$T$–analog of $q$–characters of
finite dimensional representations of
quantum affine algebras
– explicit examples

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Quantum Loop Algebras

• $g$: simple Lie algebra of type ADE /C
  (or symmetric Kac-Moody)
• $L_g = g \otimes \mathbb{C}[z, z^{-1}]$: its loop algebra
• $U_q(L_g)$: quantum loop algebra
  = quantum affine algebra $U_q(\hat{g})$
    without central extension & degree operator
  = $\mathbb{C}(q)$-algebra generated by $q^h, e_{i,r}, f_{i,r}, h_{i,n}$
    ($h \in P^*, i \in I, r \in \mathbb{Z}, n \in \mathbb{Z} \setminus \{0\}$)
  with certain relations
• $U_\varepsilon(L_g)$: specialized quantum loop algebra ($\varepsilon \in \mathbb{C}^*$)

Assume $\varepsilon \notin \sqrt{T}$ today. Write $q$ instead of $\varepsilon$ from now.
$U_q(Lh) \overset{\text{def.}}{=} \langle q^h, h_i, \pm m \rangle \subset U_q(Lg) : \text{commutative subalgebra}$

Let $p_i^\pm(z) \overset{\text{def.}}{=} \exp \left( - \sum_{m=1}^{\infty} \frac{h_i, \pm m}{[m]_{\ell}} z^m \right).$ (generating function)

### Analogy

Weights for $g$-modules $\longleftrightarrow \ell$-weights for $U_q(Lg)$-modules

- **$\ell$-weight space** $\overset{\text{def.}}{=} \text{a simultaneous generalized eigenspace}$ of $q^h, h_i, \pm m$

- **$\ell$-highest weight module $V$** $\overset{\text{def.}}{\Longleftrightarrow} \exists \text{vector } v \in V - e_{i,r} v = 0 \ \forall i, r$
  - simul. eigenv. for $q^h, h_i, \pm m$
  - $V = U_q(Lg)v$

**Slide 3**

- \{simple $\ell$-highest wt. modules\} $\longleftrightarrow$ \{\$\ell$-highest wts.\}

- **$\ell$-weight** $\overset{\text{def.}}{=} \text{eigen value (for finite dim. modules)}$
  $\longleftrightarrow \left( \frac{P_i(z)}{Q_i(z)} \right)_{i \in I}$ with $P_i(0) = Q_i(0) = 1,$ coprime
  s.t. $q^h$ acts by $q^{(h_i,\deg P_i - \deg Q_i) \varpi_i} + \text{nilp}$
  $p_i^+(z)$ acts by $\frac{P_i(z)}{Q_i(z)} + \text{nilp}$
  $p_i^-(z)$ acts by $\frac{z^{-\deg P_i} P_i(z)}{z^{-\deg Q_i} Q_i(z)} \times \text{const.} + \text{nilp}$

- **$\ell$-weight is $\ell$-dominant** $\overset{\text{def.}}{\Longleftrightarrow} \forall i \ Q_i(z) = 1,$ i.e. polynomial

- a simple $\ell$-highest wt. module is finite dim. (or integrable)
  $\longleftrightarrow$ $\ell$-highest weight is $\ell$-dominant
    (Drinfeld, Chari-Pressley)
Problem

Study the category of finite dimensional representations of $U_q(Lg)$.

New features

- No $q$-analog of the evaluation map $Lg \rightarrow g \ (z \mapsto a \in \mathbb{C}^*)$
- Therefore a $U_q(g)$-module may not be lifted to a $U_q(Lg)$-module in general.
- The category is not semisimple.
- $V \otimes W$ may not be isomorphic to $W \otimes V$.
- canonical/crystal base may not exist.

$t$–deformation of the representation ring

- $R \overset{\text{def}}{=} \text{Rep } U_q(Lg)$ : Grothendieck ring of finite dimensional $U_q(Lg)$-modules (commutative ring, i.e. $[V \otimes W]=[W \otimes V]$)
- $\{L(P)\}$ : simple finite dim. modules give a basis of $R$
- $R_t$ : its $t$-deformation (noncommutative algebra over $\mathbb{Z}[t,t^{-1}]$) can be defined. (via filtration on modules....)
- $L(P)$ has a characterization: analog of Kazhdan-Lusztig basis, PBW/canonical basis
  - $L(P) = L(P)$,
  - $L(P) \in M(P) + \sum_{Q:Q<P} t^{-1}Z[t^{-1}]M(Q)$
$L(P)$ has a characterization

\[- L(P) = L(P), \]
\[- L(P) \in M(P) + \sum_{Q : Q < P} t^{-1}Z[t^{-1}]M(Q) \]

where $M(P)$ is a standard module:

