Statements of Research accomplishment

I classify my major scientific works into several groups by their themes.

Early Works

I was guided by my supervisor Takushiro Ochiai to study nonlinear partial differential equations (PDE) on manifolds, then I chose my research theme as analytic aspects of Yang-Mills connections, which is an important examples of nonlinear PDE on manifolds.

In [1] I generalized Uhlenbeck's convergence result of Yang-Mills connections on 4-manifolds to higher dimensional manifolds. Namely a sequence of Yang-Mills connections on an n-dimensional compact manifold with L^2 curvature bounds, after taking a subsequence, converges outside a subset S with finite $(n-4)$-dimenisonal Hausdorff measure. The estimate of dimension of S is the best possible, as one can construct examples on product manifolds $X^4 \times Y^{n-4}$ which converge outside finitely many points in X^4-direction. [1] and [2] are my master thesis.

In [3] written by myself, and [4] written with Bando and Kasue, I proved a convergence result for a sequence of compact 4-manifolds with Einstein metrics whose diameter, inverse of volume, and Euler number are uniformly bounded. Then it has a subsequence converging to an orbifold with Einstein metric outside finitely many points. Moreover if we rescale metrics around a point where the convergence fails, we obtain a noncompact manifold with Einstein metric, which is approximated by a quotient of the Euclidean space by a finite subgroup of the orthogonal group. Analytic aspects of Einstein metrics are very similar to those of Yang-Mills connections, but we combine them with Gromov-Hausdoff convergences, which were hot topics in Riemannian geometry. I learned latter from Kenji Fukaya (who was an assistant professor at University of Tokyo at that time), and later from the collaborator Kasue.

Noncompact spaces with Einstein metrics appearing as the limit in the rescaled sequence above are called ALE (asymptotically locally Euclidean). The case when Einstein metrics are hyper-Kählerian is the most important. In [8] I studied moduli spaces of anti-self-dual connections (instantons) on such an ALE space. In particular, I showed that they are again ALE if their dimension are 4.

Examples and classification of hyper-Kähler ALE spaces were obtained by Kronheimer slightly earlier. I, together with him, continued study of moduli spaces of instantons on hyper-Kähler ALE spaces, and obtained their description in terms of matrices [10]. This description is a generalization of a similar description of moduli spaces of instantons on $\mathbb{R}^4$ due to Atiyah-Drinfeld-Hitchin-Manin, the famous ADHM description.

After [10], I changed my main interest from analytic aspects of Yang-Mills connections to algebraic aspects of their moduli spaces.

Quiver varieties and representation theory

The ADHM description in [10] is understood in the framework of representation theory of affine ADE quivers. It turns out moduli spaces above, in the ADHM description, make sense for arbitrary quivers, hence I named the resulted spaces *quiver varieties*. Motivated by earlier works of Ringel and Lusztig, I constructed irreducible integrable representations of Kac-Moody Lie algebras on spaces of constructible functions on quiver varieties [14], and later on homology groups of quiver varieties [21]. These constructions are very different from usual geometric constructions (i.e., Borel-Weil theory) of representations of Lie algebras. It is important to consider various quiver varieties simultaneously, as a single quiver variety just corresponds to a weight space of the representation. It is contradictory to a usual intuition that Lie algebra representations appear as differential of Lie group representations, which are *continuous* symmetry, as my construction uses *disconnected* spaces essentially.

In [22], I obtained a similar construction of a representation of the Heisenberg algebra on homology groups of Hilbert schemes of points on a complex algebraic surface. The same result was obtained by Grojnowski around the same time. Hilbert schemes of points had been studied by algebraic geometers, but this work gave a completely new way of looking at them.

I continued study of quiver varieties for several years. Papers [26,27,29,30,32,36] and [54] are about quiver varieties, but let me explain just two main papers. In [26] I constructed representations of quantum loop algebras on equivariant K-theory of quiver varieties. This construction led me a definition of Kazhdan-Lusztig type polynomials by intersection cohomology groups of torus fixed point loci of quiver varieties. These polynomials were studied in [32], and in particular, I obtained their purely combinatorial characterization, which gave an algorithm for computation.

Computation of q-characters

The algorithm for analog of Kazhdan-Lusztig type polynomials is separated into two parts, and both are rather complicated, and handle huge data. Therefore even the first step, i.e., computation of t-analog of q-characters of ℓ-fundamental representations of quantum loop algebras, requires use of supercomputer for the actual computation.

I started this project in 2002, but it finished in 2006 [47], as I waited a renovation of a supercomputer in Kyoto University so that it could handle enough data.

As a hope to get a better understanding of q-characters, I introduced a crystal structure, in the sense of Kashiwara, on monomials appearing in q-characters [34]. This initial motivation was not achieved, but the monomial realization of crystal bases has been studied in various different contexts, in my work with Hernandez [45], and others. Recent studies by Kanakubo-Nakashima gave a unexpected link with cluster algebras.

Crystal bases of extremal weight modules

Equivariant K-groups of quiver varieties, as representations of quantum loop algebras, have different, completely algebraic realization found by Kashiwara. His construction simultaneously gave crystal bases, hence I was interested in Kashiwara's approach. He posed a conjecture on their structure. I proved it for untwisted case in [33], and with Beck [38] in full generality. [38] also contained a proof of a conjecture by Lusztig on the cell structure of quantum loop algebras. All techniques used in [33], [38] are purely algebraic unlike my other works.

