Constraints on the 1PI functions

Here we derive constraints on the coefficient functions

$$\Gamma^{\mu\nu,ab}(-p,p),\ \Gamma_\psi(-p,p),\ \Gamma_{gh}(-p,p),\ \Gamma_{gh}^{\mu a}(-p-q,q,p)$$

of the 1PI effective action Γ.

(1) Lorentz (or Euclidean) invariance

Euclidean rotation transforms the fields as

$$A_\mu(p) \to A_\nu(\Lambda p)\Lambda^\nu{}_\mu$$

$$\psi(p) \to S(\Lambda)^{-1}\psi(\Lambda p),\quad \bar{\psi}(p) \to \bar{\psi}(\Lambda p)S(\Lambda)$$

$$c(p),\bar{c}(p) \to c(\Lambda p),\bar{c}(\Lambda p)$$

where $\Lambda \mapsto S(\Lambda)$ is the spin representation

$$(S(\Lambda)^{-1}\gamma^\mu S(\Lambda) = \Lambda^\mu{}_\nu\gamma^\nu).$$

Invariance of Γ under this requires

$$\Gamma^{\mu\nu,ab}(-\Lambda p,\Lambda p) = \Lambda^\mu{}_\rho\Lambda^\nu{}_\lambda\Gamma^{\rho\lambda,ab}(-p,p),$$

$$\Gamma_\psi(-\Lambda p,\Lambda p) = S(\Lambda)\Gamma_\psi(-p,p)S(\Lambda)^{-1},$$

$$\Gamma_{gh}(-\Lambda p,\Lambda p) = \Gamma_{gh}(-p,p),$$

$$\Gamma_{gh}^{\mu a}(-\Lambda p-\Lambda q,\Lambda q,\Lambda p) = \Lambda^\mu{}_\nu\Gamma_{gh}^{\nu a}(-p-q,q,p).$$

The solution to this is :

$$\Gamma^{\mu\nu,ab}(-p,p) = \alpha^{ab}(p^2)\,\delta^{\mu\nu} + \beta^{ab}(p^2)\,p^\mu p^\nu,$$

$$\Gamma_\psi(-p,p) = A(p^2)\,\not{p} + B(p^2),$$

$$\Gamma_{gh}(-p,p) = \Gamma_{gh}(p^2),$$

$$\Gamma_{gh}^{\mu a}(-p-q,q,p) = (p+q)^\mu C^a(p,q) + q^\mu D^a(p,q),$$

where $A(p^2)$ & $B(p^2)$ are identities in the spinor factor,

$C^a(p,q)$ & $D^a(p,q)$ depends on (p,q) via $p^2, q^2, p\cdot q$.

(2) Rigid G-invariance

Invariance of Γ under rigid G-transformations

$$A_\mu \to \mathrm{Ad}_g A_\mu \quad (= g A_\mu g^{-1} \text{ for matrix Lie algebra})$$

$$\psi \to g\psi,\quad \bar{\psi} \to \bar{\psi} g^{-1},\quad c \to \mathrm{Ad}_g c,\quad \bar{c} \to \bar{c}\,\mathrm{Ad}_{g^{-1}}$$

requires

$$\alpha^{ab}(p^2) = \delta^{ab}\alpha(p^2),\quad \beta^{ab}(p^2) = \delta^{ab}\beta(p^2),$$

$A(p^2)$ & $B(p^2)$ commute with G,

$$\Gamma_{gh}(p^2) = f_{gh}(p^2)\,\mathrm{id}_{\mathfrak{g}}$$

$$\mathrm{Ad}_g C^a(p,q)\,\mathrm{Ad}_g^{-1} = (\mathrm{Ad}g)_b{}^a\, C^b(p,q),$$ same for $D^a(p,q)$.

Here we have used the assumption that G is simple, which implies that the adjoint representation is irreducible.

By Schur's lemma, any $\mathbb{C}$-linear map $f : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$ that commutes with G is proportional to the identity $\mathrm{id}_{\mathfrak{g}_{\mathbb{C}}}$.

