.

4. Sheaf theoretic analysis of stable envelops

4(i). **Nearby cycle functor.** This subsection is a short detour, giving (2.7) for a symplectic resolution $\pi\colon M \to X$ without quoting Kaledin's semi-smallness result. It uses the nearby cycle functor,[11] which is important itself, and useful for sheaf theoretic understanding of stable envelops in §4(iv).

Let us recall the definition of the nearby cycle functor in [KS90, §8.6].

Let $\mathcal{X}$ be a complex manifold and $f\colon \mathcal{X} \to \mathbb{C}$ a holomorphic function. Let $X = f^{-1}(0)$ and $r\colon X \to \mathcal{X}$ be the inclusion. We take the universal covering $\tilde{\mathbb{C}}^* \to \mathbb{C}^*$ of $\mathbb{C}^* = \mathbb{C}\setminus\{0\}$. Let $c\colon \tilde{\mathbb{C}}^* \to \mathbb{C}$ be the composition of the projection and inclusion $\mathbb{C}^* \to \mathbb{C}$. We take the fiber product $\tilde{\mathcal{X}}^*$ of $\mathcal{X}$ and $\tilde{\mathbb{C}}^*$ over $\mathbb{C}$. We consider the diagram

$$\begin{array}{ccccc} & & \tilde{\mathcal{X}}^* & \longrightarrow & \tilde{\mathbb{C}}^* \\ & & \downarrow{\scriptstyle \tilde{c}} & & \downarrow{\scriptstyle c} \\ X & \xrightarrow[r]{} & \mathcal{X} & \xrightarrow[f]{} & \mathbb{C}. \end{array}$$

Then the nearby cycle functor from $D^b(\mathcal{X})$ to $D^b(X)$ is defined by

$$\psi_f(\bullet) = r^*\tilde{c}_*\tilde{c}^*(\bullet). \tag{4.1}$$

Note that it depends only on the restriction of objects in $D^b(\mathcal{X})$ to $\mathcal{X}\setminus X$.

Let $\pi\colon M \to X$ be a symplectic resolution. We use the following fact due to ***** : M and X have a 1-parameter deformation $\mathcal{M}$, $\mathcal{X}$ together with $\Pi\colon \mathcal{M} \to \mathcal{X}$ over $\mathbb{C}$ such that the original M, X are fibers

[11]The author learned usage of the nearby cycle functor for symplectic resolutions from Victor Ginzburg many years ago. He attributed it to Gaitsgory. See [Gai01].

over $0 \in \mathbb{C}$, and other fibers M^t, X^t ($t \neq 0$) are isomorphic, both are smooth and affine.

For $\widetilde{\mathcal{U}}_r^d \to \mathcal{U}_r^d$, such a deformation can be defined by using quiver description : we perturb the moment map equation as $[B_1, B_2] + IJ = t\,\mathrm{id}$. The $\mathbb{T}$-action does not preserve this equation, as it scales $t\,\mathrm{id}$. But the T-action preserves the equation. For T^*(flag variety) $\to$ (nilpotent variety), we use deformation to semisimple adjoint orbits. (See [CG97, §3.4].)

Take the projection $\mathcal{X}$, $\mathcal{M} \to \mathbb{C}$ as f. Let us denote them by f_X, f_M respectively. (Singularities of $\mathcal{X}$ causes no trouble in the following discussion: we replace $\mathcal{X}$ by an affine space to which $\mathcal{X}$ is embedded.) We consider the nearby cycle functors ψ_{f_X}, ψ_{f_M}. Since $\Pi\colon \mathcal{M} \to \mathcal{X}$ is proper, one can check

$$\psi_{f_X}\Pi_* = \pi_*\psi_{f_M} \tag{4.2}$$

from the base change.

Now we apply the both sides to the shifted constant sheaf $\mathcal{C}_{\mathcal{M}}$. Since $\mathcal{M} \to \mathbb{C}$ is a smooth fibration, we have

$$\psi_{f_M}\mathcal{C}_{\mathcal{M}} = \mathcal{C}_M[1]. \tag{4.3}$$

On the other hand, since we only need the restriction of $\Pi_*\mathcal{C}_{\mathcal{M}}$ to $\mathcal{X} \setminus X$, we may replace $\pi_*\mathcal{C}_{\mathcal{M}}$ by $\mathcal{C}_{\mathcal{X}}$ as $\mathcal{M} \setminus M \to \mathcal{X} \setminus X$ is an isomorphism. (In fact, $\Pi_*\mathcal{C}_{\mathcal{M}} = \mathrm{IC}(\mathcal{X})$ as it is known that Π is small.) We thus have

$$\psi_{f_X}\mathcal{C}_{\mathcal{X}} = \pi_*(\mathcal{C}_M)[1]. \tag{4.4}$$

We have a fundamental property of the nearby cycle functor: $\psi_f[-1]$ sends perverse sheaves on $\mathcal{X}$ to perverse sheaves on X. (See [KS90, Cor. 10.3.13].) Hence $\pi_*(\mathcal{C}_M)$ is perverse.

Exercise 7. (1) Check (4.2).
(2) Check (4.3).

4(ii). **Homology group of the Steinberg type variety.** Recall that the variety $Z_{\mathcal{A}}$ is defined as a fiber product $\mathcal{A}_M \times_{X^T} M^T$, analog of Steinberg variety. The homology group of Steinberg variety, or more generally the fiber product $M \times_X M$ nicely fits with framework of perverse sheaves by Ginzburg's theory [CG97, §8.6]. A starting point of the relationship is an algebra isomorphism (not necessarily grading preserving)

$$H_*(Z) \cong \mathrm{Ext}^*_{D^b(X)}(\pi_*(\mathcal{C}_M), \pi_*(\mathcal{C}_M)). \tag{4.5}$$

Moreover, if $M \to X$ is semi-small, $\pi_*(\mathcal{C}_M)$ is a semisimple perverse sheaf, and

$$H_{[0]}(Z) \cong \mathrm{Hom}_{D^b(X_0)}(\pi_*(\mathcal{C}_M), \pi_*(\mathcal{C}_M)). \tag{4.6}$$

See [CG97, Prop. 8.9.6].

We have the following analog for $Z_{\mathcal{A}}$:

Proposition 4.7 (cf. [Nak13, Lemma 4])**.** *We have a natural isomorphism*

$$H_*(Z_{\mathcal{A}}) \cong \operatorname{Ext}^*_{D^b(X_0^T)}(\pi^T_*(\mathcal{C}_{M^T}), p_* j^! \pi_*(\mathcal{C}_M)). \tag{4.8}$$

Recall that $H_*(Z_{\mathcal{A}})$ is an $(H^{\mathbb{T}}_{[*]}(Z), H^{\mathbb{T}}_{[*]}(Z^T))$-bimodule. Similarly the right hand side of the above isomorphism is a bimodule over

$$(\operatorname{Ext}^*_{D^b(X^T)}(\pi^T_*(\mathcal{C}_{M^T}), \pi^T_*(\mathcal{C}_{M^T})), \operatorname{Ext}^*_{D^b(X)}(\pi_*(\mathcal{C}_M), \pi_*(\mathcal{C}_M))).$$

Under (4.5) and its analog for M^T, the above isomorphism respects the bimodule structure.

