Notes on (super)conformal symmetry

1 Conformal transformations

A conformal transformation of a Riemannian manifold (M, g) is a diffeomorphism $f : M \to M$ that transforms the metric to itself up to (position dependent) scaling, $f^* g = e^\lambda g$. An infinitesimal conformal transformation is a vector field v on M such that

$$\mathscr{L}_v g = \lambda g. \tag{1.1}$$

Below, we classify conformal transformations of the Euclidean space $\mathbf{R}^d$.

1.1 Infinitesimal transformations

On the Euclidean space, (1.1) reads as

$$\partial_\mu v_\nu + \partial_\nu v_\mu = \lambda \delta_{\mu\nu}. \tag{1.2}$$

Using this in $0 = \partial_\mu(\partial_\nu v_\rho) - \partial_\nu(\partial_\mu v_\rho)$ we find

$$\partial_\rho(\partial_\nu v_\mu - \partial_\mu v_\nu) = \delta_{\rho\mu}\partial_\nu \lambda - \delta_{\rho\nu}\partial_\mu \lambda. \tag{1.3}$$

Using this last equation in $0 = \partial_\rho \partial_\sigma(\partial_\nu v_\mu - \partial_\mu v_\nu) - \partial_\sigma \partial_\rho(\partial_\nu v_\mu - \partial_\mu v_\nu)$ and contracting the $\sigma\nu$ indices, we find the equation $(d-2)\partial_\rho \partial_\mu \lambda + \delta_{\rho\mu}\partial_\sigma \partial_\sigma \lambda = 0$. If $d \neq 1$, this means $\partial_\sigma \partial_\sigma \lambda = 0$ and

$$(d-2)\partial_\rho \partial_\mu \lambda = 0.$$

Thus, if $d > 2$, λ is constant or linear in x^μ,

$$\lambda = b + c_\mu x^\mu.$$

Inserting this into (1.2) and (1.3), we find

$$\begin{aligned} \partial_\nu v_\mu + \partial_\mu v_\nu &= (b + c_\rho x^\rho)\delta_{\mu\nu}, \\ \partial_\nu v_\mu - \partial_\mu v_\nu &= \omega_{\mu\nu} + c_\nu x_\mu - c_\mu x_\nu, \end{aligned}$$

where $\omega_{\mu\nu}$ is an antisymmetric integration constant. Summing the two and integrating, we find

$$2v_\mu = a_\mu + b x_\mu + \omega_{\mu\nu} x^\nu + \left(c_\rho x^\rho x^\mu - \frac{|x|^2}{2} c_\mu \right) \tag{1.4}$$

the a, b, ω terms are (the infinitesimal versions of) translation, dilatation, and Euclidean rotation. The last c-term in the parenthsis is called the *special conformal transformations.*

The space of infinitesimal conformal transformations form a Lie algebra of dimension $d+1+\binom{d}{2}+d=\frac{(d+2)(d+1)}{2}$. We name the generators corresponding to a,b,ω,c as

$$iP_\mu = \frac{\partial}{\partial x^\mu}, \tag{1.5}$$

$$D = x^\mu \frac{\partial}{\partial x^\mu}, \tag{1.6}$$

$$iM_{\mu\nu} = x_\mu \frac{\partial}{\partial x^\nu} - x_\nu \frac{\partial}{\partial x^\mu}, \tag{1.7}$$

$$iK_\mu = |x|^2 \frac{\partial}{\partial x^\mu} - 2x_\mu x^\nu \frac{\partial}{\partial x^\nu}. \tag{1.8}$$

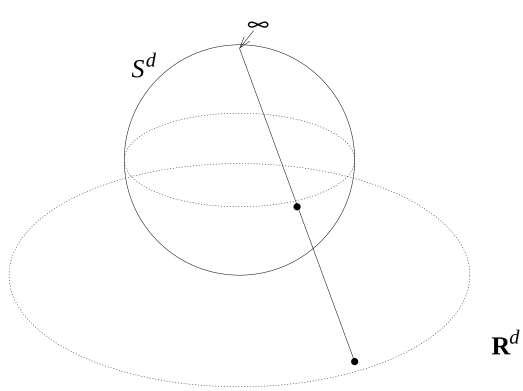
In one dimension, any diffeomorphism is a conformal transformation. In two dimensions, the equation (1.2) in the complex coordinates $z = x^1 + ix^2$, $\overline{z} = x^1 - ix^2$ reads as $\partial_{\overline{z}} v^z = \partial_z v^{\overline{z}} = 0$, which is solved by $v^z = f(z)$, $v^{\overline{z}} = \overline{f(z)}$ for any holomorphic function $f(z)$. Thus in both cases, the conformal algebra is infinite dimensional.

