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T–analog of q–characters of finite dimensional representations of quantum affine algebras

– explicit examples

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Representation Theory of Algebraic Groups and Quantum Groups 06

June 13, 2006, Nagoya Univ.

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Quantum Loop Algebras

- $\mathfrak{g}$: simple Lie algebra of type ADE $/\mathbb{C}$
  (or symmetric Kac-Moody)
- $\mathbf{L}\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[z, z^{-1}]$: its loop algebra
- $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$: quantum loop algebra
  $=$ quantum affine algebra $\mathbf{U}_q(\widehat{\mathfrak{g}})$
  without central extension & degree operator
  $=$ $\mathbb{C}(q)$-algebra generated by q^h, $e_{i,r}$, $f_{i,r}$, $h_{i,n}$
  $(h \in P^*,\ i \in I,\ r \in \mathbb{Z},\ n \in \mathbb{Z} \setminus \{0\})$
  with certain relations
- $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$: specialized quantum loop algebra ($\varepsilon \in \mathbb{C}^*$)

Assume $\varepsilon \notin \sqrt{1}$ today. Write q instead of ε from now.

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$\mathbf{U}_q(\mathbf{L}\mathfrak{h}) \stackrel{\text{def.}}{=} \langle q^h, h_{i,\pm m} \rangle \subset \mathbf{U}_q(\mathbf{L}\mathfrak{g})$: commutative subalgebra

Let $p_i^{\pm}(z) \stackrel{\text{def.}}{=} \exp\left(-\sum_{m=1}^{\infty} \frac{h_{i,\pm m}}{[m]_\varepsilon} z^{\pm m} \right)$. (generating function)

Analogy

Weights for $\mathfrak{g}$-modules $\longleftrightarrow$ ℓ-weights for $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$-modules

- ℓ-weight space $\stackrel{\text{def.}}{=}$ a simultaneous generalized eigensp. of q^h, $h_{i,\pm m}$
- ℓ-highest weight module $V \stackrel{\text{def.}}{\iff}$
  $\exists$ vector $v \in V$
  - $e_{i,r} v = 0 \quad \forall i, r$
  - simul. eigenvec. for q^h, $h_{i,m}$
  - $V = \mathbf{U}_q(\mathbf{L}\mathfrak{g}) v$

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- $\{$simple ℓ-highest wt. modules$\} \longleftrightarrow \{\ell$-highest wts.$\}$
- ℓ-weight $\stackrel{\text{def.}}{=}$ eigen value (for finite dim. modules)
  $\longleftrightarrow \left(\frac{P_i(z)}{Q_i(z)}\right)_{i \in I}$ with $P_i(0) = Q_i(0) = 1$, coprime
  s.t. q^h acts by $q^{\langle h_i, (\deg P_i - \deg Q_i)\varpi_i \rangle}$ + nilp.
  $p_i^+(z)$ acts by $\dfrac{P_i(z)}{Q_i(z)}$ + nilp.
  $p_i^-(z)$ acts by $\dfrac{z^{-\deg P_i} P_i(z)}{z^{-\deg Q_i} Q_i(z)} \times$ const. + nilp.
- ℓ-weight is *ℓ-dominant* $\stackrel{\text{def.}}{\iff} \forall i \; Q_i(z) = 1$, i.e. polynomial
- a simple ℓ-highest wt. module is finite dim. (or integrable)
  $\iff$ ℓ-highest weight is ℓ-dominant
  (Drinfeld, Chari-Pressley)

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Problem

Study the category of finite dimensional representations of $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$.

New features

- No q–analog of the evaluation map $\mathbf{L}\mathfrak{g} \to \mathfrak{g}$ ($z \mapsto a \in \mathbb{C}^*$)
- Therefore a $\mathbf{U}_q(\mathfrak{g})$-module may not be lifted to a $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$-module in general.
- The category is *not* semisimple.
- $V \otimes W$ may not be isomorphic to $W \otimes V$.
- canonical/crystal base may not exist.