In view of (4.5,4.6,4.8), it is natural to ask what happens if we replace the left hand side of (4.8) by the shifted degree 0 part $H_{[0]}(Z_{\mathcal{A}})$, where the cycle $\mathcal{L}$ in Theorem 3.11 lives. Since $\pi^T\colon M^T \to X^T$ is semi-small, $\pi^T_*(\mathcal{C}_{M^T})$ is a semisimple perverse sheaf. As we shall see in the next two subsections, the same is true for $p_* j^! \pi_*(\mathcal{C}_M)$. Then we have

$$H_{[0]}(Z_{\mathcal{A}}) \cong \operatorname{Hom}_{D^b(X_0^T)}(\pi^T_*(\mathcal{C}_{M^T}), p_* j^! \pi_*(\mathcal{C}_M)).$$

The claim that $p_* j^! \pi_*(\mathcal{C}_M)$ is a semisimple perverse sheaf is a consequence of two results:

(a) Braden's result [Bra03] on preservation purity.
(b) Dimension estimate of fibers, following an idea of Mirkovic-Vilonen [MV07].

One may formally compare these results to (a) the decomposition theorem, and (b) semi-smallness for proper pushforward homomorphisms. In fact, the actual dimension estimate (b) required for $p_* j^! \pi_*(\mathcal{C}_M)$ is rather easy to check, once we use the nearby cycle functor. The argument can be compared with one in §4(i).

4(iii). **Hyperbolic restriction.** We first treat (a).

Let $\mathcal{R}_X$ be the *repelling set*, i.e., the attracting set for the opposite chamber $-\mathcal{C}$. We have the diagram

$$X^T \underset{i_-}{\overset{p_-}{\leftrightarrows}} \mathcal{R}_X \xrightarrow{j_-} X,$$

as for the attracting set.

Theorem 4.9 ([Bra03])**.** *We have a natural isomorphism $p_* j^! \cong p_{-!} j_-^*$ on T-equivariant complexes $D^b_T(X)$.*

This theorem implies the preservation of the purity for $p_* j^! = p_{-!} j^*_-$ as p_*, $j^!$ increase weights while $p_{-!}$, j^* decrease weights. In particular, semisimple complexes are sent to semisimple ones. Thus $p_* j^! \pi_*(\mathcal{C}_M)$ is semisimple.

4(iv). **Exactness by nearby cycle functors.** Consider one parameter deformation f_M, $f_X \colon \mathcal{M}, \mathcal{X} \to \mathbb{C}$ as in §4(i). We have the diagram

$$\mathcal{X}^T \underset{i_{\mathcal{X}}}{\overset{p_{\mathcal{X}}}{\leftrightarrows}} \mathcal{A}_{\mathcal{X}} \xrightarrow{j_{\mathcal{X}}} \mathcal{X},$$

as in (3.4). We have the hyperbolic restriction functor $p_{\mathcal{X}*} j^!_{\mathcal{X}}$.

We also have a family $f_{X^T} \colon \mathcal{X}^T \to \mathbb{C}$.

The purpose of this section is to prove the following.

Proposition 4.10. (1) *The restriction of $p_{\mathcal{X}*} j^!_{\mathcal{X}} \mathcal{C}_{\mathcal{X}}$ to $\mathcal{X}^T \setminus X^T$ is canonically isomorphic to the constant sheaf $\mathcal{C}_{\mathcal{X}^T \setminus X^T}$.*

(2) *The nearby cycle functors commute with the hyperbolic restriction:*

$$p_* j^! \psi_{f_X} = \psi_{f_{X^T}} p_{\mathcal{X}*} j^!_{\mathcal{X}}.$$

Corollary 4.11. (1) *$p_* j^! \pi_*(\mathcal{C}_M)$ is perverse.*

(2) *The isomorphism $\mathcal{C}_{\mathcal{X}^T}|_{\mathcal{X}^T \setminus X^T} \xrightarrow{\cong} p_{\mathcal{X}*} j^!_{\mathcal{X}} \mathcal{C}_{\mathcal{X}}|_{\mathcal{X}^T \setminus X^T}$ induces an isomorphism $\pi^T_*(\mathcal{C}_{M^T}) \xrightarrow{\cong} p_* j^! \pi_*(\mathcal{C}_M)$.*

In fact, $\pi_*(\mathcal{C}_M) = \psi_{f_X} \mathcal{C}_{\mathcal{X}}[-1]$ by (4.4). Therefore Proposition 4.10(2) implies $p_* j^! \pi_*(\mathcal{C}_M) = \psi_{f_{X^T}} p_{\mathcal{X}*} j^!_{\mathcal{X}} \mathcal{C}_{\mathcal{X}}[-1]$. Now we can replace $p_{\mathcal{X}*} j^!_{\mathcal{X}} \mathcal{C}_{\mathcal{X}}[-1]$ by $\mathcal{C}_{\mathcal{X}^T}[-1]$ by (1) over $\mathcal{X}^T \setminus X^T$. Since $\psi_{f_{X^T}}[-1]$ sends a perverse sheaf to a perverse sheaf, we get the assertion (1).

Applying $\psi_{f_{X^T}}$ to $\mathcal{C}_{\mathcal{X}^T}|_{\mathcal{X}^T \setminus X^T} \xrightarrow{\cong} p_{\mathcal{X}*} j^!_{\mathcal{X}} \mathcal{C}_{\mathcal{X}}|_{\mathcal{X}^T \setminus X^T}$, we get an isomorphism

$$\pi^T_*(\mathcal{C}_{M^T}) \cong \psi_{f_{X^T}} \mathcal{C}_{\mathcal{X}^T} \xrightarrow{\cong} \psi_{f_{X^T}} p_{\mathcal{X}*} j^!_{\mathcal{X}} \mathcal{C}_{\mathcal{X}} \cong p_* j^! \pi_*(\mathcal{C}_M).$$

This is (2).

Moreover, the isomorphism (2) coincides with one given by the class $\mathcal{L} \in H^T_{[0]}(Z_{\mathcal{A}}) \cong \operatorname{Hom}_{D^b(X^T_0)}(\pi^T_*(\mathcal{C}_{M^T}), p_* j^! \pi_*(\mathcal{C}_M))$. This is clear from the definition, and the fact that the nearby cycle functor coincides with the specialization of Borel-Moore homology groups. *****

We have canonical homomorphisms $\mathrm{IC}(X^T) \to \pi^T_*(\mathcal{C}_{M^T})$ and $\pi_*(\mathcal{C}_M) \to \mathrm{IC}(X)$ as the inclusion and projection of direct summands. Therefore we have a canonical homomorphism

$$\mathrm{IC}(X^T) \to p_* j^! \,\mathrm{IC}(X).$$

This is a nice reformulation of a stable envelop, which makes sense even when we do not have a symplectic resolution. ******

Historical Comment 4.12. A proof of Corollary 4.11(1) was given by Varagnolo-Vasserot [VV03] for hyperbolic restrictions for quiver varieties, and it works for symplectic resolutions. The proof is based on arguments in [Lus85, 3.7] and [Lus91, 4.7], which give decomposition of restrictions of character sheaves, restrictions of perverse sheaves corresponding to canonical bases respectively. (Semisimplicity follows from a general theorem Theorem 4.9. But decomposition must be studied in a different way.)