1.2 Finite transformations

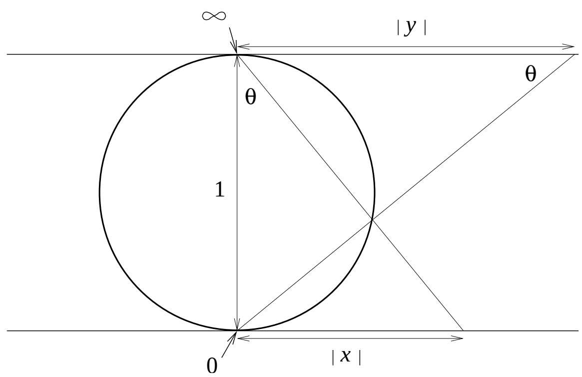
The finite versions of the translations, dilatations, and Euclidean rotations are obvious;

$$x^\mu \to x^\mu + a^\mu,$$
$$x^\mu \to e^\lambda x^\mu,$$
$$x^\mu \to O^\mu{}_\nu x^\nu,$$

where $O \in SO(d)$. To see the meaning of special conformal transformations, we consider the compactification of $\mathbf{R}^d$ by attaching one point at infinity, which yields $S^d = \mathbf{R}^d \cup \{\infty\}$.



$(x^1, ..., x^d)$ are regarded as the coordinate of S^d obtained from the stereographic projection (see figure). The coordinate of S^d near $x = \infty$ is provided by the other stereographic projection, using the rays emanating from $x = 0$ and ending on another plane $\mathbf{R}^d = \{(y^1, ..., y^d)\}$ touching S^d at ∞. As is evident from this Figure, the relation of the two



coordiates are $|x| \cdot |y| = 1$ with angles preserved. Namely,

$$(x^1, \ldots, x^d) = \frac{1}{|y|^2}(y^1, \ldots, y^d). \tag{1.9}$$

Then the infinitesimal translation of y, $\delta y^\mu = \epsilon^\mu$ results in

$$\delta x^\mu = \delta\left(\frac{y^\mu}{|y|^2}\right) = \frac{\epsilon^\mu}{|y|^2} - 2\frac{y^\mu y_\nu \epsilon^\nu}{|y|^4} = |x|^2\epsilon^\mu - 2x^\mu x_\nu \epsilon^\nu,$$

which is the infinitesimal special conformal transformation. Thus, special conformal transformations are simply the translations in the y-coordinate.

These conformal transformations form a group isomorphic to $SO(d+1,1)$. The relation is as follows. The group $SO(d+1,1)$ acts on the null cone $\mathscr{N}$;

$$-(z^0)^2 + (z^1)^1 + \cdots + (z^d)^2 + (z^{d+1})^2 = 0$$

in $(d+1)+1$ dimensional Minkowski space.[1] There is a projection from $\mathscr{N} - \{0\}$ to d-dimensional sphere S^d by $(z^0, ..., z^{d+1}) \mapsto (z^1/z_0, ..., z^{d+1}/z^0)$. The action of $SO(d+1,1)$ on $\mathscr{N} - \{0\}$ induces an action on S^d. The round metric on S^d is

$$\mathrm{d}s^2 = \sum_{I=1}^{d+1}\Big(\mathrm{d}(z^I/z^0)\Big)^2 = \frac{-(\mathrm{d}z^0)^2 + (\mathrm{d}z^1)^2 + \cdots + (\mathrm{d}z^{d+1})^2}{(z^0)^2},$$

where we have used the null cone equation. By this form, it is evident that $SO(d+1,1)$ acts on S^d as a group of conformal transformations. Note also that it has dimension $\binom{d+2}{2} = \frac{(d+2)(d+1)}{2}$. Thus, it must be that $SO(d+1,1)$ is generated by what we have seen above — translations in x, dilatation, $SO(d)$-rotations, and translations in y.

[1] $SO(d+1,1)$ also acts on the anti de Sitter space $-(z^0)^2 + (z^1)^1 + \cdots + (z^{d+1})^2 = R^2$ which approaches the null cone $\mathscr{N}$ at infinity (R is negligible for large z).