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t–deformation of the representation ring

- $\mathbf{R} \overset{\text{def.}}{=} \operatorname{Rep} \mathbf{U}_q(\mathbf{L}\mathfrak{g})$: Grothendieck ring of finite dimensional $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$-modules (commutative ring, i.e. $[V \otimes W] = [W \otimes V]$)
- $\{L(P)\}$: simple finite dim. modules give a basis of $\mathbf{R}$
- $\mathbf{R}_t$: its t-deformation (noncommutative algebra over $\mathbb{Z}[t, t^{-1}]$) can be defined. (via filtration on modules....)
- $L(P)$ has a characterization: analog of Kazhdan-Lusztig basis, PBW/canonical basis
  - $\overline{L(P)} = L(P)$,
  - $L(P) \in M(P) + \sum_{Q:Q<P} t^{-1}\mathbb{Z}[t^{-1}]M(Q)$

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$L(P)$ has a characterization

- $\overline{L(P)} = L(P)$,
- $L(P) \in M(P) + \sum_{Q:Q<P} t^{-1}\mathbb{Z}[t^{-1}]M(Q)$

where $M(P)$ is a standard module:

- a tensor product of level 0-fundamental modules w.r.t. a certain order
- (a specialization of) an extrmal weight module of Kashiwara (= a Weyl module of Chari-Pressley)

level 0-fundamental module :

$$\exists\ i \in I, a \in \mathbb{C}^* \text{ s.t. } P_i(z) = 1 - az,\ P_j(z) = 1 \text{ for } j \neq i.$$

Cf. Type A: Ginzburg-Vasserot, Arakawa

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- $\mathbf{R}_t$ can be defined by perverse sheaves on *graded quiver varieties* = (quiver varieties)$^{\text{torus fixed pts}}$.

Analogy

$\mathbf{R}_t$ via graded quiver varieties	$\longleftrightarrow$	$(\mathbf{U}^-)^*$ via quivers (Lusztig)

The affine graded quiver variety $\mathfrak{M}_0^\bullet(P)$ has a stratification:

$$\mathfrak{M}_0^\bullet(P) = \bigcup_{Q\le P} \mathfrak{M}_0^{\bullet\text{reg}}(Q,P)$$

parametrized by I-tuples of polynomials Q.

- $L(Q)$: simple module $\longleftrightarrow IC(\mathfrak{M}_0^{\bullet\text{reg}}(Q,P))$: IC sheaf
- $M(Q)$: standard module $\longleftrightarrow \mathbb{C}_{\mathfrak{M}_0^{\bullet\text{reg}}(Q,P)}$: constant sheaf extended by 0

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- $Q \leq P \overset{\text{def.}}{\Longleftrightarrow} \mathfrak{M}_0^{\bullet\mathrm{reg}}(Q,R) \subset \overline{\mathfrak{M}_0^{\bullet\mathrm{reg}}(P,R)}$ (indep. of R)

From the characterization we have

$$\Big([L(P) : M(Q)]\Big) \overset{\text{determined}}{\Longleftarrow} \Big([M(P) : \overline{M(Q)}]\Big)$$

Hence the Problem is reduced to

How to compute $\overline{M(P)}$? $\rightsquigarrow$ t-analog of q–character $\chi_{q,t}$

$$\chi_{q,t}(M(P)) = \text{the generating function of Betti numbers of smooth graded quiver varieties } \mathfrak{M}^\bullet(Q/R,P)$$

$$= \sum_{Q/R:\ell\text{-weight}} \text{gr-dim}(\ell\text{-wt. sp.}) \times e^{Q/R}$$

where $e^{Q/R} = \prod_{i\in I}\prod_a Y_{i,a}^{u_i(a)}$ for $\dfrac{Q_i(z)}{R_i(z)} = \prod_a (1-az)^{u_i(a)}$.

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Method of Computation

$\chi_{q,t}$ gives an injective homomorphism

$$\mathbf{R}_t \xrightarrow{\chi_{q,t}} \mathbb{Z}[t,t^{-1}][Y_{i,a}]_{i\in I, a\in\mathbb{C}^*}$$

- $\overline{t} = t^{-1}$, $\overline{Y_{i,a}^{\pm}} = Y_{i,a}^{\pm}$
- twisted multiplication $m_1 * m_2 = t^{d(m_1,m_2)} m_1 m_2$ and $\chi_{q,t}(M(P_1)\otimes M(P_2)) = \chi_{q,t}(M(P_1)) * \chi_{q,t}(M(P_2))$

$\star$ Finally $\chi_{q,t}$(level 0 fund. rep.) can be determined from the ℓ–highest weight vector recursively. ...... Condition $\circ$

cf. Frenkel-Mukhin : $\chi_{q,t} \in \mathrm{Ker}$(screening op.'s)

In Summary,

$\chi_{q,t}$ is "computable" theoretically, but not practically.