Instanton counting and Donaldson invariants

Quiver varieties associated with Jordan quiver are original ADHM description, but I found one new feature that they have natural resolution of singularities from general result for quiver varieties [15]. Later I realized that the resolutions are moduli spaces of framed sheaves on the projective plane (later called Gieseker spaces by other people). In 2002 Nekrasov considered the integration of the constant function 1 on the Gieseker spaces in the sense of equivariant cohomology groups, and conjectured that the leading part of the integration with respect to equivariant variables is given by period of certain hyperelliptic curves, called the Seiberg-Witten prepotential. The Seiberg-Witten prepotential was found 1994 in the study of gauge theory in 4-dimension, but its derivation remained mysterious for a while, as it is based on physical argument. Nekrasov's conjecture provided us a mathematical approach, and it was proved in my joint work with Yoshioka [40], and independently by Nekrasov-Okounkov, Braverman-Etingof by completely different methods.

Nekrasov's framework gave us a better approach to Donaldson invariants for 4-manifolds, whose computation was usually very difficult. In [46] Göttsche, Yoshioka and I gave a proof of wallcrossing formula of Donaldson invariants for a 4-manifold X with $b_+ = 1$, when X is a complex projective surface. This result was proved earlier by Göttsche, assumping Kotschieck-Morgan conjecture. Our proof is based on earlier works: wallcrossing formula is given by certain integral over Hilbert schemes of points on X (Ellingsrud-Göttsche), and this integral is 'universal' in X (Ellingsrud-Göttsche-Lehn). These results, in particular, imply that it is enough to compute wallcrossing formula for projective toric surfaces. They are expressed by combinatorial formula by localization theorem of equivariant cohomology groups, but one needs to set the equivariant variables zero. The expected result means that there is a myracle of cancellation of poles in the localization formula.

In [58], a joint work with Göttsche and Yoshioka, I proved the so called Witten conjecture, relating Donaldson invariants and Seiberg-Witten invariants for a complex projective surface. By a result of Mochizuki, these invariants are related via certain integral over Hilbert schemes of points. We then use a similar technique as in our proof of wallcrossing formula, we compute the integral.

Perverse coherent sheaves on blow-up

This is a side project of instanton counting with Yoshioka. Perverse coherent sheaves on blow-up are certain sheaves which live between a complex surface X and its blow-up $\widehat{X}$. The definition is motivated by a work of Bridgeland, who considered similar sheaves for a 3-fold flop. We studied moduli spaces of perverse coherent sheaves, and showed that they connect a moduli space of stable torsion free sheaves on X and that on $\widehat{X}$ by a sequence of wallcrossing [53]. There is a quiver description when X is the affine plane $\mathbb{C}^2$ [52]. Using a sequence of wallcrossing, we compare Nekrasov partition function on $\mathbb{C}^2$ and the blow-up $\widehat{\mathbb{C}^2}$. This method is more powerful than an earlier approach used in [40], as we can handle the partition functions with matters [57].

Instanton moduli spaces and $\mathcal{W}$-algebras

Motivated by a work of physicists, Alday-Gaiotto-Tachikawa (AGT), I, together with Braverman and Finkleberg [68], constructed a representation of the $\mathcal{W}$-algebra on the direct sum of equivariant intersection cohomology groups of Uhlenbeck partial compactification of instanton moduli spaces on $\mathbb{R}^4$ (Uhlenbeck spaces, in short) of type ADE, where the direct sum is taken over all instanton numbers. This is based on an earlier work by Maulik-Okounkov, which proves a similar result for type A case. More precisely they consider equivariant cohomology groups of Gieseker spaces, which are resolution of singularities of Uhlenbeck spaces. Since Gieseker spaces are available only in type A, a general case requires a new technique: We reduced general ADE to the A_2-case by studying hyperbolic restriction on Uhlenbeck spaces.

A mathematical definition of Coulomb branches of 3-dimensional SUSY gauge theories

This is a on-going project. For a given pair of a compact Lie group G and its quaternionic representation $\mathbb{M}$, physicists consider a SUSY (supersymmetric) gauge theory in dimension 3, and associate its Coulomb branch, which is a noncompact hyperKähler manifold possibly with singularities. The physical definition involves quantum corrections, which are difficult to justify mathematically. In [68], I propose an approach to a rigorous mathematical definition, as an affine algebraic symplectic variety. This approach was worked out in detail in [69], written with Braverman, Finkelberg when $\mathbb{M}$ is of a form $\mathbb{N}\oplus\mathbb{N}^*$. Then I determine Coulomb branches of quiver gauge theories of type ADE with known spaces, the so-called (generalized) slices in the affine Grassmannian in [70] (with Braverman-Finkelberg). In appendix of [70] (with Braverman-Finkelberg-Kamnitzer-Kodera-Webster-Weekes) we study quantization of Coulomb branches of quiver gauge theories, and relate them to shifted Yangian. In [72], together with Takayama, I determine Coulomb branches of quiver gauge theories of affine type A. They are Cherkis bow varieties. In order to show this statement, we study Cherkis bow varieties as affine algebraic varieties, using quiver type description. In [73], together with Kodera, I study quantized Coulomb branches of Jordan quiver gauge theories, and identify them with cyclotomic rational Cherednik algebras.