2 Radial Quantization and Hermiticity

Euclidean space $\mathbf{R}^d$ is conformaly equivalent to the cylinder $\Sigma^d = S^{d-1} \times \mathbf{R}$,

$$\mathrm{d}s^2_{\mathbf{R}^d} = \mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2_{S^{d-1}} = \mathrm{e}^{2\tau}\left(\mathrm{d}\tau^2 + \mathrm{d}\Omega^2_{S^{d-1}}\right) = \mathrm{e}^{2\tau}\mathrm{d}s^2_{\Sigma^d},$$

where we used the polar coordinates $x^\mu = r\Omega^\mu = \mathrm{e}^\tau \Omega^\mu$ (where $\sum_{\mu=1}^d (\Omega^\mu)^2 = 1$). We use this cylinder geometry to quantize the system in operator formalism. Namely, we consider the S^{d-1} as the space and τ as the Euclidean time, or $t = -i\tau$ as the real time. From the point of view of the original Euclidean space, time is the (logarithm of) the radial coordinate. Thus, this is often called the radial quantization.

Let us express the conformal transformations $P_\mu, D, M_{\mu\nu}, K_\mu$ in the coordinate (τ, σ) or (t, σ), where $\sigma = (\sigma^1, ..., \sigma^{d-1})$ are (local) coordinates of S^{d-1}, $\Omega^\mu = \Omega^\mu(\sigma)$. We have

$$\frac{\partial}{\partial \tau} = \mathrm{e}^\tau \Omega^\mu(\sigma)\frac{\partial}{\partial x^\mu}, \quad \frac{\partial}{\partial \sigma^i} = \mathrm{e}^\tau \frac{\partial \Omega^\mu(\sigma)}{\partial \sigma^i}\frac{\partial}{\partial x^\mu}.$$

Inverting this, we have

$$\frac{\partial}{\partial x^\mu} = \mathrm{e}^{-\tau}\left[\Omega^\mu(\sigma)\frac{\partial}{\partial \tau} + J^i_\mu(\sigma)\frac{\partial}{\partial \sigma^i}\right],$$

where $J^i_\mu(\sigma)$ is some $(d-1)\times d$ matrix. Using the last expression, we find the following expressions of the conformal generators,

$$iD = i\tfrac{\partial}{\partial \tau} = \frac{\partial}{\partial t}, \tag{2.1}$$

$$iM_{\mu\nu} = \left[\Omega_\mu J^i_\nu - \Omega_\nu J^i_\mu\right]\tfrac{\partial}{\partial \sigma^i} = \left[\Omega_\mu J^i_\nu - \Omega_\nu J^i_\mu\right]\frac{\partial}{\partial \sigma^i}, \tag{2.2}$$

$$iP_\mu = \mathrm{e}^{-\tau}\left[\Omega^\mu(\sigma)\tfrac{\partial}{\partial \tau} + J^i_\mu(\sigma)\tfrac{\partial}{\partial \sigma^i}\right] = \mathrm{e}^{-it}\left[-i\Omega^\mu(\sigma)\frac{\partial}{\partial t} + J^i_\mu(\sigma)\frac{\partial}{\partial \sigma^i}\right], \tag{2.3}$$

$$iK_\mu = \mathrm{e}^{\tau}\left[-\Omega^\mu(\sigma)\tfrac{\partial}{\partial \tau} + J^i_\mu(\sigma)\tfrac{\partial}{\partial \sigma^i}\right] = \mathrm{e}^{it}\left[i\Omega^\mu(\sigma)\frac{\partial}{\partial t} + J^i_\mu(\sigma)\frac{\partial}{\partial \sigma^i}\right]. \tag{2.4}$$

The middle are the expressions in the Euclidean cylinder and the right hand sides are the expressions in the Minkowski cylinder.