Proof of Proposition 4.10. (1) The key is that $\mathcal{M} \setminus M$ is isomorphic to $\mathcal{X} \setminus X$. We have the decomposition $\mathcal{M}^T = \bigsqcup \mathcal{F}_\alpha$ corresponding to $M^T = \bigsqcup F_\alpha$ to connected components. Then we have the induced decomposition

$$\mathcal{A}_{\mathcal{X}} \setminus \mathcal{A}_X = \bigsqcup p_{\mathcal{X}}^{-1}(\mathcal{F}_\alpha \setminus F_\alpha)$$

to connected components, such that each $p_{\mathcal{X}}^{-1}(\mathcal{F}_\alpha \setminus F_\alpha)$ is a vector bundle over $\mathcal{F}_\alpha \setminus F_\alpha$. The same holds for $\mathcal{R}_{\mathcal{X}} \setminus \mathcal{R}_X = \bigsqcup p_{\mathcal{X}-}^{-1}(\mathcal{F}_\alpha \setminus F_\alpha)$.

Moreover $p_{\mathcal{X}}^{-1}(\mathcal{F}_\alpha \setminus F_\alpha)$ is a smooth closed subvariety of $\mathcal{X} \setminus X$. Its codimension is equal to the rank of $p_{\mathcal{X}-}^{-1}(\mathcal{F}_\alpha \setminus F_\alpha)$.

Thus the hyperbolic restriction $p_{\mathcal{X}*} j_{\mathcal{X}}^! \mathcal{C}_{\mathcal{X}}$ is the constant sheaf $\mathcal{C}_{\mathcal{X}^T \setminus X^T}$ up to shifts. (Shifts could possibly different on components.)

Now $p_{\mathcal{X}}^{-1}(\mathcal{F}_\alpha \setminus F_\alpha)$ and $p_{\mathcal{X}-}^{-1}(\mathcal{F}_\alpha \setminus F_\alpha)$ are dual vector bundles with respect to the symplectic structure. In particular their ranks are equal. This observation implies that shifts are unnecessary, $p_{\mathcal{X}*} j_{\mathcal{X}}^! \mathcal{C}_{\mathcal{X}}$ is equal to $\mathcal{C}_{\mathcal{X}^T \setminus X^T}$. Moreover the isomorphism is given by the restriction of the constant sheaves and the Thom isomorphism for the cohomology group of a vector bundle. Therefore it is canonical.

(2) Recall that the nearby cycle functor ψ_{f_X} is given by $r_X^* \tilde{c}_{X*} \tilde{c}_X^*$ as in (4.1), where we put subscripts X to names of maps to indicate we are considering the family $\mathcal{X} \to \mathbb{C}$.

We replace $p_* j^!$ by $p_{-!} j_-^*$ by Theorem 4.9. We have the commutative diagram

$$\begin{CD} X^T @<{p_-}<< \mathcal{R}_X @>{j_-}>> X \\ @V{r_{X^T}}VV @VV{r_{\mathcal{R}_X}}V @VV{r_X}V \\ \mathcal{X}^T @<<{p_{\mathcal{X}-}}< \mathcal{R}_{\mathcal{X}} @>>{j_{\mathcal{X}-}}> \mathcal{X}, \end{CD}$$

where both squares are cartesian. Thus $p_{-!} j_-^* r_X^* = p_{-!} r_{\mathcal{R}_X}^* j_{\mathcal{X}-}^* = r_{X^T}^* p_{\mathcal{X}-!} j_{\mathcal{X}-}^*$. Namely the hyperbolic restriction commutes with the restrictions r_X^*, $r_{X^T}^*$ to 0-fibers.

Next we replace back $p_{\mathfrak{X}-!}j^*_{\mathfrak{X}-}$ to $p_{\mathfrak{X}*}j^!_{\mathfrak{X}}$ and consider the diagram

$$\begin{array}{ccccc}
\mathfrak{X}^T & \xleftarrow{p_{\mathfrak{X}}} & \mathcal{A}_{\mathfrak{X}} & \xrightarrow{j_{\mathfrak{X}}} & \mathfrak{X} \\
{\scriptstyle \tilde{c}_{X^T}}\uparrow & & \uparrow{\scriptstyle \tilde{c}_{\mathcal{A}_X}} & & \uparrow{\scriptstyle \tilde{c}_X} \\
\tilde{\mathfrak{X}}^{T*} & \xleftarrow[p_{\tilde{\mathfrak{X}}^*}]{} & \tilde{\mathcal{A}}^*_{\mathfrak{X}} & \xrightarrow[j_{\tilde{\mathfrak{X}}^*}]{} & \tilde{\mathfrak{X}}^*.
\end{array}$$

The bottom row is pull back to the universal cover $\tilde{\mathbb{C}}^*$ of the upper row. This is commutative and both squares are cartesian. Thus we have $p_{\mathfrak{X}*}j^!_{\mathfrak{X}}\tilde{c}_{X*} = p_{\mathfrak{X}*}\tilde{c}_{\mathcal{A}_X*}j^!_{\tilde{\mathfrak{X}}^*} = \tilde{c}_{X^T*}(p_{\tilde{\mathfrak{X}}^*})_*j^!_{\tilde{\mathfrak{X}}^*}$. Thus the hyperbolic restriction commutes with the pushforward for the coverings $\tilde{c}_X$, $\tilde{c}_{X^T}$.

Finally we commute the hyperbolic restriction with pullbacks for $\tilde{c}_X$, $\tilde{c}_{X^T}$. We do not need to use Theorem 4.9 as we saw $j_{\tilde{\mathfrak{X}}^*}$, $p_{\tilde{\mathfrak{X}}^*}$ are (union of) embedding of smooth closed subvarieties and projections of vector bundles. Therefore $*$ and ! are the same up to shift. (Also Theorem 4.9 is proved for algebraic varieties, and is not clear whether the proof works for $\tilde{\mathfrak{X}}^*$ in general. This problem disappears if we consider the nearby cycle functor in algebraic context.) This finishes the proof of (2). □

4(v). **Hyperbolic semi-smallness.** Looking back the proof of Proposition 4.10, we see that a key observation is the equalities $\operatorname{rank}\mathcal{A}_X = \operatorname{rank}\mathcal{R}_X = \operatorname{codim}_X X^T/2$. (More precisely restriction to each component of X^T.)

In order to prove the exactness of the hyperbolic restriction functor in more general situation, in particular, when we do not have symplectic resolution, we will introduce the notion of hyperbolic semi-smallness in this subsection.

The terminology is introduced in [BFN14], but the concept itself has appeared in [MV07] in the context of the geometric Satake correspondence.

Let $X = \bigsqcup X_\alpha$ be a stratification of X such that $i^!_\alpha \operatorname{IC}(X)$, $i^*_\alpha \operatorname{IC}(X)$ are locally constant sheaves. Here i_α denotes the inclusion $X_\alpha \to X$. We suppose that X_0 is the smooth locus of X as a convention.

