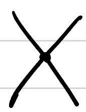
proof of ①

$$-W(J) = -\frac{1}{2}\log\left(\det\frac{A}{2\pi}\right) + \sum D$$

D : connected vacuum diagram
↳ "c.v.d." below

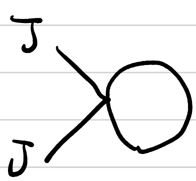
of the perturbation theory with

propagator $\quad i \text{———} j \quad = \overbrace{\phi_i \phi_j} = A^{-1}_{ij}$

vertices  etc $\quad = -\frac{\lambda}{4!}\phi^4$ etc

$J\text{——} \quad = J \cdot \phi$

e.g. $\quad J\text{——}J \quad = \frac{1}{2!} J\overbrace{\phi\ J\phi} = \frac{1}{2} J^i A^{-1}_{ij} J^j$

 $\quad = \frac{1}{2!} J\phi\ J\phi\ -\frac{\lambda}{4!}\phi\phi\phi\phi \times \binom{4}{2} \cdot 2$

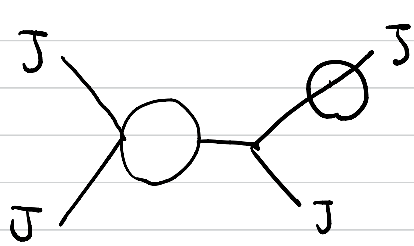
$N_J(D) :=$ # J's in D

$N_{1PI}(D) :=$ # 1PI blocks in D

$N_{s.l.}(D) :=$ # separating lines in D

a separating line := a line s.t. the diagram is disconnected if cut

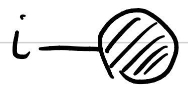
e.g.

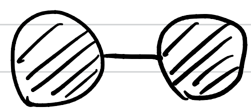


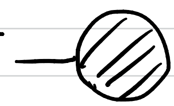
$N_J = 4$

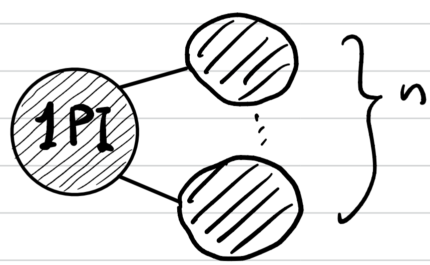
$N_{1PI} = 3$

$N_{sl} = 6$

$\langle \phi_i \rangle_J =$  sum of connected diagrams with one external line ending at i.

- $\displaystyle\sum_{D:\,\text{c.v.d.}} N_{\text{s.l.}}(D)\cdot D =$  $\displaystyle = \frac{1}{2}\langle \phi_i \rangle_J A^{ij} \langle \phi_j \rangle_J$

- $\displaystyle\sum_{D:\,\text{c.v.d.}} N_J(D)\cdot D =$  $= J^i \langle \phi_i \rangle_J$

- $\displaystyle\sum_{D:\,\text{c.v.d.}} N_{1PI}(D)\cdot D = \sum_{n=0}^{\infty}$ 

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{i_1 \cdots i_n} \lambda_{1PI}^{i_1 \cdots i_n} \langle \phi_{i_1} \rangle_J \cdots \langle \phi_{i_n} \rangle_J$$

If we regard 1PI blocks as vertices, then a c.v.d. D is a tree diagram (a diagram without a loop) s.t.

$$E = N_J, \quad V = N_{1PI}, \quad P = N_{\text{s.l.}}$$

$$\therefore\ 0 = L = P - E - V + 1 = N_{\text{s.l.}} - N_J - N_{1PI} + 1$$

$$\therefore\ -N_{\text{s.l.}}(D) + N_J(D) + N_{1PI}(D) = 1.$$

$$-W(J)+\frac{1}{2}\log\det\left(\frac{A}{2\pi}\right) = \sum_{D:\text{c.v.d}} D$$

$$= \sum_{D:\text{c.v.d.}} \left(-N_{s.l.}(D)+N_J(D)+N_{1PI}(D)\right)\cdot D$$

$$= -\frac{1}{2}\langle\phi_i\rangle_J A^{ij}\langle\phi_j\rangle_J + J^i\langle\phi_i\rangle_J + \sum_{n=0}^{\infty}\frac{1}{n!}\lambda_{1PI}^{i_1\cdots i_n}\langle\phi_{i_1}\rangle_J\cdots\langle\phi_{i_n}\rangle_J$$

—— (☆)

Set $J=J(\phi)$ here : $\langle\phi_i\rangle_{J(\phi)}=\phi_i$

$$-W(J(\phi))+\frac{1}{2}\log\det\left(\frac{A}{2\pi}\right)$$

$$= -\frac{1}{2}\phi_i A^{ij}\phi_j + J^i(\phi)\phi_i + \sum_{n=0}^{\infty}\frac{1}{n!}\lambda_{1PI}^{i_1\cdots i_n}\phi_{i_1}\cdots\phi_{i_n}$$

$$\therefore \Gamma(\phi) = W(J(\phi))+J(\phi)\cdot\phi$$

$$= \frac{1}{2}\log\det\left(\frac{A}{2\pi}\right)+\frac{1}{2}\phi_i A^{ij}\phi_j - \sum_{n=0}^{\infty}\frac{1}{n!}\lambda_{1PI}^{i_1\cdots i_n}\phi_{i_1}\cdots\phi_{i_n}$$