- a tensor product of level 0-fundamental modules w.r.t. a certain order
- (a specialization of) an extrmal weight module of Kashiwara (= a Weyl module of Chari-Pressley)

level 0-fundamental module:

\[ \exists i \in I, a \in \mathbb{C}^* \text{ s.t. } P_i(z) = 1 - az, P_j(z) = 1 \text{ for } j \neq i. \]

Cf. Type A: Ginzburg-Vasserot, Arakawa

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- $R_t$ can be defined by perverse sheaves on \textit{graded quiver varieties} = (quiver varieties)$^{\text{torus fixed pts}}$.

**Analogy**

$R_t$ via graded quiver varieties $\iff$ $(U^-)^*$ via quivers (Lusztig)

The affine graded quiver variety $\mathcal{M}_0^*(P)$ has a stratification:

\[ \mathcal{M}_0^*(P) = \bigcup_{Q \leq P} \mathcal{M}_0^{\text{reg}}(Q, P) \]

parametrized by $I$-tuples of polynomials $Q$.

- $L(Q)$ : simple module $\iff$ $IC(\mathcal{M}_0^{\text{reg}}(Q, P)) :$ IC sheaf
- $M(Q)$ : standard module $\iff$ $\mathcal{C}_{\mathcal{M}_0^{\text{reg}}(Q, P)} :$ constant sheaf extended by 0
\[ Q \leq P \overset{\text{def}}{\iff} \mathcal{M}_0^{\text{reg}}(Q, R) \subseteq \mathcal{M}_0^{\text{reg}}(P, R) \text{ (indep. of } R) \]

From the characterization we have
\[
\left( [L(P) : M(Q)] \right)^{\text{determined}} \overset{\iff}{=} \left( [M(P) : M(Q)] \right)
\]
Hence the Problem is reduced to

\[ \text{How to compute } M(P) ? \overset{\sim}{\iff} t\text{-analog of } q\text{-character} \]

\[ \chi_{q,t}(M(P)) = \text{the generating function of Betti numbers of smooth graded quiver varieties } \mathcal{M}^*(Q/R, P) \]

\[ = \sum_{Q/R:\ell\text{-weight}} \text{gr-dim}(\ell\text{-wt. sp.}) \times e^{Q/R} \]

where \( e^{Q/R} = \prod_i \prod_a Y_{i,a}^{u_i(a)} \) for \( Q_i(z) = \prod_a (1 - az)^{u_i(a)} \).

\[ \chi_{q,t}(M(P)) = \chi_{q,t}(M(P_1) \otimes M(P_2)) = \chi_{q,t}(M(P_1)) \ast \chi_{q,t}(M(P_2)) \]

\[ \chi_{q,t}(\text{level 0 fund. rep.}) \text{ can be determined from the } \ell \text{-highest weight vector recursively. . . . . . Condition } \]

\[ \text{cf. Frenkel-Mukhin : } \chi_{q,t} \in \text{Ker(screening op.'s)} \]

\[ \text{In Summary,} \]

\[ \chi_{q,t} \text{ is "computable" theoretically, but not practically.} \]
$q$–character (or $\ell$–character)

Recall the character of a $g$-module:

$$\text{ch}(V) = \sum_{\lambda: \text{weight}} \dim(\text{wt. sp.}) \times e^\lambda$$

**Definition.** (Frenkel-Reshetikhin)

$$\chi_q(M) \overset{\text{def.}}{=} \sum_{P(z) \atop Q(z)} \dim(\ell\text{-wt. sp.}) \times e^{P(z)/Q(z)}$$

where $e^{P(z)/Q(z)} = \prod_{i \in I} \prod_{a} Y_{i,a}^{u_i(a)}$ for $P_i(z) = \prod_a (1 - az)^{u_i(a)}$.