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q–character (or ℓ–character)

Recall the character of a $\mathfrak{g}$-module:

$$\mathrm{ch}(V) = \sum_{\lambda:\text{weight}} \dim(\text{wt. sp.}) \times e^{\lambda}$$

Definition. (Frenkel-Reshetikhin)

$$\chi_q(M) \overset{\text{def.}}{=} \sum_{\frac{P(z)}{Q(z)}:\ell\text{-weight}} \dim(\ell\text{-wt. sp.}) \times e^{P/Q}$$

where $e^{P/Q} = \prod_{i\in I}\prod_{a} Y_{i,a}^{u_i(a)}$ for $\dfrac{P_i(z)}{Q_i(z)} = \prod_a (1-az)^{u_i(a)}$.

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Example. $\mathfrak{g} = \mathfrak{sl}_{n+1}$: V = vector representation

It can be lifted to a $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$-module.

$$\begin{array}{cccccc}
 & \bullet & \xrightarrow{f_{1,0}} \bullet & \xrightarrow{f_{2,0}} \cdots\cdots \xrightarrow{f_{n,0}} & \bullet \\
\text{weight} & q^{\varpi_1} & q^{\varpi_2-\varpi_1} & & q^{-\varpi_n} \\
\ell\text{-weight} & Y_{1,1} & Y_{1,q^2}^{-1}Y_{2,q} & & Y_{n,q^{n+1}}^{-1} \\
 & \| & \| & & \| \\
 & \boxed{1} & \boxed{2} & & \boxed{n+1}
\end{array}$$

$$\therefore\ \chi_q(V) = \sum_{i=1}^{n+1} \boxed{i} \qquad \text{(tableaux sum expression)}$$

For later purpose, let us introduce

$$\boxed{1}_k = Y_{1,q^k}, \quad \boxed{2}_k = Y_{1,q^{k+2}}^{-1}Y_{2,q^{k+1}}, \quad \text{etc.}$$

Example. $\mathfrak{g} = \mathfrak{sl}_2$, ℓ-highest wt. $P(z)=(1-z)(1-zq)\cdots(1-zq^{2(n-1)})$

a lift of the irreducible n-dim. rep. of $\mathbf{U}_q(\mathfrak{sl}_2)$ (evaluation module)

$$\bullet \xrightarrow{f_{1,0}} \bullet \xrightarrow{f_{1,0}} \cdots\cdots \xrightarrow{f_{1,0}} \bullet$$

weight	q^n	q^{n-2}	q^{-n}
ℓ-weight	$(1-z)(1-zq^2)\cdots(1-zq^{2(n-1)})$	$(1-z)\cdots(1-zq^{2(n-2)})\times(1-zq^{2n})^{-1}$	$(1-zq^2)^{-1}(1-zq^4)^{-1}\cdots(1-zq^{2n})^{-1}$
tableau	$\boxed{1}_0\boxed{1}_2\cdots\boxed{1}_{2(n-1)}$	$\boxed{1}_0\cdots\boxed{1}_{2(n-2)}\boxed{2}_{2(n-1)}$	$\boxed{2}_0\boxed{2}_2\cdots\boxed{2}_{2(n-1)}$

$$\therefore \qquad \chi_q(L(P)) \quad = \quad \sum_{1\le i_1\le i_2\le\cdots\le i_n\le 2} \boxed{i_1}_0 \cdots \boxed{i_n}_{2(n-1)}$$

Let us prove this by our method !

For example, $n=4$:

$$\begin{aligned}
\chi_{q,t}(M(P)) = {} & \boxed{1}_0\boxed{1}_2\boxed{1}_4\boxed{1}_6 (= Y_{1,1}Y_{1,q^2}Y_{1,q^4}Y_{1,q^6}) + \dots \\
& + t^{-1}\boxed{2}_0\boxed{1}_2\boxed{1}_4\boxed{1}_6 (= Y_{1,q^4}Y_{1,q^6}) + \cdots \\
& + t^{-1}\boxed{1}_0\boxed{2}_2\boxed{1}_4\boxed{1}_6 (= Y_{1,1}Y_{1,q^6}) + \cdots \\
& + t^{-1}\boxed{1}_0\boxed{1}_2\boxed{2}_4\boxed{1}_6 (= Y_{1,1}Y_{1,q^2}) + \cdots \\
& + t^{-2}\boxed{2}_0\boxed{1}_2\boxed{2}_4\boxed{1}_6 (= 1) + \cdots
\end{aligned}$$

$$\therefore \chi_{q,t}(L(P)) = \chi_{q,t}(M(P)) - t^{-1}\chi_{q,t}(M(Y_{1,q^4}Y_{1,q^6}))$$
$$-t^{-1}\chi_{q,t}(M(Y_{1,1}Y_{1,q^6})) - t^{-1}\chi_{q,t}(M(Y_{1,1}Y_{1,q^2})) + t^{-2}\chi_{q,t}(M(1))$$