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proof of ②

$$e^{-\Gamma(\phi)} = e^{-W(J(\phi)) - J(\phi)\cdot\phi}$$

$$= \int d\phi' \, e^{-S(\phi') + J(\phi)\cdot\phi' - J(\phi)\cdot\phi}$$

$\phi' = \phi + \xi$

$$= \int d_\phi \xi \, e^{-S_\phi(\xi) + J(\phi)\cdot\xi} = e^{-W^\phi(J(\phi))}.$$

$$W^\phi, \ \langle\cdots\rangle^\phi_J \ \text{etc} := W, \ \langle\cdots\rangle_J \ \text{etc for } \underbrace{\mathcal{J}(\phi)}_{\text{theory with background } \phi}.$$

$$Z^\phi \cdot \langle \xi_i \rangle^\phi_{J(\phi)} = \int d_\phi \xi \, e^{-S_\phi(\xi) + J(\phi)\cdot\xi} \, \xi_i$$

$$= \int d\phi' \, e^{-S(\phi') + J(\phi)\cdot(\phi' - \phi)} (\phi'_i - \phi_i)$$

$$= e^{-J(\phi)\cdot\phi} \left(\langle \phi_i \rangle_{J(\phi)} - \phi_i \right) = 0$$

Apply (☆) to $\mathcal{J}(\phi)$ with $J = J(\phi)$:

$$-\Gamma(\phi) = -W^\phi(J(\phi))$$

$$\overset{(☆)}{=} -\frac{1}{2}\log\det\left(\frac{A}{2\pi}\right) - \frac{1}{2} \overset{0}{\langle \xi_i \rangle^\phi_{J(\phi)}} A^{ij} \overset{0}{\langle \xi_j \rangle^\phi_{J(\phi)}}$$

$$+ J^i(\phi) \overset{0}{\langle \xi_i \rangle^\phi_{J(\phi)}} + \sum_{n=0}^{\infty} \frac{1}{n!} \lambda^{\phi \, i_1 \cdots i_n}_{1PI} \overset{0}{\langle \xi_{i_1} \rangle^\phi_{J(\phi)}} \cdots \overset{0}{\langle \xi_{i_n} \rangle^\phi_{J(\phi)}}$$

Only $n=0$ remains!

$$= -\frac{1}{2}\log\det\left(\frac{A}{2\pi}\right) + \underbrace{\lambda^{\mathrm{vac}\,\phi}_{1PI}}_{}$$

sum of 1PI vacuum diagrams of $J(\varphi)$. //

<u>proof of (3)</u> (The case $N=1$ for simplicity)

$$S(\varphi) = \frac{1}{2}(a+b)\varphi^2 - P(\lambda,\varphi)$$

polynomial of φ with parameter λ

Perturbation theory P1 : $S = \underbrace{\frac{1}{2}(a+b)\varphi^2}_{\text{free}} - \underbrace{P(\lambda,\varphi)}_{\text{interaction}}$

vs

Perturbation theory P2 : $S = \underbrace{\frac{1}{2}a\varphi^2}_{\text{free}} + \underbrace{\frac{1}{2}b\varphi^2 - P(\lambda,\varphi)}_{\text{interaction}}$

In P1, $a+b \in \mathbb{C}$, $\mathrm{Re}(a+b)>0$, λ : formal parameter

In P2, $a \in \mathbb{C}$, $\mathrm{Re}(a)>0$, b & λ : formal parameters.

For comparison to make sense, $a, b \in \mathbb{C}$, $|b| < \mathrm{Re}\, a$,

$f(a+b) = (a+b)^{-r}$ in P1 is regarded as

$\sum_{m=0}^{\infty} \frac{1}{m!} f^{(m)}(a)\, b^m$ in P2.

With this understanding

$$Z_{P1} = \sum_{n=0}^{\infty} \frac{1}{n!} \int d\phi \; e^{-\frac{1}{2}(a+b)\phi^2} P(\lambda,\phi)^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \underline{\int d\phi \; e^{-\frac{1}{2}a\phi^2} \sum_{m=0}^{\infty} \frac{1}{m!} \left(-\frac{1}{2}b\phi^2\right)^m P(\lambda,\phi)^n}$$

absolutely convergent for $|b| < \mathrm{Re}\, a$

$$= \sum_{n,m=0}^{\infty} \frac{1}{n!}\frac{1}{m!} \int d\phi \; e^{-\frac{1}{2}a\phi^2} \left(-\frac{1}{2}b\phi^2\right)^m P(\lambda,\phi)^n$$

$$= \sum_{K=0}^{\infty} \frac{1}{K!} \sum_{m=0} \binom{K}{m} \int d\phi \; e^{-\frac{1}{2}a\phi^2} \left(-\frac{1}{2}b\phi^2\right)^m P(\lambda,\phi)^{K-m}$$

$$= \sum_{K=0}^{\infty} \frac{1}{K!} \int d\phi \; e^{-\frac{1}{2}a\phi^2} \underbrace{\sum_{m=0} \binom{K}{m} \left(-\frac{1}{2}b\phi^2\right)^m P(\lambda,\phi)^{K-m}}_{\left(-\frac{1}{2}b\phi^2 + P(\lambda,\phi)\right)^K}$$

$$= Z_{P2}$$

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