We also suppose that the fixed point set X^T has a stratification $X^T = \bigsqcup Y_\beta$ such that the restriction of p to $p^{-1}(Y_\beta) \cap X_\alpha$ is a topologically locally trivial fibration over Y_β for any α, β (if it is nonempty). We assume the same is true for p_-.

Definition 4.13. We say Φ is *hyperbolic semi-small* if the following two estimates hold

$$\begin{aligned} \dim p^{-1}(y_\beta) \cap X_\alpha &\le \frac{1}{2}(\dim X_\alpha - \dim Y_\beta), \\ \dim p_-^{-1}(y_\beta) \cap X_\alpha &\le \frac{1}{2}(\dim X_\alpha - \dim Y_\beta). \end{aligned} \tag{4.14}$$

Suppose X is smooth (and $X = X_0$). Then X^T is also smooth. We decompose $X^T = \bigsqcup Y_\beta$ into connected components as usual. Then the above inequalities must be equalities, i.e., $\operatorname{rank} \mathcal{A}_X|_{Y_\beta} = \operatorname{rank} \mathcal{R}_X|_{Y_\beta} = \operatorname{codim}_X Y_\beta/2$. They are the condition which we have mentioned in the beginning of this subsection.

Note that $p^{-1}(y_\beta) \cap X_0$ and $p_-^{-1}(y_\beta) \cap X_0$ are at most $(\dim X - \dim Y_\beta)/2$-dimensional if Φ is hyperbolic semi-small. In this case, cohomology groups have bases given by $(\dim X - \dim Y_\beta)/2$-dimensional irreducible components of $p^{-1}(y_\beta) \cap X_0$ and $p_-^{-1}(y_\beta) \cap X_0$ respectively. Let $H_{\dim X - \dim Y_\beta}(p^{-1}(y_\beta) \cap X_0)_\chi$ and $H_c^{\dim X - \dim Y_\beta}(p_-^{-1}(y_\beta) \cap X_0)_\chi$ denote the components corresponding to a simple local system χ on Y_β.

Theorem 4.15. *Suppose Φ is hyperbolic semi-small. Then $\Phi(\mathrm{IC}(X))$ is perverse and it is isomorphic to*

$$\bigoplus_{\beta,\chi} \mathrm{IC}(Y_\beta, \chi) \otimes H_{\dim X - \dim Y_\beta}(p^{-1}(y_\beta) \cap X_0)_\chi.$$

Moreover, we have an isomorphism

$$H_{\dim X - \dim Y_\beta}(p^{-1}(y_\beta) \cap X_0)_\chi \cong H_c^{\dim X - \dim Y_\beta}(p_-^{-1}(y_\beta) \cap X_0)_\chi.$$

The proof is similar to one in [MV07, Theorem 3.5], hence the detail is left as an exercise for the reader. In fact, we only use the case when X^T is a point, and the argument in detail for that case was given in [BFN14, Th. A.5].

4(vi). **Hyperbolic restriction for affine Grassmannian.** Recall the affine Grassmannian $\mathrm{Gr}_G = G((z))/G[[z]]$ is defined for a finite dimensional group G. Perverse sheaves on Gr_G are related to finite dimensional representations of $G^\vee$, the Langlands dual of G by the geometric Satake correspondence.

To be expanded.

5. R-matrix

We continue the study of stable envelops for symplectic resolutions.

5(i). **Definition of R-matrix.** Let $\iota\colon M^T \to M$ be the inclusion. Then $\iota^* \circ \mathrm{Stab}_{\mathcal{C}} \in \mathrm{End}(H^{[*]}_{\mathbb{T}}(M^T))$ is upper triangular, and the diagonal entries are multiplication by $e(\mathrm{Leaf}^-_\alpha)$. Since $e(\mathrm{Leaf}^-_\alpha)|_{H^*_T(\mathrm{pt})}$ is nonzero, $\iota^* \circ \mathrm{Stab}_{\mathcal{C}}$ is invertible over $\mathbb{C}(\mathrm{Lie}\,\mathbb{T})$. By the localization theorem for equivariant cohomology groups, ι^* is also invertible. Therefore $\mathrm{Stab}_{\mathcal{C}}$ is invertible over $\mathbb{C}(\mathrm{Lie}\,\mathbb{T})$.

Definition 5.1. Let $\mathcal{C}_1$, $\mathcal{C}_2$ be two chambers. We define the *R-matrix* by

$$R_{\mathcal{C}_1,\mathcal{C}_2} = \mathrm{Stab}^{-1}_{\mathcal{C}_1} \circ \mathrm{Stab}_{\mathcal{C}_2} \in \mathrm{End}(H^{[*]}_{\mathbb{T}}(M^T)) \otimes \mathbb{C}(\mathrm{Lie}\,\mathbb{T}).$$

Example 5.2. Consider $M = T^*\mathbb{P}^1$ with $T = \mathbb{C}^\times$, $\mathbb{T} = \mathbb{C}^\times \times \mathbb{C}^\times$ as before. We denote the corresponding equivariant variables by u, $\hbar$ respectively. (So $H^*_T(\mathrm{pt}) = \mathbb{C}[\mathrm{Lie}\,T] = \mathbb{C}[u]$, $H^*_{\mathbb{T}}(\mathrm{pt}) = \mathbb{C}[\mathrm{Lie}\,\mathbb{T}] = \mathbb{C}[u,\hbar]$.) Choose a chamber $\mathcal{C} = \{u > 0\}$. Then $R_{-\mathcal{C},\mathcal{C}}$ is the middle block of Yang's R-matrix

$$R = 1 - \frac{\hbar P}{u}, \qquad P = \sum_{i,j=1}^{2} e_{ij} \otimes e_{ji},$$

where e_{ij} is the matrix element acting on $\mathbb{C}^2$ up to normalization.

Exercise 8. Check this example.

It is customary to write the R-matrix as $R(u)$ to emphasize its dependence on u. Yang's R-matrix *******

This variable u is called a *spectral parameter* in the context of representation theory of Yangian.

5(ii). **Yang-Baxter equation.** Suppose that T is two dimensional such that $\mathrm{Lie}\,T = \{a_1 + a_2 + a_3 = 0\}$. We suppose that there are six chambers given by hyperplanes $a_1 = a_2$, $a_2 = a_3$, $a_3 = a_1$ as for Weyl chambers for $\mathfrak{sl}(3)$. The cotangent bundle of the flag variety of SL(3) and $\widetilde{\mathcal{U}}^d_3$ are such examples by Exercise 5. We factorize the R-matrix from $\mathcal{C}$ to $-\mathcal{C}$ in two ways to get the Yang-Baxter equation

$$\begin{aligned}R_{12}(a_1 - a_2)&R_{13}(a_1 - a_3)R_{23}(a_2 - a_3)\\ &= R_{23}(a_2 - a_3)R_{13}(a_1 - a_3)R_{12}(a_1 - a_2).\end{aligned}$$

See Figure 3.

5(iii). **R-matrix and Virasoro algebra.** In this section, we study the R-matrix for the case $X = \widetilde{\mathcal{U}}^d_r$.