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type D_n

Example. V = vector representation

$$
\begin{tikzcd}[column sep=small]
 & & & & \boxed{n} \arrow[dr, "f_{n,0}"] & & & & \\
\boxed{1} \arrow[r, "f_{1,0}"] & \boxed{2} \arrow[r, "f_{2,0}"] & \cdots \arrow[r, "f_{n-2,0}"] & \boxed{n-1} \arrow[ur, "f_{n-1,0}"] \arrow[dr, "f_{n,0}"'] & & \boxed{\overline{n-1}} \arrow[r, "f_{n-2,0}"] & \cdots \arrow[r, "f_{2,0}"] & \boxed{\overline{2}} \arrow[r, "f_{1,0}"] & \boxed{\overline{1}} \\
 & & & & \boxed{\overline{n}} \arrow[ur, "f_{n-1,0}"'] & & & &
\end{tikzcd}
$$

We define the ordering $\prec$ on the set $\mathbf{B} = \{1, \ldots, n, \overline{n}, \ldots, \overline{1}\}$ by above. Remark that there is no order between n and $\overline{n}$.

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Theorem. Let $1 \le \ell \le n$.

$$
\chi_{q,t}(\ell^{\text{th}} \text{ level 0 fund. rep.}) = \sum_{i_1 \not\succeq i_2 \not\succeq \cdots \not\succeq i_\ell} t^{-l(T)} \begin{array}{l} \boxed{i_1}_{\ell-1} \\ \boxed{i_2}_{\ell-3} \\ \boxed{\vdots} \\ \boxed{i_\ell}_{1-\ell} \end{array}
$$

where $l(T) = \#\{p \mid i_p = i, i_{p+n-1-i} = \overline{i}, i = 1, \ldots, n-2\}$.

Remark. (1) Proof is given by checking the RHS satisfies the Condition $\circ$.

(2) $t = 1$ case : shown by Kuniba-Suzuki

(3) $\operatorname{Res} L(P) = L(\varpi_\ell) \oplus L(\varpi_{\ell-2}) \oplus \cdots \oplus \begin{cases} L(\varpi_1) & \text{if } \ell \text{ is odd} \\ L(0) & \text{if } \ell \text{ is even} \end{cases}$

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Recall that a general $\chi_{q,t}(M(P))$ is given as a twisted product of $\chi_{q,t}$(level 0 fund. rep.)'s. It can be given by a tableaux sum expression.

(See 't–analog of q–characters of quantum affine algebras of type A_n, D_n, Cont. Math. **325**, 2003.)

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Exceptional types

Our condition $\circ$ can be implemented into a computer program. (I have used the programing language **C**.)

Write $i_n = Y_{i,q^n}$ (and replace t by t^{-1}).

Example.

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Two things to overcome

- We must deal with a huge number of data. For example, $\mathfrak{g} = E_8$,

  $\chi_{q,t}$(5^{th} level 0 fund. rep.) consists of 6899079264 monomials.

  We need to avoid a memory overflow.

  - I store the data in a recursive way to save the memory.
  - I used a supercomputer with a huge memory ($>$ 120 GByte). (Kyoto Univ. did not have one in 2002.)

- We need a budget to use a supercomputer.

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E_8, 5^{th} level 0 fund. rep.

- Require 120GB memory
- More than 300 hours
- Final result $>$ 450GB $\approx$ 100 DVD's

For example, the monomial with the highest coeff. is

$$(1 + 4t^2 + 10t^4 + 20t^6 + 33t^8 + 47t^{10} + 59t^{12} + 66t^{14} + 66t^{16} + 59t^{18} + 47t^{20} + 33t^{22} + 20t^{24} + 10t^{26} + 4t^{28} + t^{30}) \times 1_{14}1_{16}^{-1}3_{14}^{2}3_{16}^{-2}5_{14}^{3}5_{16}^{-3}7_{14}7_{16}^{-1}.$$

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Fermionic formula

- Kirillov-Reshetikhin made a conjecture on

$$\chi(\mathrm{Res}(\mathrm{KR\ module})),$$

where χ is the ordinary character of a $\mathbf{U}_q(\mathfrak{g})$-module.

- Hatayama et al. introduced its graded version with a conjectural crystal theoretic interepretation of the grading.
- Lusztig conjectured the grading = the cohomology grading for the case a KR module = a standard module.
- There are many partial results. (e.g., true for classical types, T-system giving a recursion (unknown initial term, Nakajima, Hernandez, etc.)