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**Example.** $g = sl_{n+1} : V = \text{vector representation}$

It can be lifted to a $U_q(Lg)$-module.

$$\begin{array}{cccc}
\bullet & \overset{f_{1,0}}{\longrightarrow} & \overset{f_{2,0}}{\longrightarrow} & \ldots \longrightarrow \\
\text{weight} & q^{\omega_1} & q^{\omega_2 - \omega_1} & q^{\omega_n} \\
\ell\text{-weight} & Y_{1,1} & Y_{1,q^{2}Y_{2,q}}^{-1} & Y_{1,q^{n+1}}^{-1} \\
\mid & \mid & \mid \\
1 & 2 & \ldots \\
\end{array}$$

\[ \therefore \chi_q(V) = \sum_{\square} \overset{n+1}{\text{(tableaux sum expression)}} \]

For later purpose, let us introduce

$$\begin{align*}
\square_k &= Y_{1,q^k}, \\
\square_{k+1}^2 &= Y_{1,q^{k+2}}^{-1}Y_{2,q^{k+1}}, \text{ etc.}
\end{align*}$$
Example. $g = \mathfrak{sl}_2$, $\ell$-highest wt. $P(z) = (1-z)(1-zq) \cdots (1-zq^{2(n-1)})$

a lift of the irreducible $n$-dim. rep. of $U_q(\mathfrak{sl}_2)$ (evaluation module)

\[
\begin{array}{cccc}
\cdot & f_{1,0} & \cdot & f_{1,0}
\end{array}
\quad \begin{array}{cccc}
\cdot & f_{1,0} & \ldots & f_{1,0} & \cdot
\end{array}
\quad \begin{array}{c}
q^n
\end{array}
\quad \begin{array}{c}
q^{n-2}
\end{array}
\quad \begin{array}{c}
q^{-n}
\end{array}
\]

\[
\begin{array}{cccc}
(1-z)(1-zq) & \cdots & (1-zq^{n-1}) & \cdots
\end{array}
\quad \begin{array}{cccc}
(1-z)(1-zq) & \cdots & (1-zq^{n-1}) & \cdots
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(1-z)(1-zq)
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\quad \begin{array}{c}
(1-z)(1-zq)
\end{array}
\quad \begin{array}{c}
(1-z)(1-zq)
\end{array}
\]

tableau

\[
\begin{array}{cccc}
1 & 1 & 1 & \ldots
\end{array}
\quad \begin{array}{cccc}
2 & 2 & 2 & \ldots
\end{array}
\quad \begin{array}{c}
1 & \ldots
\end{array}
\quad \begin{array}{c}
2 & \ldots
\end{array}
\quad \begin{array}{c}
(n-1)
\end{array}
\]

\[
\therefore \quad \chi_q(L(P)) = \sum_{1 \leq i_1 \leq i_2 \leq \ldots \leq i_n \leq 2} \text{tableau}
\]

Let us prove this by our method!

For example, $n = 4$:

\[
\begin{align*}
\chi_{q,t}(M(P)) &= 1 \cdot 1 \cdot 1 \cdot 1 = Y_{1,1}Y_{1,q}Y_{1,q^2}Y_{1,q^4} + \ldots \\
&\quad + t^{-1} 2 \cdot 1 \cdot 1 \cdot 1 = Y_{1,q}Y_{1,q^4} + \ldots \\
&\quad + t^{-1} 2 \cdot 1 \cdot 1 \cdot 1 = Y_{1,1}Y_{1,q} + \ldots \\
&\quad + t^{-2} 2 \cdot 1 \cdot 1 \cdot 1 = Y_{1,1}Y_{1,q^2} + \ldots \\
&\quad + t^{-2} 2 \cdot 1 \cdot 1 \cdot 1 = 1 + \ldots
\end{align*}
\]

\[
\therefore \chi_{q,t}(L(P)) = \chi_{q,t}(M(P)) - t^{-1} \chi_{q,t}(M(Y_{1,q}Y_{1,q^4})) - t^{-1} \chi_{q,t}(M(Y_{1,1}Y_{1,q})) - t^{-1} \chi_{q,t}(M(Y_{1,1}Y_{1,q^2})) + t^{-2} \chi_{q,t}(M(1))
\]
**Example.** \( V = \text{vector representation} \)

We define the ordering \( < \) on the set \( B = \{1, \ldots, n, \overline{n}, \ldots, \overline{1}\} \) by above. Remark that there is no order between \( n \) and \( \overline{n} \).

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**Theorem.** Let \( 1 \leq \ell \leq n \).

\[
\chi_{q,t}(\ell^{\text{th}} \text{ level 0 fund. rep.}) = \sum_{i_1 \neq i_2 \neq \cdots \neq i_\ell} t^{-l(T)}
\]

where \( l(T) = \#\{p \mid i_p = i, i_{p+n-1-i} = \overline{i}, i = 1, \ldots, n-2\} \).