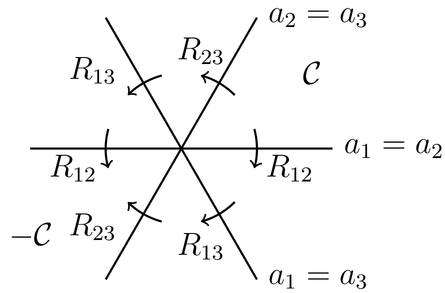

FIGURE 3. Yang-Baxter equation

(a). *Heisenberg operators.* Let us observe that the Heisenberg operators $P^\Delta_{-m}(\alpha)$ sends $H^{\mathbb{T}}_{[*]}(\mathcal{A}_{\widetilde{\mathcal{U}}^d_r})$ to $H^{\mathbb{T}}_{[*-\deg\alpha]}(\mathcal{A}_{\widetilde{\mathcal{U}}^{d+m}_r})$. This is because $q_1(q_2^{-1}(\mathcal{A}_{\widetilde{\mathcal{U}}^d_r}\times\mathbb{C}^2))\subset\mathcal{A}_{\widetilde{\mathcal{U}}^{d+m}_r}$. Therefore the direct sum $\bigoplus_d H^{\mathbb{T}}_{[*]}(\mathcal{A}_{\widetilde{\mathcal{U}}^d_r})$ is a module over the Heisenberg algebra. Via the isomorphism (3.13) we have the Heisenberg algebra module structure on

$$\begin{aligned}\bigoplus_d H^{[*]}_{\mathbb{T}}((\widetilde{\mathcal{U}}^d_r)^T) &= \bigoplus_{d_1,\dots,d_r} H^{[*]}_{\mathbb{T}}((\mathbb{C}^2)^{[d_1]}\times\cdots\times(\mathbb{C}^2)^{[d_r]}) \\ &= \bigotimes_{i=1}^r \bigoplus_{d_i=0}^\infty H^{[*]}_{\mathbb{T}}((\mathbb{C}^2)^{[d_i]}).\end{aligned} \tag{5.3}$$

Note that this is the tensor product of r copies of the Fock space. Therefore it is a representation of the product of r copies of the Heisenberg algebra.

Proposition 5.4. *In the above isomorphism, the operator $P^\Delta_{-m}(\alpha)$ is mapped to the diagonal Heisenberg operator*

$$\sum_{i=1}^r \mathrm{id}\otimes\cdots\otimes\mathrm{id}\otimes\underbrace{P_{-m}(\alpha)}_{i^{\text{th}}\text{ factor}}\otimes\mathrm{id}\otimes\cdots\otimes\mathrm{id}\,.$$

(b). *Virasoro algebra.* Consider the R-matrix for $\widetilde{\mathcal{U}}^d_r$. By ****, it is enough to consider the $r = 2$ case. Then $\bigoplus H^{[*]}_{\mathbb{T}}((\widetilde{\mathcal{U}}^d_2)^T)$ is isomorphic to the tensor product of two copies of Fock space. Let us denote the Heisenberg generator for the first and second factors by $P^{(1)}_m(\alpha)$, $P^{(2)}_m(\alpha)$ respectively. We have $P^\Delta_m(\alpha) = P^{(1)}_m(\alpha) + P^{(2)}_m(\alpha)$. Since $P^\Delta_m(\alpha)$ is defined a correspondence which makes sense without going to fixed points $(\widetilde{\mathcal{U}}^d_2)^T$, it commutes the R-matrix. Therefore the R-matrix should be described by anti-diagonal Heisenberg generators $P^{(1)}_m(\alpha)-P^{(2)}_m(\alpha)$. Let us denote them by $P^-_m(\alpha)$.

Let us denote the corresponding Fock spaces by F^Δ and F^-. Therefore we have $\bigoplus H^{[*]}_{\mathbb{T}}((\widetilde{\mathcal{U}}^d_2)^T) \cong F^\Delta \otimes F^-$. The above observation means that the R-matrix is of a form $\mathrm{id}_{F^\Delta} \otimes R'$ for some operator R' on F^-.

We now characterize R' in terms of the Virasoro algebra, acting on F^- by the well-known Feigin-Fuchs construction.

Let us recall the Feigin-Fucks construction. We take the tensor product $H^{[*]}_{\mathbb{T}}((\widetilde{\mathcal{U}}^d_2)^T) \otimes_{H^*_{\mathbb{T}}(\mathrm{pt})} \mathbb{C}(\varepsilon_1, \varepsilon_2, \vec{a})$ from now. It means that we consider variables ε_1, ε_2, $\vec{a}$ take *generic value*.

Let us write P^-_n for $P^-_n(1)$ for brevity, where $1 \in H^{[*]}_{\mathbb{T}}(\mathbb{C}^2)$ is the unit element. Note that we have $\langle 1, 1\rangle = -1/\varepsilon_1\varepsilon_2$ in the commutation relation. Thus we will consider the localized equivariant cohomology groups hereafter. We will comment the integral form at the end ******.

We put

$$P^-_0 \overset{\text{def.}}{=} \frac{1}{\varepsilon_1\varepsilon_2}\left(a_1 - a_2 - (\varepsilon_1 + \varepsilon_2)\right).$$

This element is central, i.e., it commutes with all other P^-_m. In particular, the Heisenberg relation (2.3) remains true.

We then define

$$L_n \overset{\text{def.}}{=} -\frac{\varepsilon_1\varepsilon_2}{4}\sum_m {:}P^-_m P^-_{n-m}{:} - \frac{n+1}{2}(\varepsilon_1+\varepsilon_2)P^-_n. \tag{5.5}$$

These L_n satisfy the Virasoro relation

$$[L_m, L_n] = (m-n)L_{m+n} + \left(1 + \frac{6(\varepsilon_1+\varepsilon_2)^2}{\varepsilon_1\varepsilon_2}\right)\delta_{m,-n}\frac{m^3-m}{12}$$

with the central charge $1 + 6(\varepsilon_1+\varepsilon_2)^2/\varepsilon_1\varepsilon_2$.

The vacuum vector $|\mathrm{vac}\rangle = 1_{\widetilde{\mathcal{U}}^0_2} \in H^{[*]}(\widetilde{\mathcal{U}}^d_2)$ is a *highest weight vector*, it is killed by L_n $(n > 0)$ and satisfies

$$L_0|\mathrm{vac}\rangle = -\frac{1}{4}\left(\frac{(a_1-a_2)^2}{\varepsilon_1\varepsilon_2} - \frac{(\varepsilon_1+\varepsilon_2)^2}{\varepsilon_1\varepsilon_2}\right)|\mathrm{vac}\rangle.$$

Here we have used the normal ordering $:\ :$, which is defined by moving all annihilation operators to the right. See [Nak99, Def. 9.34] for more detail.

It is known that the Fock space, as a representation of the Virasoro algebra, irreducible if its highest weight is generic. Moreover its isomorphism class is determined by its highest weight (and the central charge).