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As we have

$$\chi_t(\mathrm{Res}(\ell^{\mathrm{th}}\text{ level 0 fund.})) = \chi_{q,t}(\ell^{\mathrm{th}}\text{ level 0 fund.})\Big|_{Y_{i,a}\mapsto y_i},$$

we can compute the LHS.

Theorem. The (graded version of the) fermionic formula (of the original form) is true for all level 0 fundamental representations.

For example, the restriction of the 5^{th} level 0 fund. rep. of E_8 is

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ϖ_5	$\varpi_3+\varpi_7$	$\varpi_2+\varpi_8$	$\varpi_1+2\varpi_7$	$\varpi_1+\varpi_6$	$2\varpi_2$	$\varpi_7+\varpi_8$
1	t^2	t^2+t^4	t^4	$t^2+t^4+t^6$	t^6	$t^2+2t^4+2t^6+t^8$

$\varpi_1+\varpi_3$	ϖ_4	$2\varpi_1+\varpi_7$	$\varpi_2+\varpi_7$	$\varpi_1+\varpi_8$
$2t^4+t^6+t^8$	$t^2+2t^4+2t^6+t^8+t^{10}$	$2t^6+t^8+t^{10}$	$3t^4+4t^6+4t^8+2t^{10}+t^{12}$	$2t^4+5t^6+5t^8+3t^{10}+2t^{12}+t^{14}$

$2\varpi_7$	ϖ_6	$3\varpi_1$	$\varpi_1+\varpi_2$	ϖ_3	$\varpi_1+\varpi_7$
$3t^6+2t^8+3t^{10}+t^{12}+t^{14}$	$2t^4+4t^6+6t^8+4t^{10}+3t^{12}+t^{14}+t^{16}$	t^8+t^{12}	$2t^6+5t^8+5t^{10}+3t^{12}+2t^{14}+t^{16}$	$5t^6+5t^8+7t^{10}+4t^{12}+3t^{14}+t^{16}+t^{18}$	$2t^6+9t^8+10t^{10}+10t^{12}+6t^{14}+4t^{16}+2t^{18}+t^{20}$

ϖ_8	$2\varpi_1$	ϖ_2	ϖ_7	ϖ_1	0
$t^6+5t^8+8t^{10}+7t^{12}+6t^{14}+4t^{16}+2t^{18}+t^{20}+t^{22}$	$5t^{10}+4t^{12}+6t^{14}+3t^{16}+3t^{18}+t^{20}+t^{22}$	$3t^8+6t^{10}+11t^{12}+8t^{14}+7t^{16}+4t^{18}+3t^{20}+t^{22}+t^{24}$	$5t^{10}+6t^{12}+9t^{14}+6t^{16}+6t^{18}+3t^{20}+2t^{22}+t^{24}+t^{26}$	$4t^{12}+5t^{14}+8t^{16}+5t^{18}+5t^{20}+3t^{22}+2t^{24}+t^{26}+t^{28}$	$t^{14}+3t^{18}+t^{20}+2t^{22}+t^{24}+t^{26}+t^{30}$

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Quantum toroidal algebras ?

'Theoretical' results remain true.

Remark. The coproduct does not make sense, but the standard modules = 'tensor products' of level 0 fund. rep. can be defined.

But the computer program does not stop forever......

Example (Varagnoro-Vasserot). $\mathfrak{g} = A_n^{(1)} = \widehat{\mathfrak{sl}}_{n+1}$.

$$\mathrm{Res}(0^{\mathrm{th}} \text{ level 0 fund. rep.}) = \text{the Fock space rep.}$$

The corresponding quiver variety

$$\text{smooth one} : \bigsqcup_{N \geq 0} \mathrm{Hilb}^N(\mathbb{C}^2)^{\mathbb{Z}/n+1}$$

$$\text{affine one} : \bigcup_{N \geq 0} S^N(\mathbb{C}^2/(\mathbb{Z}/n+1))$$

There is a T^2-action on $\mathrm{Hilb}^N(\mathbb{C}^2)^{\mathbb{Z}/n+1}$ coming from $T^2 \curvearrowright \mathbb{C}^2$.
$\rightsquigarrow$ two parameter (q_1, q_2) deformation of toroidal Lie alg.

Fixed points are parametrized by Young diagram ($\leftrightarrow$ monomial ideals). This gives an explicit formula for $\chi_{q,t}$.