By the standard Noether procedure in Minkowski cylinder, we find the conserved charges for these transformations, which we denote again by $D, M_{\mu\nu}, P_\mu$ and K_μ. The right hand sides of (2.1)-(2.4) dictates the reality of the charges: $D^* = D$, $M^*_{\mu\nu} = M_{\mu\nu}$, $P^*_\mu = K_\mu$ and $K^*_\mu = P_\mu$. In the quantum theory, these become the operators that generate the correspponding transformations. Note in particular that the dilatation D corresponds to the time translation and plays the role of the Hamiltonian. They obey the commutation relations

$$[D, P_\mu] = P_\mu, \quad [D, K_\mu] = -K_\mu, \quad [D, M_{\mu\nu}] = 0, \tag{2.5}$$

$$[K_\mu, P_\nu] = 2(\delta_{\mu\nu} D + i M_{\mu\nu}), \tag{2.6}$$

$$i[M_{\mu\nu}, P_\rho] = \delta_{\rho\mu} P_\nu - \delta_{\rho\nu} P_\mu, \quad i[M_{\mu\nu}, K_\rho] = \delta_{\rho\mu} K_\nu - \delta_{\rho\nu} K_\mu, \tag{2.7}$$

which are inherited from the commutation relations of the vector fields (1.5)-(1.8), with the standard sign flip. The reality of the Noether charges determines now the Hermiticity of the operators,

$$D^\dagger = D, \quad M_{\mu\nu}^\dagger = M_{\mu\nu}, \tag{2.8}$$

$$P_\mu^\dagger = K_\mu, \quad K_\mu^\dagger = P_\mu. \tag{2.9}$$

Note the difference from the Hermiticity $P_\mu^\dagger = P_\mu$ in ordinary quantum field theories. Also, it is easy to see that the commutation relations (2.5)-(2.7) are compatible with this Hermiticity.

3 Superconformal symmetry in four dimensions

In what follows, we focus on four spacetime dimensions. $\sigma^\mu_{\alpha\dot\beta}$ are the Euclidean version of the σ-matrices. Explicitly, $\sigma^4 = i\sigma^0 = -i\mathbf{1}$ and $\sigma^{1,2,3}$ are the Pauli matrices. Others are the same as in Wess-Bagger, $\overline{\sigma}^{\mu\dot\alpha\alpha} = \epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta}\sigma^\mu_{\beta\dot\beta}$, $\sigma^{\mu\nu} = \frac{1}{4}(\sigma^\mu\overline{\sigma}^\nu - \sigma^\nu\overline{\sigma}^\mu)$ and $\overline{\sigma}^{\mu\nu} = \frac{1}{4}(\overline{\sigma}^\mu\sigma^\nu - \overline{\sigma}^\nu\sigma^\mu)$.

Superconformal symmetry extends the ordinary supersymmetry

$$\{Q_\alpha, \overline{Q}_{\dot\beta}\} = 2\sigma^\mu_{\alpha\dot\beta} P_\mu, \quad \{Q_\alpha, Q_\beta\} = \{\overline{Q}_{\dot\alpha}, \overline{Q}_{\dot\beta}\} = 0, \tag{3.1}$$

by the conformal symmetry. It includes "special supersymmetry" $S^\alpha, \overline{S}_{\dot\alpha}$ which obey

$$\{S_\alpha, \overline{S}_{\dot\beta}\} = 2\sigma^\mu_{\alpha\dot\beta} K_\mu, \quad \{S_\alpha, S_\beta\} = \{\overline{S}_{\dot\alpha}, \overline{S}_{\dot\beta}\} = 0. \tag{3.2}$$

Other (anti)commutation relations, including the R-charge R, are given as follows;

$$[D, Q_\alpha] = \tfrac{1}{2} Q_\alpha, \quad [D, \overline{Q}_{\dot\alpha}] = \tfrac{1}{2}\overline{Q}_{\dot\alpha}, \quad [D, S_\alpha] = -\tfrac{1}{2} S_\alpha, \quad [D, \overline{S}_{\dot\alpha}] = -\tfrac{1}{2}\overline{S}_{\dot\alpha}, \tag{3.3}$$

$$[R, Q_\alpha] = -Q_\alpha, \quad [R, \overline{Q}_{\dot\alpha}] = \overline{Q}_{\dot\alpha}, \quad [R, S_\alpha] = S_\alpha, \quad [R, \overline{S}_{\dot\alpha}] = -\overline{S}_{\dot\alpha}, \tag{3.4}$$