Thus the G-invariance of Γ, which implies that $(\alpha^{ab}(p^2))$, $(\beta^{ab}(p^2))$, $\Gamma_{gh}(p^2)$ commute with $\mathrm{Ad}\,G$,

requires $\alpha^{ab}(p^2) \propto \delta^{ab}$, $\beta^{ab}(p^2) \propto \delta^{ab}$, $\Gamma_{gh}(p^2) \propto \mathrm{id}_{\mathfrak{g}}$

(3) BRST symmetry

Recall the Slavnov-Taylor identity:

If the system has a symmetry $\phi \mapsto \phi + \delta\phi$,
the 1PI effective action $\Gamma[\varphi]$ is invariant under

$$\varphi \mapsto \varphi + \hat{\delta}\varphi \; ; \quad \hat{\delta}\varphi = \langle \delta\varphi \rangle_{J(\varphi)}$$

Let us apply this to the BRST symmetry:

$$\hat{\delta}_B \Gamma[X] = 0.$$

(This is nothing but the Zinn-Justin eqn at $K=0$.)

By ghost # symmetry, Lorentz invariance, and fermion # symmetry, we know possible forms of $\hat{\delta}_B X$:

$$\hat{\delta}_B A_\mu(p) = f(p^2)\, P_\mu C(p) + \text{quadratic or higher in fields}$$

$$\hat{\delta}_B \psi(p) = \text{quadratic or higher in fields}$$

$$\hat{\delta}_B C(p) = \text{quadratic or higher in fields}$$

$$\hat{\delta}_B \bar{C}(p) \propto i B(p)$$

$$\hat{\delta}_B B(p) = 0$$

(To be precise, these are for bare fields, but by the form of renormalization, these hold also for the renormalized fields.)

Then, the terms of $\hat{\delta}_B \Gamma$ which are linear in A_μ & C come only from $\hat{\delta}_B \int \frac{d^4p}{(2\pi)^4} \frac{1}{2} A_{\mu a}(-p)\, \Gamma^{\mu\nu,ab}(-p,p)\, A_{\nu b}(p)$

and that is

$$\int \frac{d^4p}{(2\pi)^4} f(p^2)\,(-p)_\mu\, C_a(-p)\, \Gamma^{\mu\nu,ab}(-p,p)\, A_{\nu b}(p).$$

$\Big($ We used $\Gamma^{\mu\nu,ab}(-p,p) = \Gamma^{\nu\mu,ba}(p,-p)$ which we may assume by definition. $\Big)$

Vanishing of this term requires $p_\mu \Gamma^{\mu\nu,ab}(-p,p) = 0$

For $\Gamma^{\mu\nu,ab}(-p,p) = \delta^{ab}\left(\alpha(p^2)\delta^{\mu\nu} + \beta(p^2) p^\mu p^\nu\right)$, this $\Rightarrow$

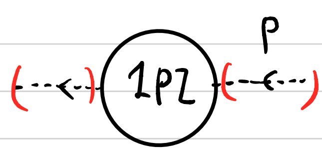
$$\alpha(p^2) p^\nu + \beta(p^2) p^2 p^\nu = 0$$

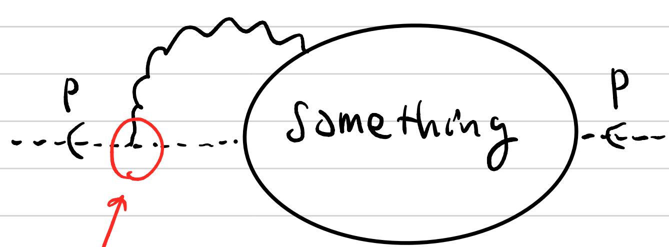
i.e. $\quad \alpha(p^2) = -p^2 \beta(p^2)$.

Thus, we find

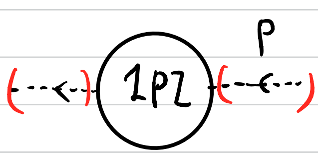
$$\Gamma^{\mu\nu,ab}(-p,p) = \delta^{ab}\left(p^2\delta^{\mu\nu} - p^\mu p^\nu\right)\Pi(p^2).$$

(4) A diagramatic constraint on $\Gamma_{gh}(-p,p)$

$\Gamma_{gh}(-p,p) = -p^2 -$ 

Any 1PI diagram is of the form 

proportional to p_μ

$\therefore$  vanishes if $p_\mu = 0$

$\therefore \ \Gamma_{gh}(-p,p) = f_{gh}(p^2)\,\mathrm{id}_g \Rightarrow f_{gh}(p^2) = 0$ at $p^2 = 0$

$\therefore \ \Gamma_{gh}(-p,p) = p^2\,\Pi_{gh}(p^2)\,\mathrm{id}_g$.