Looking at the above formula for the highest weight, we see that it is unchanged under the exchange $a_1 \leftrightarrow a_2$. Therefore there exists the unique automorphism on F^- (over $\mathbb{C}(\varepsilon_1, \varepsilon_2, \vec{a})$) sending $|\mathrm{vac}\rangle$ to itself, and intertwining L_n and L_n with $a_1 \leftrightarrow a_2$. It is called the *reflection operator*.

Now a fundamental observation due to Maulik-Okounkov is the following result:

Theorem 5.6 ([MO12]). *The R-matrix is* $\mathrm{id}_{F^\Delta} \otimes(\text{reflection operator})$.

In the remainder of this section, we explain several key points of the proof of Theorem 5.6.

(c). *Virasoro algebra and Hilbert schemes.* The first link between the Virasoro algebra and cohomology groups of instanton moduli spaces was found for the rank 1 case by Lehn [Leh99]. Lehn's result holds for an arbitrary nonsingular complex quasiprojective surface, but let us specialize to the case $\mathbb{C}^2$.

Let $\mathcal{V}$ be the tautological bundle over the Hilbert scheme $(\mathbb{C}^2)^{[d]}$. It is a rank d vector bundle whose fiber at $I \subset \mathbb{C}[x,y]$ is $\mathbb{C}[x,y]/I$. In the quiver variety description, it is the vector bundle associated with the principal $\mathrm{GL}(d)$-bundle $\mu^{-1}(0)^{\mathrm{stable}} \to (\mathbb{C}^2)^{[d]}$. We consider its first Chern class $c_1(\mathcal{V})$.

Let P_n denote the Heisenberg operator $P_n^\Delta(1)$ for the $r = 1$ case.

Theorem 5.7 ([Leh99]). *We have*

$$c_1(\mathcal{V}) \cup \bullet = -\frac{(\varepsilon_1\varepsilon_2)^2}{3!} \sum_{m_1+m_2+m_3=0} :P_{m_1} P_{m_2} P_{m_3}: \\ - \frac{\varepsilon_1\varepsilon_2(\varepsilon_1+\varepsilon_2)}{4} \sum_m (|m|-1) :P_{-m} P_m:.$$

Taking the commutator with P_n, we have

$$[c_1(\mathcal{V}) \cup \bullet, P_n] = \frac{n\varepsilon_1\varepsilon_2}{2} \sum_{l+m=n} :P_l P_m: - \frac{n(|n|-1)}{2} (\varepsilon_1+\varepsilon_2) P_n.$$

If we compare this with (5.5), we find that this looks very similar to nL_n, except a mysterious expression $|n|$.

A different proof, which works only for $\mathbb{C}^2$, was given in [Nak14].

(d). *R-matrix at the minimal element.*

To be expanded.

6. Perverse sheaves on Uhlenbeck partial compactification

We now turn to $\mathcal{U}_G^d$ for general G.

6(i). **Hyperbolic restriction on Uhlenbeck partial compactification.** Let $\rho\colon \mathbb{C}^\times \to T$ be a one parameter subgroup. We have associated Levi and parabolic subgroups

$$L = G^{\rho(\mathbb{C}^\times)}, \qquad P = \left\{ g \in G \,\middle|\, \exists \lim_{t\to 0} \rho(t) g \rho(t)^{-1} \right\}.$$

Unlike before, here we allow nongeneric ρ so that $G^{\rho(\mathbb{C}^\times)}$ could be different from T. This is not an actual generalization. We can replace T by $Z(L)^0$, the connected center of L. Then ρ above can be considered as a generic one parameter subgroup in $Z(L)^0$.

We consider the induced $\mathbb{C}^\times$-action on $\mathcal{U}_G^d$. Let us introduce the following notation for the diagram (3.4):

$$\mathcal{U}_L^d \overset{\text{def.}}{=} (\mathcal{U}_G^d)^{\rho(\mathbb{C}^\times)} \underset{i}{\overset{p}{\leftrightarrows}} \mathcal{U}_P^d \overset{\text{def.}}{=} \mathcal{A}_{\mathcal{U}_G^d} \xrightarrow{j} \mathcal{U}_G^d. \tag{6.1}$$

Let us explain how these notation can be justified. If we restrict our concern to the open subscheme Bun_G^d, a framed G-bundle $(\mathcal{F}, \varphi)$ is fixed by $\rho(\mathbb{C}^\times)$ if and only if we have an L-reduction $\mathcal{F}_L$ of $\mathcal{F}$ (i.e., $\mathcal{F} = \mathcal{F}_L \times_L G$) so that $\mathcal{F}_L|_{\ell_\infty}$ is sent to $\ell_\infty \times L$ by the trivialization ρ. Thus $(\operatorname{Bun}_G^d)^{\rho(\mathbb{C}^\times)}$ is the moduli space of framed L-bundles, which we could write Bun_L^d.[12] The definition of Uhlenbeck partial compactification is a little delicate, and is defined for almost simple groups. Nevertheless it is still known that $\mathcal{U}_L^d$ is homeomorphic to the Uhlenbeck partial compactification for $[L, L]$ when it has only one simple factor ([BFN14, 4.7]), though we do not know they are the same as schemes or not. We will actually use this fact later, therefore the same notation for fixed point subschemes and genuine Uhlenbeck partial compactifications are natural for us.

Let us turn to the notation $\mathcal{U}_P^d$. If we have a framed P-bundle $(\mathcal{F}_P, \varphi)$, the associated framed G-bundle $(\mathcal{F}_P \times_P G, \varphi \times_P G)$ is actually point in the attracting set $\mathcal{A}_{\mathcal{U}_G^d}$. This is the reason why we use the notation $\mathcal{U}_P^d$. However a point in $\mathcal{U}_P^d \cap \operatorname{Bun}_G^d$ is not necessarily coming from a framed P-bundle like this. See Exercise **??** below. Nevertheless we believe that it is safe to use the notation $\mathcal{U}_P^d$, as we never consider genuine Uhlenbeck partial compactifications for the parabolic subgroup P.

Exercise 9 (cf. an example in [BFN14, §4(iv)])**.** Consider the case $G = \mathrm{SL}(r)$. Suppose (E, φ) is a framed vector bundle which fits in an

[12] Here we use the assumption G is of type ADE. The instanton number is defined via an invariant bilinear form on $\mathfrak{g}$. For almost simple groups, we normalize it so that the square length of the highest root θ is 2. If G is of type ADE, instanton numbers are preserved for fixed point sets, but not in general. See [BFN14, 2.1].

exact sequence $0 \to E_1 \to E \to E_2 \to 0$ compatible with the framing. Here we merely assume E_1, E_2 are torsion-free sheaves. Are E_1, E_2 locally free ?

Definition 6.2. Now we consider the hyperbolic restriction functor $p_* j^! \colon D^b_{\mathbb{T}}(\mathcal{U}^d_G) \to D^b_{\mathbb{T}}(\mathcal{U}^d_L)$ and denote it by $\Phi^P_{L,G}$. If groups are clear from the context, we simply denote it by Φ.

We have the following associativity of hyperbolic restrictions.