**Remark.** (1) Proof is given by checking the RHS satisfies the Condition \( \circ \).

(2) \( t = 1 \) case : shown by Kuniba-Suzuki

(3) \( \text{Res } L(P) = L(\varpi_\ell) \oplus L(\varpi_{\ell-2}) \oplus \cdots \oplus \begin{cases} L(\varpi_1) & \text{if } \ell \text{ is odd} \\ L(0) & \text{if } \ell \text{ is even} \end{cases} \)
Recall that a general $\chi_{q,t}(M(P))$ is given as a twisted product of $\chi_{q,t}$ (level 0 fund. rep.)’s. It can be given by a tableaux sum expression.

(See ‘$t$–analog of $q$–characters of quantum affine algebras of type $A_n$, $D_n$, Cont. Math. 325, 2003.’)

Our condition $\circ$ can be implemented into a computer program. (I have used the programing language C.)

Write $i_n = Y_{i,q^n}$ (and replace $t$ by $t^{-1}$).

Example.
Two things to overcome

- We must deal with a huge number of data. For example, $g = E_8$.
  $\chi_{q,t}(5^{\text{th}} \text{ level 0 fund. rep.})$ consists of $6899079264$ monomials.

  We need to avoid a memory overflow.
  - I store the data in a recursive way to save the memory.
  - I used a supercomputer with a huge memory (> 120 GByte). (Kyoto Univ. did not have one in 2002.)

- We need a budget to use a supercomputer.

$E_8$, $5^{\text{th}}$ level 0 fund. rep.

- Require 120GB memory
- More than 300 hours
- Final result > 450GB $\approx$ 100 DVD’s

For example, the monomial with the highest coeff. is

$$(1 + 4t^2 + 10t^4 + 20t^6 + 33t^8 + 47t^{10} + 59t^{12} + 66t^{14} + 66t^{16} + 59t^{18} + 47t^{20} + 33t^{22} + 20t^{24} + 10t^{26} + 4t^{28} + t^{30}) \times 1_{14}^{-1} 1_{16}^{3} 3_{14}^{-2} 3_{16}^{2} 5_{14}^{-3} 5_{16}^{3} 7_{14}^{-1} 7_{16}.$$
Kirillov-Reshetikhin made a conjecture on
\[ \chi(\text{Res}(\text{KR module})) \]
where \( \chi \) is the ordinary character of a \( U_q(\mathfrak{g}) \)-module.

Hatayama et al. introduced its graded version with a conjectural crystal theoretic interpretation of the grading.

Lusztig conjectured the grading = the cohomology grading for the case a KR module = a standard module.

There are many partial results. (e.g., true for classical types, \( T \)-system giving a recursion (unknown initial term, Nakajima, Hernandez, etc.)

As we have
\[ \chi_t(\text{Res}(\ell^{\text{th}} \text{ level 0 fund.})) = \chi_{q,t}(\ell^{\text{th}} \text{ level 0 fund.})_{Y_i,a \rightarrow y_i} \]
we can compute the LHS.

**Theorem.** The (graded version of the) fermionic formula (of the original form) is true for all level 0 fundamental representations.

For example, the restriction of the 5\(^{\text{th}}\) level 0 fund. rep. of \( E_8 \) is
Quantum toroidal algebras?

'Theoretical' results remain true.

**Remark.** The coproduct does not make sense, but the standard modules = 'tensor products' of level 0 fund. rep. can be defined.

But the computer program does not stop forever......
Example (Varagnoro-Vasserot). $\mathfrak{g} = A_n^{(1)} = \widehat{\mathfrak{sl}}_{n+1}$.

Res($0^{th}$ level 0 fund. rep.) = the Fock space rep.

The corresponding quiver variety

- smooth one: $\bigcup_{N \geq 0} \text{Hilb}^N(\mathbb{C}^2)^{\mathbb{Z}/n+1}$
- affine one: $\bigcup_{N \geq 0} S^N(\mathbb{C}^2/(\mathbb{Z}/n + 1))$

There is a $T^2$-action on $\text{Hilb}^N(\mathbb{C}^2)^{\mathbb{Z}/n+1}$ coming from $T^2 \acts \mathbb{C}^2$.

$\leadsto$ two parameter $(q_1, q_2)$ deformation of toroidal Lie alg.

Fixed points are parametrized by Young diagram ($\leftrightarrow$ monomial ideals). This gives an explicit formula for $\chi_{q,t}$. 