$$i[M^{\mu\nu}, Q_\alpha] = -\sigma^{\mu\nu}{}_\alpha{}^\beta Q_\beta, \quad i[M^{\mu\nu}, \overline{Q}_{\dot\alpha}] = -\overline{\sigma}^{\mu\nu\dot\alpha}{}_{\dot\beta}\overline{Q}^{\dot\beta},$$
$$i[M^{\mu\nu}, S^\alpha] = S^\beta\sigma^{\mu\nu}{}_\beta{}^\alpha, \quad i[M^{\mu\nu}, \overline{S}_{\dot\alpha}] = \overline{S}_{\dot\beta}\overline{\sigma}^{\mu\nu\dot\beta}{}_{\dot\alpha}, \tag{3.5}$$
$$[P^\mu, S_\alpha] = -\sigma^\mu_{\alpha\dot\alpha}\overline{Q}^{\dot\alpha}, \;\; [P^\mu, \overline{Q}^{\dot\alpha}] = 0, \quad [P^\mu, \overline{S}_{\dot\alpha}] = Q^\alpha\sigma^\mu_{\alpha\dot\alpha}, \;\; [P^\mu, Q^\alpha] = 0, \tag{3.6}$$
$$[K^\mu, Q_\alpha] = -\sigma^\mu_{\alpha\dot\alpha}\overline{S}^{\dot\alpha} \;\; [K^\mu, \overline{S}^{\dot\alpha}] = 0, \quad [K^\mu, \overline{Q}_{\dot\alpha}] = S^\alpha\sigma^\mu_{\alpha\dot\alpha}, \;\; [K^\mu, S^\alpha] = 0, \tag{3.7}$$

and

$$\{Q_\alpha, S^\beta\} = 2i\sigma^{\mu\nu}{}_\alpha{}^\beta M_{\mu\nu} + \delta_\alpha^\beta(2D + 3R), \tag{3.8}$$
$$\{\overline{Q}^{\dot\alpha}, \overline{S}_{\dot\beta}\} = 2i\overline{\sigma}^{\mu\nu\dot\alpha}{}_{\dot\beta}M_{\mu\nu} + \delta^{\dot\alpha}_{\dot\beta}(2D - 3R), \tag{3.9}$$

Eqns (3.3), (3.4) and (3.5) show the dimension, R-charge and spin of the fermionic generators. Eqns (3.6) are the ascent series by the operator P_μ increasing the dimension by 1, while Eqns (3.7) are the descent series by K_μ. Eqns (3.8) and (3.9) are the supersymmetric version of (2.6), and will be important for the unitarity bound.

The generators in a radially-quantized system obey the Hermiticity relations

$$Q_\alpha^\dagger = S^\alpha, \quad \overline{S}_{\dot\alpha}^\dagger = \overline{Q}^{\dot\alpha}. \tag{3.10}$$

and $R^\dagger = R$, in addition to (2.8)-(2.9). This is again significantly different from the relation $Q_\alpha^\dagger = \overline{Q}_{\dot\alpha}$ in ordinary supersymmetric field theories in Minkowski space. Check that the (anti)commutation relations are compatible with this Hermiticity (be careful with the minus sign in, say, $(Q^\alpha)^\dagger = -S_\alpha$).

4 Representation theory — unitarity bounds

Here we study representation of (super)conformal algebra. We are interested in *unitary* representations, namely, those with positive-definite Hermitian inner product with respect to which the Hermiticity (2.8), (2.9), (3.10) holds.

4.1 Conformal algebra

We classify the states with respect to the dimenions = D-eigenvalues. Since $M_{\mu\nu}$ commute with D, each D-level is a representation of $SO(4) \simeq SU(2) \times SU(2)$. By the commutation relation $[D, P_\mu] = P_\mu$ and $[D, K_\mu] = -K_\mu$, P_μ raises the D-level by 1 and K_μ lowers the D-level by 1. A state $|\psi\rangle$ is called a *primary state* if it cannot be lowered by K_μ,

$$K_\mu|\psi\rangle = 0.$$

Unitarity impose the constraints on possible values of the dimensions. By the Hermiticity $K_\mu = P_\mu^\dagger$, we find for a primary state $|\psi\rangle$,

$$0 \leq \|P_\mu|\psi\rangle\|^2 = \langle\psi|K_\mu P_\mu|\psi\rangle = \langle\psi|[K_\mu, P_\mu]|\psi\rangle = 2\langle\psi|D|\psi\rangle, \tag{4.1}$$

where we have used (2.6). Thus the dimension of the primary state must be non-negative. The primary state with $D = 0$ is annihilated by P_μ and hence the operator corresponding to it must be position independent. Let us next consider the norm-squared of $P^\mu P_\mu|\psi\rangle$ (μ-index summed) for a spin-0 primary state $|\psi\rangle$:

$$\begin{aligned}
0 &\leq \|P^\mu P_\mu|\psi\rangle\|^2 = \langle\psi|K^\mu K_\mu P^\nu P_\nu|\psi\rangle \\
&= \langle\psi|K^\mu[K_\mu, P^\nu]P_\nu|\psi\rangle + \langle\psi|K^\mu P^\nu[K_\mu, P_\nu]|\psi\rangle \\
&= 2\langle\psi|K^\mu(\delta_{\mu\nu}D + iM_{\mu\nu})P_\nu|\psi\rangle + 2\langle\psi|K^\mu P^\nu(\delta_{\mu\nu}D + iM_{\mu\nu})|\psi\rangle \\
&= 2\langle\psi|K^\mu(\delta_{\mu\nu}P_\nu(1+D) + \delta_{\nu\mu}P_\nu - \delta_{\nu\nu}P_\mu)|\psi\rangle + 2\delta_{\mu\nu}\langle\psi|K_\mu P_\nu D|\psi\rangle \\
&= 2((1+\Delta) + 1 - 4 + \Delta)\langle\psi|K_\mu P_\mu|\psi\rangle \\
&= 2\cdot 2(\Delta - 1)\Delta \cdot 2 \cdot 4\psi|\psi\rangle
\end{aligned}$$

where Δ is the dimension of $|\psi\rangle$. We see that

$$\Delta = 0, \quad \text{or} \quad \Delta \geq 1. \tag{4.2}$$

A primary state with $\Delta = 1$ corresponds to a conconstant field annihilated by $P^\mu P_\mu \sim \partial^\mu\partial_\mu$. Namely, it must be a free field.

Applying (4.1) for more general primary, one can also show that the primary state in representation V has dimension bounded as

$$\Delta \geq \frac{1}{2}\Big(C_2(V) + C_2(\tfrac{1}{2}, \tfrac{1}{2}) - C_2(V \otimes (\tfrac{1}{2}, \tfrac{1}{2}))_{\min}\Big),$$

where $C_2(V)$ is the quadratic Casimir of the representation V, and $(\frac{1}{2}, \frac{1}{2})$ is the vector representation $\mathbf{R}^4$ of $SO(4)$. Also, $C_2(V \otimes (\frac{1}{2}, \frac{1}{2}))$ may depend on the irreducible components, and $C_2(V \otimes (\frac{1}{2}, \frac{1}{2}))_{\min}$ is the minimum among them. For $V = (j_1, j_2)$ with $j_1, j_2 > 0$, the bound is $\Delta \geq j_1 + j_2 + 2$. For $V = (j, 0)$ or $(0, j)$ with $j > 0$, it is $\Delta \geq j + 1$.

4.2 Superconformal algebra

Q_α and $\overline{Q}^{\dot\alpha}$ raise the dimension by $\frac{1}{2}$, while S^α and $\overline{S}_{\dot\alpha}$ lower the dimension by $\frac{1}{2}$. A state $|\psi\rangle$ is called a *primary state* if it cannot be lowered,

$$S^\alpha|\psi\rangle = \overline{S}_{\dot\alpha}|\psi\rangle = 0.$$

As in ordinary supersymmetric quantum field theory, a state $|\psi\rangle$ is *chiral* (resp. *antichiral*) if it is annihilated by $\overline{Q}^{\dot\alpha}$ (res. Q_α).

Unitarity again constrains the dimension. Let $|\psi\rangle$ be a state with spin 0. Then, we have

$$\|Q_\alpha|\psi\rangle\|^2 + \|S^\alpha|\psi\rangle\|^2 = \langle\psi|(2D+3R)|\psi\rangle,$$
$$\|\overline{Q}^{\dot\alpha}|\psi\rangle\|^2 + \|\overline{S}_{\dot\alpha}|\psi\rangle\|^2 = \langle\psi|(2D-3R)|\psi\rangle.$$

From this we find the bound

$$D \geq \frac{3}{2}|R| \tag{4.3}$$

for any state of spin 0. A spin 0 state with $D = \frac{3}{2}R$ must be annihilated by $\overline{Q}^{\dot\alpha}$ and $\overline{S}_{\dot\alpha}$ and hence must be chiral (one can also show that it must be chiral primary i.e. annihilated also by S^α – see below). Conversely a spin-0 chiral primary state must have $D = \frac{3}{2}R$. A spin 0 state with $D = -\frac{3}{2}R$ must be annihilated by Q_α and $S\alpha$ and hence must be antichiral (it is actually antichiral primary), and a spin-0 antichiral primary must have $D = -\frac{3}{2}R$.