Proposition 6.3. *Let Q be another parabolic subgroup of G, contained in P and let M denote its Levi subgroup. Let Q_L be the image of Q in L and we identify M with the corresponding Levi group. Then we have a natural isomorphism of functors*

$$\Phi^{Q_L}_{M,L} \circ \Phi^P_{L,G} \cong \Phi^Q_{M,G}.$$

Proof. It is enough to show that

$$\mathcal{U}^d_P \times_{\mathcal{U}^d_L} \mathcal{U}^d_{Q_L} = \mathcal{U}^d_Q.$$

This is easy to check. See [BFN14, 4.16]. □

6(ii). **Exactness.** For a partition λ, let $S_\lambda\mathbb{C}^2$ be a stratum of a symmetric product as before. Let $\operatorname{Stab}(\lambda) = S_{\alpha_1} \times S_{\alpha_2} \times \dots$ if we write $\lambda = (1^{\alpha_1} 2^{\alpha_2} \dots)$. We consider an associated covering

$$(\mathbb{C}^2)^{\alpha_1} \times (\mathbb{C}^2)^{\alpha_2} \times \cdots \setminus \text{diagonal} \to S_\lambda(\mathbb{C}^2).$$

Let ρ be a simple local system over $S_\lambda\mathbb{C}^2$ corresponding to an irreducible representation of $\operatorname{Stab}(\lambda)$.

We consider the following class of perverse sheaves:

Definition 6.4. Let $\operatorname{Perv}(\mathcal{U}^d_L)$ be the additive subcategory of the abelian category of semisimple perverse sheaves on $\mathcal{U}^d_L$, consisting of finite direct sum of $\operatorname{IC}(\operatorname{Bun}^{d'}_L \times S_\lambda(\mathbb{C}^2), 1 \boxtimes \rho)$ for various d', λ, ρ.

Here we consider the stratification of $\mathcal{U}^d_L$ as in (1.2):

$$\mathcal{U}^d_L = \bigsqcup_{d=|\lambda|+d'} \operatorname{Bun}^{d'}_L \times S_\lambda(\mathbb{C}^2).$$

It is the restriction of the stratification (1.2) to $\mathcal{U}^d_L$.

Let us explain why we need to consider nontrivial local systems, even though our primary interest will be on $\operatorname{IC}(\mathcal{U}^d_G)$: When we analyze $\operatorname{IC}(\mathcal{U}^d_G)$ through hyperbolic restriction functor, IC sheaves for nontrivial local systems occur. This phenomenon can be seen for type A as follows.

Let us take the Gieseker space $\widetilde{\mathcal{U}}^d_r$ and consider the hyperbolic restriction for a chamber $\mathcal{C}$ for the T-action. By Corollary 4.11(2), we have $p_* j^! \pi_*(\mathcal{C}_{\widetilde{\mathcal{U}}_r^d}) \cong \pi_*^T(\mathcal{C}_{(\widetilde{\mathcal{U}}_r^d)^T})$, where $(\widetilde{\mathcal{U}}_r^d)^T$ is the fixed point set and $\pi^T \colon (\widetilde{\mathcal{U}}_r^d)^T \to (\mathcal{U}_r^d)^T$ is the restriction of π. The fixed point sets are given by Hilbert schemes and symmetric products, and π^T factors as

$$(\widetilde{\mathcal{U}}_r^d)^T = \bigsqcup_{d_1+\cdots+d_r=d} (\mathbb{C}^2)^{[d_1]} \times \cdots \times (\mathbb{C}^2)^{[d_r]} \xrightarrow{\pi\times\cdots\times\pi} \bigsqcup S^{d_1}\mathbb{C}^2 \times \cdots \times S^{d_r}\mathbb{C}^2 \xrightarrow{\kappa} S^d\mathbb{C}^2 = (\mathcal{U}_r^d)^T,$$

where κ is the 'sum map', defined by $\kappa(C_1, \cdots, C_r) = C_1 + \cdots + C_r$ if we use the 'sum notation' for points in symmetric products, like $x_1 + x_2 + \cdots + x_d$. The pushforward for the first factor $\pi \times \cdots \times \pi$ is

$$\bigoplus_{|\lambda^1|+\cdots+|\lambda^r|=d} \mathcal{C}_{\overline{S_{\lambda^1}(\mathbb{C}^2)}} \boxtimes \cdots \boxtimes \mathcal{C}_{\overline{S_{\lambda^r}(\mathbb{C}^2)}}$$

by discussion in §2(iii), where $\lambda^1, \dots, \lambda^r$ are partitions. Therefore we do not have nontrivial local systems. But κ produces nontrivial local systems. Since κ is a finite morphism, in order to calculate its pushforward, we need to study how κ restricts to covering on strata. For example, for $\lambda = (1^r)$, $d_1 = d_2 = \cdots = d_r = 1$, it is a standard S_r-covering $(\mathbb{C}^2)^r \setminus \text{diagonal} \to S_{(1^r)}(\mathbb{C}^2)$.

We have the following

Theorem 6.5. *$\Phi_{L,G}^P$ sends $\operatorname{Perv}(\mathcal{U}_G^d)$ to $\operatorname{Perv}(\mathcal{U}_L^d)$.*

By the remark after Theorem 4.9 we know that $\Phi_{L,G}^P$ send $\operatorname{Perv}(\mathcal{U}_G^d)$ to semisimple complexes. From the factorization, it is more or less clear that they must be direct sum of shifts of simple perverse sheaves in $\operatorname{Perv}(\mathcal{U}_L^d)$. Therefore the actual content of this theorem is the t-exactness, that is *shifts are unnecessary.* For type A, it is a consequence of Corollary 4.11.

For general G, we use Theorem 4.15. The detail is given in [BFN14, Appendix A], and is a little complicated to be reproduced here. Let us mention only several key points: By a recursive nature and the factorization property of Uhlenbeck partial compactifications, it is enough to estimate dimension of the extreme fibers, i.e., $p^{-1}(d \cdot 0)$, $p_-^{-1}(d \cdot 0)$ in (4.14). Furthermore using the associativity of hyperbolic restriction (Proposition 6.3), it is enough to prove the case $L = T$. In fact, we can further reduce to the case of the hyperbolic restriction for the larger torus $T \times \mathbb{C}^\times_{\text{hyp}}$. The associativity (Proposition 6.3) remains true for the larger torus. Finally, we consider affine Zastava spaces, i.e., Uhlenbeck partial compactificationss corresponding to moduli spaces of framed parabolic bundles. In some sense, these spaces behave well, and we have required dimension estimate there.

6(iii). **Calculation of the hyperbolic restriction.**

(a). *The space* U^d. Our next task is to compute $\Phi_{L,G}^P(\mathrm{IC}(\mathcal{U}_G^d))$. We have two most extreme simple direct summands in it:

(a) $\mathrm{IC}(\mathcal{U}_L^d)$,
(b) $\mathcal{C}_{S_{(d)}(\mathbb{C}^2)}$.