(5) More on $\Gamma^{\mu a}_{gh}(-p-q, q, p)$

Recall (1) & (2) : $\Gamma^{\mu a}_{gh}(-p-q, q, p) = (p+q)^\mu C^a(p,q) + q^\mu D^a(p,q)$,

where $X^a = C^a(p,q)$ & $D^a(p,q) \in \mathrm{End}\, \mathfrak{g}_\mathbb{C}$ obey

$$\mathrm{Ad}_g \circ X^a \circ \mathrm{Ad}_g^{-1} = X^b (\mathrm{Ad}_g)_b{}^a \qquad (\star)$$

Recall $(\mathrm{Ad}_g)_b{}^a$ is the representing matrix of $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$ with respect to a basis $\{e^a\} \subset \mathfrak{g}$, $\mathrm{Ad}_g\, e^a = e^b (\mathrm{Ad}_g)_b{}^a$.

Note that $X^a = \mathrm{ad}\, e^a$ obey $(\star)$:

$$\mathrm{Ad}_g\, \mathrm{ad}\, e^a\, \mathrm{Ad}_g^{-1} X = \mathrm{Ad}_g [e^a, \mathrm{Ad}_g^{-1} X] = [\mathrm{Ad}_g e^a, X]$$

$$= [e^b (\mathrm{Ad}_g)_b{}^a, X] = \mathrm{ad}\, e^b X (\mathrm{Ad}_g)_b{}^a.$$

Are there others ?

$\mathrm{End}\, \mathfrak{g}_\mathbb{C}$ is a representation of G by

$$g \in G : f \in \mathrm{End}_\mathbb{C}\, \mathfrak{g}_\mathbb{C} \longmapsto \mathrm{Ad}_g \circ f \circ \mathrm{Ad}_g^{-1},$$

and it can be decomposed into irreducible representations.

Suppose this includes m-copies of the adjoint representation

$$\mathrm{End}\, \mathfrak{g}_\mathbb{C} \cong \underbrace{\mathfrak{g}_\mathbb{C} \oplus \cdots \oplus \mathfrak{g}_\mathbb{C}}_{m} \oplus \text{other representations}$$

<u>Examples</u> · $G = SU(2)$

$$\mathrm{End}\,\mathfrak{g}_{\mathbb{C}} = \mathrm{End}\,\mathbf{3} \cong \mathbf{3} \otimes \mathbf{3} \cong \mathbf{1} \oplus \mathbf{3} \oplus \mathbf{5}.$$

$m = 1$

· $G = SU(3)$

$$\mathrm{End}\,\mathfrak{g}_{\mathbb{C}} \cong \ydiagram{2,1} \otimes \ydiagram{2,1} \cong \ydiagram{4,2} \oplus \ydiagram{3} \oplus \ydiagram{3,3} \oplus \ydiagram{2,1} \oplus \ydiagram{2,1} \oplus \mathbb{C}.$$

$m = 2$

Let $X_I^a \in \mathrm{End}\,\mathfrak{g}_{\mathbb{C}}$ be the image of $e^a \in \mathfrak{g}$ into the I-th copy ($I = 0, 1, \cdots, m-1$). We regard $\mathrm{ad}\,\mathfrak{g}_{\mathbb{C}}$ as the 0-th copy, so that $X_0^a = \mathrm{ad}\,e^a$. Then, we can expand

$$C^a(p,q) = \sum_I X_I^a C_I(p,q), \quad D^a(p,q) = \sum_I X_I^a D_I(p,q).$$

<u>Comment on renormalization condition</u>

Renormalization condition is imposed just on $C_0(p,q)$:

$$C_0(p,q)\Big|_{p^2 = q^2 = (p+q)^2 = \mu^2} = -i.$$

We should not impose any condition on

$C_I(p,q)$ with $I > 0$ & $D_I(p,q)$ $\forall I$.

Thus, the condition presented at the class (= the one in the note)

$$C^a(p,q)\Big|_{p^2=q^2=(p+q)^2=\mu^2} = -ie^a$$

was too strong in general (it was OK if $m=1$ eg. $G=SU(2)$).

That is corrected in the revised version of the note.