For a state in $SO(4)$ representation V, the bounds are

$$D \geq \frac{3}{2}R + \frac{1}{2}\Big(C_2(V) + C_2(0,\tfrac{1}{2}) - C_2(V\otimes(0,\tfrac{1}{2}))_{\min}\Big),$$
$$D \geq -\frac{3}{2}R + \frac{1}{2}\Big(C_2(V) + C_2(\tfrac{1}{2},0) - C_2(V\otimes(\tfrac{1}{2},0))_{\min}\Big).$$

We now show that a state $|\psi\rangle$ with $D = 3R/2$ (which we know must be annihilated by $\overline{Q}^{\dot\alpha}$ and $\overline{S}_{\dot\alpha}$) must also be annihilated by S^α, and hence is a primary state. First note that $D(S^\alpha\psi) = D(\psi) - \frac{1}{2}$ and $R(S^\alpha\psi) = R(\psi) + 1$. Then, $D(S^\alpha\psi) - 3R(S^\alpha\psi)/2 = D(\psi) - 3R(\psi)/2 - 2 = -2$ which violates the first bound above. Thus it must be that $S^\alpha|\psi\rangle = 0$.

5 Chiral rings

Let us consider the set of chiral operators modulo $\overline{Q}^{\dot\alpha}$-exact ones,

$$\mathcal{R} = \left\{ \mathcal{O} \,\middle|\, \overline{Q}^{\dot\alpha}\mathcal{O} = 0 \right\} \Big/ {\overline{Q}^{\dot\alpha}\Lambda \sim 0}.$$

By the Hermiticity relation $(\overline{Q}^{\dot\alpha})^\dagger = \overline{S}_{\dot\alpha}$, elements of $\mathcal{R}$ are in one-to-one correspondence with the operators annihilated by $\overline{Q}^{\dot\alpha}$ and $\overline{S}_{\dot\alpha}$, namely, chiral primary fields. Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be spin-0 chiral primary fields of dimensions D_1 and D_2 (which must be equal to $\frac{3R(\mathcal{O}_1)}{2}$

and $\frac{3R(\mathcal{O}_2)}{2}$ respectively). Their operator product expansion (OPE) will take the form

$$\mathcal{O}_1(x)\mathcal{O}_2(0) = \sum_I \frac{1}{x^{D_1+D_2-D(\mathcal{U}_I)}}\mathcal{U}_I(0).$$

By the unitarity bound, we know

$$D(\mathcal{U}_I) \geq \frac{3}{2}R(\mathcal{U}_I) = \frac{3}{2}\Big(R(\mathcal{O}_1) + R(\mathcal{O}_2)\Big) = D_1 + D_2.$$

Namely, the powers of x in OPE are all non-negative. Thus, the limit $x \to 0$ is non-singular, and we can define the product of the two operators by simply taking $x = 0$. If it is not zero, it leads to a new chiral primary operator of dimension $D_1 + D_2$. Thus the space of chiral primary fields and hence $\mathcal{R}$ form a ring. We call this the *chiral ring.*

References

For the relation of scale invariance and conformal invariance, see

S. Coleman, "Dilatations," In: *Aspects of Symmetry*, Cambridge University Press - 1985; J. Polchinski, "Scale And Conformal Invariance In Quantum Field Theory," Nucl. Phys. B **303** (1988) 226.

For superconformal invariance in four-dimension, see for example,

T. E. Clark, O. Piguet and K. Sibold, "Supercurrents, Renormalization And Anomalies," Nucl. Phys. B **143** (1978) 445.

For unitarity bounds in superconformal field theories in d=3,4,5,6, see

S. Minwalla, "Restrictions imposed by superconformal invariance on quantum field theories," Adv. Theor. Math. Phys. **2** (1998) 781 [arXiv:hep-th/9712074].

There is a parallel (and simpler in some sense) story for $(2,2)$ superconformal symmetry in two-dimensions. For unitarity bounds and chiral rings in that context, see

W. Lerche, C. Vafa and N. P. Warner, "Chiral Rings In N=2 Superconformal Theories," Nucl. Phys. B **324** (1989) 427.