Other direct summands are basically products of type (a) and (b). Let us first consider (a). Let us restrict the diagram (6.1) to Bun_L^d. Note that a point in $p^{-1}(\mathrm{Bun}_L^d)$ is a genuine bundle, cannot have singularities, as singularities are equal or increased under p. Thus the diagram sit in moduli spaces of genuine bundles as

$$\mathrm{Bun}_L^d \underset{i}{\overset{p}{\leftrightarrows}} \mathrm{Bun}_P^d \xrightarrow{j} \mathrm{Bun}_G^d .$$

Then Bun_P^d is a vector bundle over Bun_L^d. This can be seen as follows. For $\mathcal{F} \in \mathrm{Bun}_G^d$, the tangent space $T_{\mathcal{F}} \mathrm{Bun}_G^d$ is $H^1(\mathbb{P}^2, \mathfrak{g}_{\mathcal{F}}(-\ell_\infty))$, where $\mathfrak{g}_{\mathcal{F}}$ is the associated bundle $\mathcal{F} \times_G \mathfrak{g}$. If $\mathcal{F} \in \mathrm{Bun}_L^d$, we have the decomposition

$$\begin{aligned} &H^1(\mathbb{P}^2, \mathfrak{g}_{\mathcal{F}}(-\ell_\infty)) \\ &\qquad \cong H^1(\mathbb{P}^2, \mathfrak{l}_{\mathcal{F}}(-\ell_\infty)) \oplus H^1(\mathbb{P}^2, \mathfrak{n}^+_{\mathcal{F}}(-\ell_\infty)) \oplus H^1(\mathbb{P}^2, \mathfrak{n}^-_{\mathcal{F}}(-\ell_\infty)), \end{aligned}$$

according to the decomposition $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{n}^+ \oplus \mathfrak{n}^-$. Here $\mathfrak{n}^+$ is the nilradical of $\mathfrak{p}$, and $\mathfrak{n}^-$ is the nilradical of the opposite parabolic. The first summand gives the tangent bundle of Bun_L^d. Thus the normal bundle is given by sum of the second and third summands. Moreover, the second and third summands are Leaf and Leaf^- respectively.

Therefore we have the Thom isomorphism $p_* j^! \mathcal{C}_{\mathrm{Bun}_G^d} \cong \mathcal{C}_{\mathrm{Bun}_L^d}$. It extends to a canonical isomorphism

$$\mathrm{Hom}(\mathrm{IC}(\mathrm{Bun}_L^d), p_* j^! \,\mathrm{IC}(\mathcal{U}_G^d)) \cong \mathbb{C}.$$

In other words, we have $p^{-1}(y_\beta) \cap X_0$ in Theorem 4.15 is the fiber of the vector bundle $\mathrm{Bun}_P^d \to \mathrm{Bun}_L^d$, hence we have the canonical isomorphism $H_{\dim X - \dim Y_\beta}(p^{-1}(y_\beta \cap X_0)_\chi \cong \mathbb{C}$, given by its fundamental class.

On the other hand, the multiplicity of the direct summand $\mathcal{C}_{S_{(d)}(\mathbb{C}^2)}$ of type (b) has no simple description. Therefore let us introduce the following space:

$$U^d \equiv U_{L,G}^{d,P} \overset{\text{def.}}{=} \mathrm{Hom}(\mathcal{C}_{S_{(d)}(\mathbb{C}^2)}, \Phi_{L,G}^P(\mathrm{IC}(\mathcal{U}_G^d))).$$

Thus $U^d \otimes \mathcal{C}_{S_{(d)}(\mathbb{C}^2)}$ is the isotropic component of $\Phi_{L,G}^P(\mathrm{IC}(\mathcal{U}_G^d))$ for $\mathcal{C}_{S_{(d)}(\mathbb{C}^2)}$.

From the factorization, we have the canonical isomorphism

$$\Phi^P_{L,G}(\mathrm{IC}(\mathcal{U}^d_G)) \cong \bigoplus_{d=|\lambda|+d'} \mathrm{IC}(\mathrm{Bun}^{d'}_L \times S_\lambda(\mathbb{C}^2), (U^1)^{\otimes\alpha_1} \otimes (U^2)^{\otimes\alpha_2} \otimes \cdots),$$

where $\lambda = (1^{\alpha_1} 2^{\alpha_2} \dots)$ and $(U^1)^{\otimes\alpha_1} \otimes (U^2)^{\otimes\alpha_2} \otimes \cdots$ is a representation of $\mathrm{Stab}(\lambda) = S_{\alpha_1} \times S_{\alpha_2} \times \cdots$. Though $(U^1)^{\otimes\alpha_1} \otimes (U^2)^{\otimes\alpha_2} \otimes \cdots$ may not be irreducible, it is always semisimple. We understand the associated IC-sheaf as the direct sum of IC-sheaves of simple constitutes.

If we take the global cohomology group, contributions of nontrivial local systems vanish. (See [BFN14, Lemma 4.31].) Therefore

$$\bigoplus_d H^*_{\mathbb{T}}(\Phi^P_{L,G}(\mathrm{IC}(\mathcal{U}^d_G)) \cong \bigoplus_d H^*_{\mathbb{T}}(\mathrm{IC}(\mathrm{Bun}^d_L)) \otimes \mathrm{Sym}\left(U^1 \oplus U^2 \oplus \cdots\right).$$

(b). *Affine Grassmannian for an affine Lie algebra.* By [BFN14, 4.39], we have

$$\dim U^d = \operatorname{rank} G - \operatorname{rank}[L, L],$$

in particular $\dim U^d = \operatorname{rank} G$ if $L = T$. This result was proved as

1. (1) a reduction to the case $L = T$ via the associativity (Proposition 6.3),
2. (2) a reduction to a computation of the ordinary restriction to $S_{(d)}\mathbb{A}^2$ thanks to a theorem of Laumon [Lau81], and
3. (3) the computation of the ordinary restriction to $S_{(d)}\mathbb{A}^2$ in [BFG06, Theorem 7.10].

Instead of explaining the detail of this argument, we explain the result in view of the double affine Grassmannian proposed by Braverman-Finkelberg [BF10] in the remainder of this subsection. For this purpose, it is more natural to drop the assumption G is of type ADE.
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By [Kac90, Prop. 12.13] we have

$$\mathrm{mult}_{L(\Lambda_0)}(\Lambda_0 - k\delta) = p^A(k),$$

where

$$\sum_{k\geq 0} p^A(k) q^k = \prod_{n\geq 1} (1-q^n)^{-\operatorname{mult} n\delta}.$$

Moreover $\operatorname{mult} n\delta = \ell$ if $\mathfrak{g} = X^{(1)}_\ell$ where X is of type ADE (hence $\mathfrak{g}^\vee = \mathfrak{g}$).[13]

[13] Suppose $\mathfrak{g} = X^{(1)}_\ell$ where X is of type $BCFG$. Let $r^\vee$ be the lacing number of $\mathfrak{g}$, i.e., the maximum number of edges connecting two vertices of the Dynkin diagram of $\mathfrak{g}$ (and $\mathfrak{g}^\vee$). Then $\operatorname{mult} n\delta = \ell$ if n is a multiple of r, and equals to the number of long simple roots in the finite dimensional Lie algebra X_ℓ otherwise. Explicitly $\ell - 1$ for B_ℓ, 1 for C_ℓ and G_2, and 2 for F_4. See [Kac90, Cor. 8.3].

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Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan

E-mail address: nakajima@kurims.kyoto-u.ac.jp